JOURNAL OF COMBINATORIAL THEORY, Series B 52, 191–218 (1991)

Permutation–Partition Pairs. III. Embedding Distributions of Linear Families of Graphs

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Communicated by the Editors

Received January 4, 1988

For any fixed graph H, an H-linear family of graphs is a sequence $\{G_n\}_{n=1}^{\infty}$ of graphs in which G_n consists of n copies of H that have been linked in a consistent manner so as to form a chain. Generating functions for the region distribution of any such family are found. It is also shown that the minimum genus and the average genus of G_n are essentially linear functions of n. © 1991 Academic Press, Inc.

INTRODUCTION

The genus distribution of the orientable 2-cell embeddings of various families of graphs has been the subject of a considerable amount of attention in recent years [1, 4, 5, 13–16, 18, 19]. This article is concerned with the distributions of graphs that, loosely speaking, consist of a chain of links that are joined to each other in a consistent manner. It is reasonable to expect that the genus distribution of such a chain should be a function of its length and the genus distribution of the link. We produce recursion matrices that depend only on the link and the linking scheme and make this dependence explicit.

In Chapter 1 we introduce the notion of a permutation–partition pair. These pairs are generalizations of graphs and are the formalization of the concept of a graph with a partially specified embedding. They have proven useful elsewhere [2, 3, 11, 12] and are essential to this paper. A reduction procedure is described here that shows how to translate distribution problems regarding a given permutation–partition pair into questions regarding smaller pairs. This sets the stage for the recursion of Chapter 2. In this second chapter the above mentioned chains are formalized. Recursion matrices are associated with these chains and these matrices are in turn used to obtain explicit generating functions for the distributions of the chains. In Chapter 3 the Peron–Frobenius theory of stochastic matrices is

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used to show that the average genus of a chain is an essentially linear function of its length. Finally, the recursion matrices are applied in Chapter 4 to the derivation of the minimum and maximum genera of a chain, and linear programming techniques are employed to prove a stability theorem.

For general graph theoretical background and information the reader is referred to [7, 8, 17].

1. Permutation–Partition Pairs

The composition of permutations is to be read here from left to right and the number of factors in the disjoint cycle factorization of the permutation P is denoted by $\|P\|$.

A *permutation–partition pair* (P, Π) consists of a permutation P and a partition $\Pi = \{\Pi_i\}_{i=1}^{k}$, both defined over a common underlying finite set S. This notion was first introduced in [11] and then independently rediscovered, in a more geometrical form, in [2, 3], where such a pair is called an IDS. To every pseudograph G we can associate a permutation–partition pair (P_G, Π_G) in the following manner. Let D_G be the symmetric digraph obtained by replacing each edge e of G with a pair if oppositely oriented arcs e_1, e_2. Let

$$S_G = \bigcup_{e \in E(G)} \{e_1, e_2\}$$

$$P_G = \prod_{e \in E(G)} (e_1 e_2)$$

$$\Pi_v = \{\text{arcs of } D_G \text{ emanating from } v \in V(G)\}$$

$$\Pi_G = \{\Pi_v\}_{v \in V(G)}.$$

We refer to the arcs of D_G as the *bits* of G.

It is clear that P_G is necessarily a fixed-point free involution and that every pair (P, Π) wherein P is a fixed point free involution arises in the same manner from some graph. Given a pair (P, Π) we denote by $S(\Pi)$ the set of all permutations Q of the underlying set S such that the orbits (cyclic factors) of Q agree with the members of the partition Π. If $Q \in S(\Pi)$ the permutation pair (P, Q) is called an *embedding* of the permutation–partition pair (P, Π). In the case of a pair (P_G, Π_G) that arises from a graph G the above defined embeddings of (P_G, Π_G) are in a one-to-one correspondence with the orientable embeddings of G. In this correspondence an embedding of G on an oriented surface Σ determines the corresponding embedding (P_G, Q_G) of (P_G, Π_G) by using the counterclockwise sense at each vertex $v \in V(G)$ to endow the corresponding member $(\Pi_G)_v$ of Π_G with a cyclic order. For example, if G is the graph K_4 of Fig. 1, then $P_G = (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ a)(b\ c)$ and $\Pi_G = \{\{1, 3, 5\}, \{2, 7, b\}, \{4, 8, 9\}, \{6, a, c\}\}$. This correspondence is such that the cyclic factors of $P_G Q_G$ describe the regions of the embedding. Figure 1 contains two embeddings of $G = K_4$ and displays the corresponding embeddings of (P_G, Π_G). Embeddings of arbitrary pairs (P, Π), where P is not necessarily a fixed point free involution, can also be given a geometrical interpretation (see [2, 3, 16]). As this interpretation is irrelevant to the subsequent discussion it will be ignored here.

We now go on to show that arbitrary permutation–partition pairs have a natural interpretation as graphs with embedding constraints. If $a, b \in S$ are in the same member of Π we shall write

$$a \equiv b \qquad (\text{mod } \Pi).$$

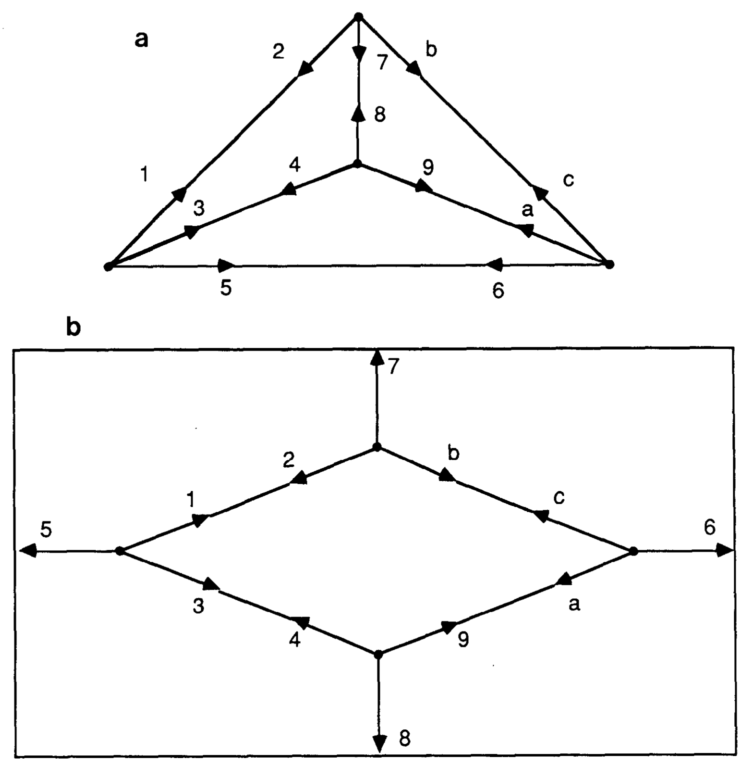

FIG. 1. (a) A plane embedding of K_4 with $Q_G = (531)(ca6)(849)(27b)$, $P_G Q_G = (174)(25c)(396)(8ba)$; (b) a toroid embedding of K_4 with $Q_G = (531)(ca6)(894)(2b7)$, $P_G Q_G = (1ba4)(25c79638)$.

A constraint on the pair (P, Π) is an ordered pair, denoted by $a \to b$, such that $a \equiv b \pmod{\Pi}$. A *constraint set* C on (P, Π) is a set

$$C = \{a_i \to b_i\}_{i=1}^{c},$$

where the $a_i \to b_i$ are constraints, the a_i are all distinct, and the b_i are also all distinct. If C is a constraint set on (P, Π), then we define

$$S(\Pi, C) = \{Q \in S(\Pi) \mid a_i Q = b_i,\ i = 1, 2, ..., c\}.$$

Thus it is the function of a constraint set of (P, Π) to prespecify some of the action of the otherwise arbitrary permutation Q of $S(\Pi)$. It is clear that each $b \in S$ induces a partition

$$S(\Pi) = \bigcup_{a \equiv b,\ a \neq b} S(\Pi, a \to b).$$

For any permutation P on S let P/b denote the permutation of $S - \{b\}$ obtained by deleting b from the disjoint cycle factorization of P. Thus, if $P = (1\ 2\ 3)(4\ 5)(6)$, then $P/1 = (2\ 3)(4\ 5)(6)$, $P/5 = (1\ 2\ 3)(4)(6)$, and $P/6 = (1\ 2\ 3)(4\ 5)$. Similarly, for any partition Π of S, Π/b denotes the partition of $S - \{b\}$ obtained by deleting b from its containing member in Π. For any constraint $a \to b$ of (P, Π) we denote by $(P, \Pi)/a \to b$ the pair $(\bar{P}, \bar{\Pi})$ defined over the set $\bar{S} = S - \{b\}$, where

$$\bar{\Pi} = \Pi/b$$

$$\bar{P} = \begin{cases} P(b\ \ a\ \ bP)/b & \text{if}\ \ a \neq bP,\ a \neq b,\ b \neq bP \\ P(b\ \ a)/b = P/b & \text{if}\ \ a = bP \neq b \text{ or } a = b \neq bP \\ P/b & \text{if}\ \ b = bP \end{cases}$$

For example, if

$$(P, \Pi) = ((1\ 2\ 3\ 4)(5\ 6\ 7)(8), \{\{1, 3, 5, 6\}, \{2, 4, 7, 8\}\}),$$

then

$$(P, \Pi)/3 \to 1 = ((2)(3\ 4)(5\ 6\ 7)(8), \{\{3, 5, 6\}, \{2, 4, 7, 8\}\}),$$

$$(P, \Pi)/5 \to 1 = ((2\ 3\ 4\ 5\ 6\ 7)(8), \{\{3, 5, 6\}, \{2, 4, 7, 8\}\}),$$

$$(P, \Pi)/6 \to 1 = ((2\ 3\ 4\ 6\ 7\ 5)(8), \{\{3, 5, 6\}, \{2, 4, 7, 8\}\}),$$

$$(P, \Pi)/2 \to 4 = ((2\ 3)(1)(5\ 6\ 7)(8), \{\{1, 3, 5, 6\}, \{2, 7, 8\}\}),$$

$$(P, \Pi)/7 \to 2 = ((3\ 4\ 1\ 7\ 5\ 6)(8), \{\{1, 3, 5, 6\}, \{4, 7, 8\}\}),$$

$$(P, \Pi)/8 \to 2 = ((3\ 4\ 1\ 8)(5\ 6\ 7), \{\{1, 3, 5, 6\}, \{4, 7, 8\}\}),$$

$$(P, \Pi)/6 \to 5 = ((1\ 2\ 3\ 4)(6\ 7)(8), \{\{1, 3, 6\}, \{2, 4, 7, 8\}\}).$$

$$(P, \Pi)/2 \to 8 = ((1\ 2\ 3\ 4)(5\ 6\ 7), \{\{1, 3, 5, 6\}, \{2, 4, 7\}\}).$$

The reader may find the following observations regarding the derivation of $\bar{P}$ from P helpful. If a and b are in the same orbit $B=(a\ d\ \cdots\ e\ b\ f\ \cdots\ g)$ of P, then $\bar{P}$ is obtained from P by splitting B (at a and at b) into two cycles, and suppressing b in one, so as to obtain $B_1 B_2=(a\ d\ \cdots\ e)(f\ \cdots\ g)$; all the other orbits of P are inherited intact by $\bar{P}$. If a and b are in distinct orbits $B_1=(a\ d\ \cdots\ e)$ and $B_2=(b\ f\ \cdots\ g)$ of P, then $\bar{P}$ is obtained by coalescing B_1 and B_2 into a single cycle and deleting b so as to get the cycle $B=(a\ d\ \cdots\ e\ f\ \cdots\ g)$; again, all the other orbits of P are inherited intact by $\bar{P}$.

Our strategy will be to describe a reduction procedure that replaces an arbitrary pair (P, Π) by a set of smaller pairs, thus allowing for standard recursion procedures to be applied. Specifically, if (P, Π) is an arbitrary pair and b is an arbitrary element of the underlying set S, then (P, Π) will be replaced by

$$\{(P, \Pi)/a \to b \mid a \equiv b (\operatorname{mod} \Pi),\ a \neq b\}.$$

LEMMA 1.1. *Let (P, Π) be a pair with bits $a \neq b$, and let $(\bar{P}, \bar{\Pi})=(P, \Pi)/a \to b$. Then the function*

$$f\colon S(\Pi; a \to b) \to S(\bar{\Pi})$$

defined by

$$f(Q)=(a\ b)Q/b \overset{\text{def}}{=} \bar{Q}$$

is a bijection such that

$$\|PQ\| = \begin{cases} \|\bar{P}\bar{Q}\|+1 & \text{if } a=bP\neq b \\ \|\bar{P}\bar{Q}\| & \text{otherwise.} \end{cases}$$

Proof. Note that for any $Q \in S(\Pi; a \to b)$, $(a\ b)Q$ has (b) as a singleton, and hence f is clearly injective. It is surjective because for any $\bar{Q} \in S(\bar{\Pi})$, $(a\ b)\bar{Q}$ maps a to b and

$$f((a\ b)\bar{Q})=\bar{Q}.$$

Note that

$$P=\begin{cases} \bar{P}(bP\ a\ b) & \text{if } a\neq bP\neq b \\ \bar{P}(a\ b) & \text{if } a=bP\neq b \\ \bar{P}(b) & \text{if } b=bP. \end{cases}$$

Hence, if $a \neq bP \neq b$, since b is in the domain of neither $\bar{P}$ nor $\bar{Q}$,

$$\begin{aligned} \|PQ\| &= \|\bar{P}(bP\ a\ b)(a\ b)\bar{Q}\| \\ &= \|\bar{P}(bP\ b)\bar{Q}\| = \|\bar{P}\bar{Q}\|. \end{aligned}$$

On the other hand, if $a = bP \neq b$,

$$\begin{aligned}\|PQ\| &= \|\bar{P}(a\ \ b)(a\ \ b)\bar{Q}\| = \|\bar{P}(b)\bar{Q}\| \\ &= \|\bar{P}\bar{Q}\| + 1.\end{aligned}$$

If $b = bP$ then $\|PQ\| = \|\bar{P}(b)(a\ \ b)\bar{Q}\| = \|\bar{P}\bar{Q}\|$. Q.E.D.

The *orbit distribution* of the pair (P, Π) is the function

$$r_{(P,\Pi)}(k) = |\{Q \in S(\Pi) \mid \|PQ\| = k\}|.$$

Note that a simple parity argument implies that if $r_{(P,\Pi)}(k) \neq 0$ for some k, then $r_{(P,\Pi)}(k \pm 1) = 0$.

COROLLARY 1.2. *If (P, Π) is a permutation–partition pair, $b \in S$, and $\{b\} \notin \Pi$, then*

$$r_{(P,\Pi)}(k) = \sum_{a \equiv b(\mathrm{mod}\ \Pi),\, a \neq b} r_{(P,\Pi)/a \to b}(k - \delta_{a,bP}),$$

where

$$\delta_{a,bP} = \begin{cases} 0 & \text{if } a \neq bP \\ 1 & \text{if } a = bP. \end{cases}$$

It is also necessary to have a version of the above reduction process that disposes of the degenerate case, where $\{b\}$ is a singleton member of Π. Accordingly, if $\Pi = \{\Pi_i\}_{i=1}^k$ and $\{b\} = \Pi_{i_0}$ we define

$$\Pi/b = \{\Pi - \{\Pi_{i_0}\}\}$$
$$(\bar{P}, \bar{\Pi})/b \to b = (P/b, \Pi/b).$$

LEMMA 1.3. *Let (P, Π) be a pair such that $\{b\} \in \Pi$, and let $(\bar{P}, \bar{\Pi}) = (P, \Pi)/b \to b$. Then the function*

$$f\colon S(\Pi) \to S(\bar{\Pi})$$

defined by

$$f(Q) = Q/b \overset{\text{def}}{=} \bar{Q}$$

is a bijection such that

$$\|PQ\| = \begin{cases} \|\bar{P}\bar{Q}\| & \text{if } b \neq bP \\ \|\bar{P}\bar{Q}\| + 1 & \text{if } b = bP. \end{cases}$$

Proof. The function f is clearly a bijection, since it merely deletes the orbit (b) from Q. Note that if $b \neq bP$, then $\bar{P} = P/b = P(b\ \ bP)/b$, where (b) is an orbit of $P(b\ \ bP)$. Hence, in this case,

$$\|PQ\| = \|\bar{P}(b\ \ bP) \circ (b)\bar{Q}\| = \|\bar{P}\bar{Q}\|$$

since b is in the domain of neither $\bar{P}$ nor $\bar{Q}$. On the other hand, if $b = bP$, then (b) is an orbit of both P and Q, and so clearly

$$\|PQ\| = \|\bar{P}\ (b)\ \bar{Q}\| = \|\bar{P}\bar{Q}\| + 1. \qquad \text{Q.E.D.}$$

COROLLARY 1.4. *If (P, Π) is a permutation–partition pair such that $\{b\} \in \Pi$, then*

$$r_{(P,\Pi)}(k) = r_{(P,\Pi)/b \to b}(k - \delta_{b,bP}),$$

where

$$\delta_{b,bP} = \begin{cases} 0 & \text{if } b \neq bP \\ 1 & \text{if } b = bP. \end{cases}$$

For any graph G let $r_G(k)$ denote the number of 2-cell orientable embeddings of G that have exactly k regions. This function is the region distribution of G. If (P_G, Π_G) is the pair associated above with G, then

$$r_G(k) = r_{(P_G, \Pi_G)}(k).$$

The average $\mu_{(P,\Pi)}$ of the orbit distribution $r_{(P,\Pi)}$, where $\Pi = \{\Pi_i\}_{i=1}^{k}$ and $p_i = |\Pi_i|$ is

$$\mu_{(P,\Pi)} = \left[\prod_{i=1}^{k} (p_i - 1)!\right]^{-1} \sum_{k=1}^{\infty} k r_{(P,\Pi)}(k)$$

and we set

$$\mu_G = \mu_{(P_G, \Pi_G)}.$$

The complete reduction diagram $\mathscr{T}_{(P,\Pi)}$ of the pair (P, Π) is a weighted directed graph that pictures the above reduction process and is constructed as follows. Let $\tilde{S} = (b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n)$ be any linear ordering of the set S that underlies the pair (P, Π) and let $\mathscr{T}^{(0)}_{(P,\Pi)} = \{(P, \Pi)\}$. Assuming that the vertex set $\mathscr{T}^{(i)}_{(P,\Pi)}$ $(0 \leqslant i < n)$ has been defined, let $(\bar{P}, \bar{\Pi})$ be any vertex in this set $\mathscr{T}^{(i)}_{(P,\Pi)}$. If $\{b_{i+1}\}$ is a singleton member of $\bar{\Pi}$, then $(\bar{P}, \bar{\Pi})$ has only one descendent, namely, $(\bar{P}, \bar{H})/b_{i+1} \to b_{i+1}$. If on the other hand $\{b_{i+1}\}$ is not a singleton member of $\bar{\Pi}$, then each of the pairs $(\bar{P}, \bar{\Pi})/a \to b_{i+1}$ $(a \equiv b_{i+1},\ a \neq b_{i+1})$ is a descendent of $(\bar{P}, \bar{\Pi})$. Each branch from $(\bar{P}, \bar{\Pi})$ to any of its descendent $(\bar{P}, \bar{\Pi})/a \to b_{i+1}$ is assigned the weight $\delta_{a, b_{i+1}P}$. The vertex set $\mathscr{T}^{(i+1)}_{(P,\Pi)}$ consists of the set of all the descendents of all the vertices in $\mathscr{T}^{(i)}_{(P,\Pi)}$. It is clear that each pair of vertices $(\bar{P}, \bar{\Pi})$ and $(\bar{\bar{P}}, \bar{\bar{\Pi}})$ in $\mathscr{T}^{(i)}_{(P,\Pi)}$ have $\bar{\Pi} = \bar{\bar{\Pi}}$ and also have the same underlying set $\bar{S} = \{b_{i+1}, b_{i+2}, ..., b_n\}$. The set $\mathscr{T}^{(n)}_{(P,\Pi)}$ consists of only the trivial pair (ϕ, ϕ). The next lemma follows immediately from Corollaries 1.2 and 1.4.

LEMMA 1.5. *The embeddings of the pair (P, Π) are in a one-to-one correspondence with the directed paths of $\mathscr{T}_{(P,\Pi)}$ that start from (P, Π) and end at (ϕ, ϕ). This correspondence is such that the number of regions of the embedding (P, Q) (i.e. $\|PQ\|$) is given by the sum of the weights along the corresponding path.*

If (P, Π) is the pair that corresponds to the graph G, then $\mathscr{T}_{(P,\Pi)}$ will be written as $\mathscr{T}_G$. By the partial reduction diagram $\mathscr{T}^{(\leqslant i)}_{(P,\Pi)}$ we shall mean the portion of $\mathscr{T}_{(P,\Pi)}$ that is induced by $\bigcup_{j \leqslant i} \mathscr{T}^{(i)}_{(P,\Pi)}$. Examples of such diagrams appear in Figs. 2, 3, where $\bar{\Pi}$ has been suppressed.

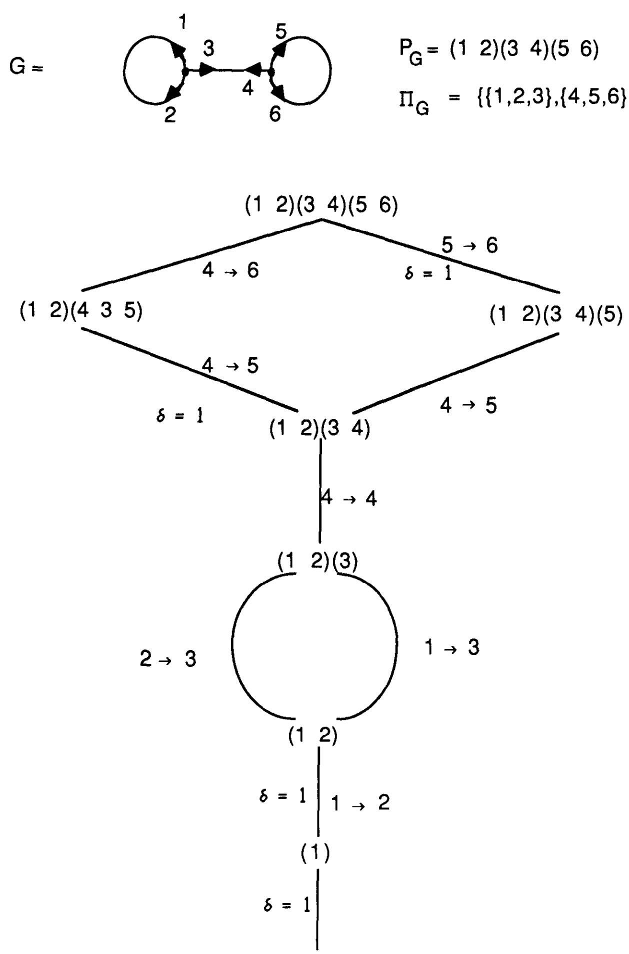

FIGURE 2

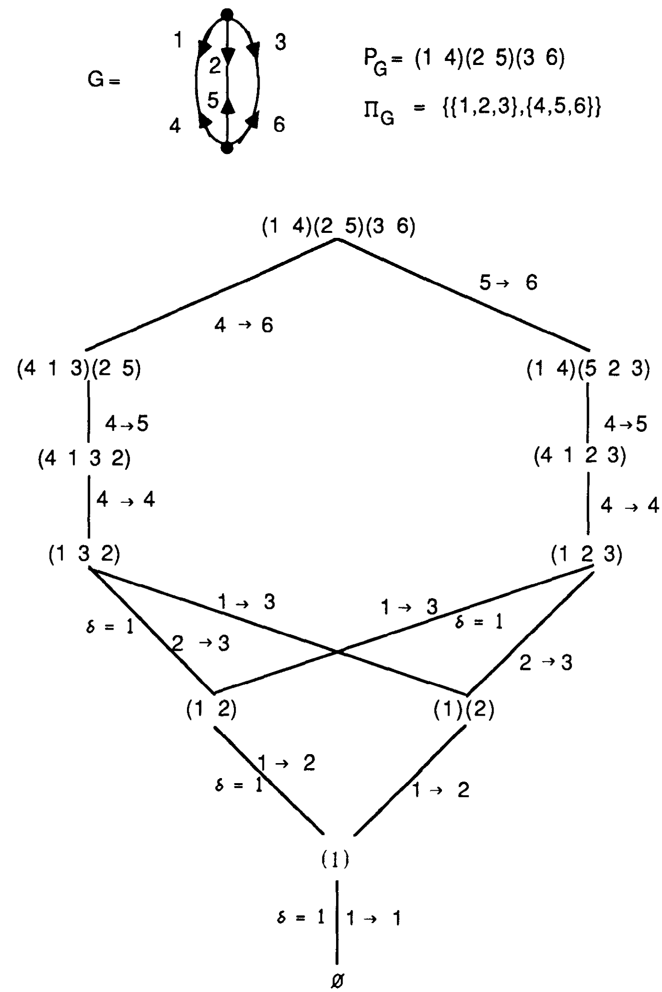

FIGURE 3

2. Generating Functions

Let H be a graph. Loosely speaking, a family of graphs $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ is said to be H-linear if each G_n consists of a chain of n copies of H each of which is linked to the next in a consistent manner. More formally, let U_1 and U_2 be two disjoint subsets of $V(H)$ and ι: $U_2 \to U_1$ be a bijection. For each $n \geqslant 1$ let H_n be a copy of H with an isomorphism f_n: $H_n \to H$ which is read as simultaneously mapping the vertices, edges, and bits of H_n onto those of H. Finally, G_n is constructed from $\{H_1, H_2, ..., H_n\}$ by identifying each vertex v of $(U_2)f_i^{-1} \subseteq H_i$ with the vertex $(v)f_i \iota f_{i+1}^{-1}$ of $(U_1)f_{i+1}^{-1} \subseteq H_{i+1}$, for each $i = 1, 2, ..., n-1$.

EXAMPLE 2.1. The cobblestone path [4] of Fig. 4. Here $U_i = \{u_i\}$, $i = 1, 2$ and $\iota(u_2) = u_1$.

EXAMPLE 2.2. The diamond band of Fig. 5. Here $U_i = \{u_i\}$, $i = 1, 2$ and $\iota(u_2) = u_1$.

EXAMPLE 2.3. The ladder [4] of Fig. 6. Here $U_i = \{u_i, v_i\}$, $i = 1, 2$, $\iota(u_2) = u_1$, $\iota(v_2) = v_1$.

Let the H-linear family $\mathscr{G} = \{G_n\}_{n=1}^{\infty}$ be fixed. We go on to associate with this family a sequence $\{T^{(\varepsilon)}\}_{\varepsilon=0}^{\infty}$ of nonnegative integer matrices which describe a recursion between G_n and those of G_{n-1}. Let (P, Π) be the permutation–partition pair associated with H with the underlying set S consisting of the bits of H. For each permutation α of S let $(P_{G_n}^{\alpha}, \Pi_{G_n}^{\alpha})$ be the permutation–partition pair such that

$$S_{G_n}^{\alpha} = S_{G_n}$$
$$P_{G_n}^{\alpha} = P_{G_{n-1}} \circ (f_n \alpha f_n^{-1})$$
$$\Pi_{G_n}^{\alpha} = \Pi_{G_n}.$$

Let $\tilde{S} = (b_1 \leqslant b_2 \leqslant \cdots \leqslant b_h)$ be a linear ordering of the set $S = D(H)$. This order induces the following linear order on S_{G_n},

$$x \leqslant y \begin{cases} \text{if} & x \in D(H_i), \quad y \in D(H_j), \quad \text{and} \quad i > j, \\ \text{if} & x, y \in D(H_i) \quad \text{and} \quad (x)f_i \leqslant (y)f_i. \end{cases}$$

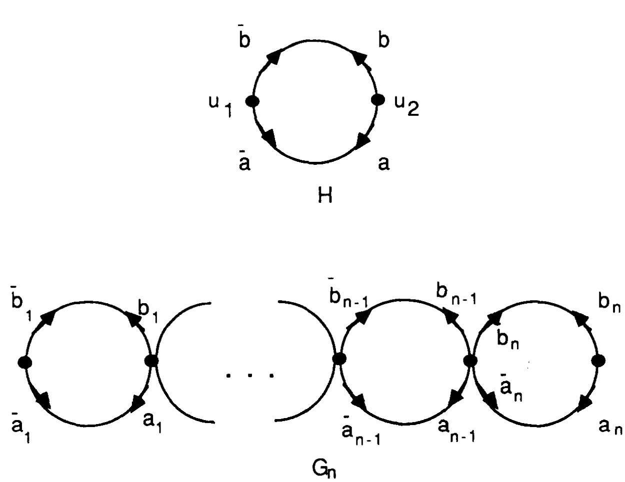

FIGURE 4

Given any permutation α of $S=D(H)$, we shall say that β is an immediate copermutation of α if there exists a pair

$$(P^{\beta}_{G_{n-1}}, \Pi_{G_{n-1}}) \in \mathscr{T}^{(h)}_{(P^{\alpha}_{G_n}, \Pi_{G_n})}.$$

In other words, β is an immediate copermutation of α if there is a sequence of reductions (i.e., a path in the corresponding reduction diagram) that leads from $P^{\alpha}_{G_n}$ to $P^{\beta}_{G_{n-1}}$ for some n. Since $P^{\alpha}_{G_n}$ is obtained by replacing the edges of the nth copy of H with α, this means that, loosely speaking, the permutation β is a "descendent" of α in this reduction proces. The reader may find Examples 2.4 and 2.5 below to be helpful. In general, the permutation β of S will be called a *copermutation* of α if there is a sequence $\alpha=\beta_1, \beta_2, ..., \beta_t=\beta$ wherein each β_{i+1} is an immediate copermutation of β_i $(1 \leqslant i < t)$, or else β_i is an immediate copermutation of β_{i+1}. Set $\alpha_1 = P$, the graphical involution on S, and let $\{\alpha_1, \alpha_2, ..., \alpha_d\}$ be the set of all the copermutations of α_1. For the integers $\varepsilon \geqslant 0$, $1 \leqslant i, j \leqslant c$, define $T^{(\varepsilon)}_{ij}=0$ if α_j is not an immediate copermutation of α_i. On the other hand, if α_j is an immediate copermutation of α_i, then $T^{(\varepsilon)}_{ij}$ is the number of directed paths in

$$\mathscr{T}_{(P^{\alpha_i}_{G_n}, \Pi_{G_n})} \quad \text{from} \quad (P^{\alpha_i}_{G_n}, \Pi_{G_n}) \quad \text{to} \quad (P^{\alpha_j}_{G_{n-1}}, \Pi_{G_{n-1}})$$

whose weights add up to ε.

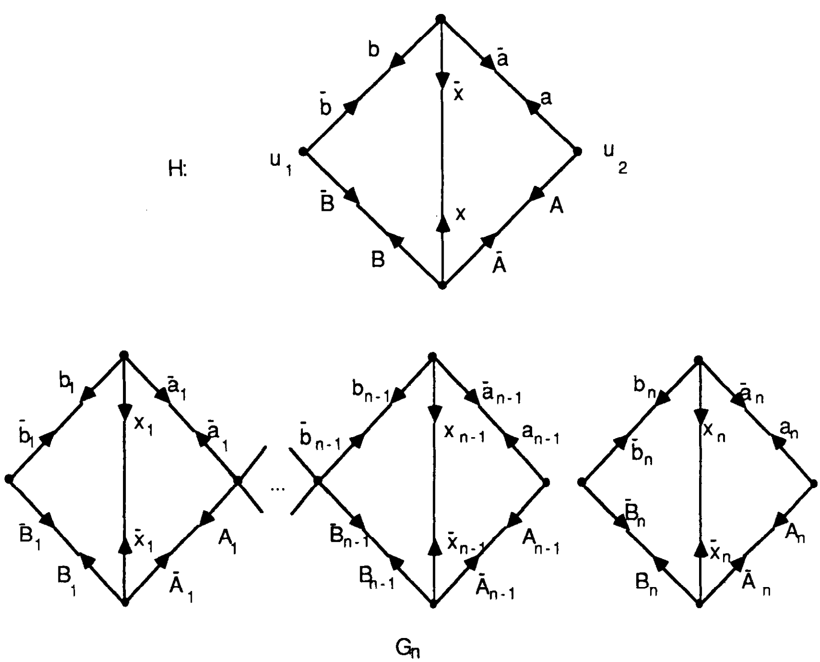

FIGURE 5

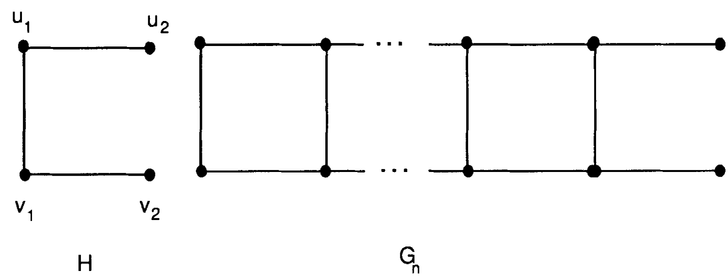

FIGURE 6

By Lemma 1.5

$$r_{(P^{\alpha_i}_{G_n},\, \Pi_{G_n})}(k) = \sum_{\varepsilon=0}^{h} \sum_{j=1}^{d} T^{(\varepsilon)}_{ij}\, r_{(P^{\alpha_j}_{G_{n-1}},\, \Pi_{G_{n-1}})}(k-\varepsilon).$$

Or, if we let $\mathbf{r}^{(n)}(k)$ be the d-vector whose jth component is

$$r^{(n)}_j(k) = r_{(P^{\alpha_j}_{G_n},\, \Pi_{G_n})}(k), \qquad 1 \leqslant j \leqslant d,$$

and also set

$$T^{(\varepsilon)} = (T^{(\varepsilon)}_{ij}), \qquad 1 \leqslant i, j \leqslant d$$

then

$$\mathbf{r}^{(n)}(k) = \sum_{\varepsilon=0}^{h} T^{(\varepsilon)} \mathbf{r}^{(n-1)}(k-\varepsilon). \tag{2.1}$$

Of course, $r^{(n)}_1(k)$ is the component of $\mathbf{r}^{(n)}(k)$ that is of most interest to us, since it is the one that refers to the given graph G_n, whereas the other components of $\mathbf{r}^{(n)}(k)$ provide information about the auxiliary permutation–partition pairs.

EXAMPLE 2.4. The cobblestone path. With the bits of H labelled as in Fig. 4, we choose $\tilde{S} = (b \leqslant a \leqslant \bar{b} \leqslant \bar{a})$ and write $(x) f_n^{-1}$ as x_n for every $x \in \tilde{S}$. With $\alpha_1 = (a\ \bar{a})(b\ \bar{b})$, it follows from the reductions of Figs. 7 and 8 that $\alpha_2 = (a\ \bar{a}\ b\ \bar{b})$ is the only other copermutation of α_1 and that $T^{(0)}_{1\,1} = 0$, $T^{(0)}_{1\,2} = 2$, $T^{(1)}_{1\,1} = 4$, $T^{(1)}_{1\,2} = 0$, $T^{(0)}_{2\,1} = 6$, $T^{(0)}_{2\,2} = T^{(1)}_{2\,1} = T^{(1)}_{2\,2} = 0$. Clearly, $T^{(\varepsilon)}_{ij} = 0$ for all $\varepsilon > 1$. Thus,

$$T^{(0)} = \begin{pmatrix} 0 & 2 \\ 6 & 0 \end{pmatrix}, \qquad T^{(1)} = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}.$$

EXAMPLE 2.5. The diamond band. With the bits of H labelled as in Fig. 5, we choose $S = \{A \leqslant a \leqslant \bar{A} \leqslant \bar{a} \leqslant \bar{x} \leqslant x \leqslant B \leqslant b \leqslant \bar{B} \leqslant \bar{b}\}$. With $\alpha_1 =$

$(a\ \bar{a})(b\ \bar{b})(x\ \bar{x})(A\ \bar{A})(B\ \bar{B})$, it can be shown that $\alpha_2 = (a\ \bar{a}\ A\ \bar{A})(b\ \bar{b})$ $(x\ \bar{x})(B\ \bar{B})$ is the only other copermutation of α_1 and

$$T^{(0)} = \begin{pmatrix} 12 & 0 \\ 0 & 8 \end{pmatrix}, \qquad T^{(1)} = \begin{pmatrix} 0 & 4 \\ 16 & 0 \end{pmatrix}, \qquad T^{(2)} = \begin{pmatrix} 8 & 0 \\ 0 & 0 \end{pmatrix}.$$

For each of the copermutations α_i define a generating function

$$F_i(x, y) = \sum_{k=0,\, n \geqslant 1}^{\infty} r_i^{(n)}(k) x^k y^n.$$

These generating functions, of which $F_1(x, y)$ is, of course, the generating function for the region distribution of the family $\mathscr{G}$ can be easily derived in

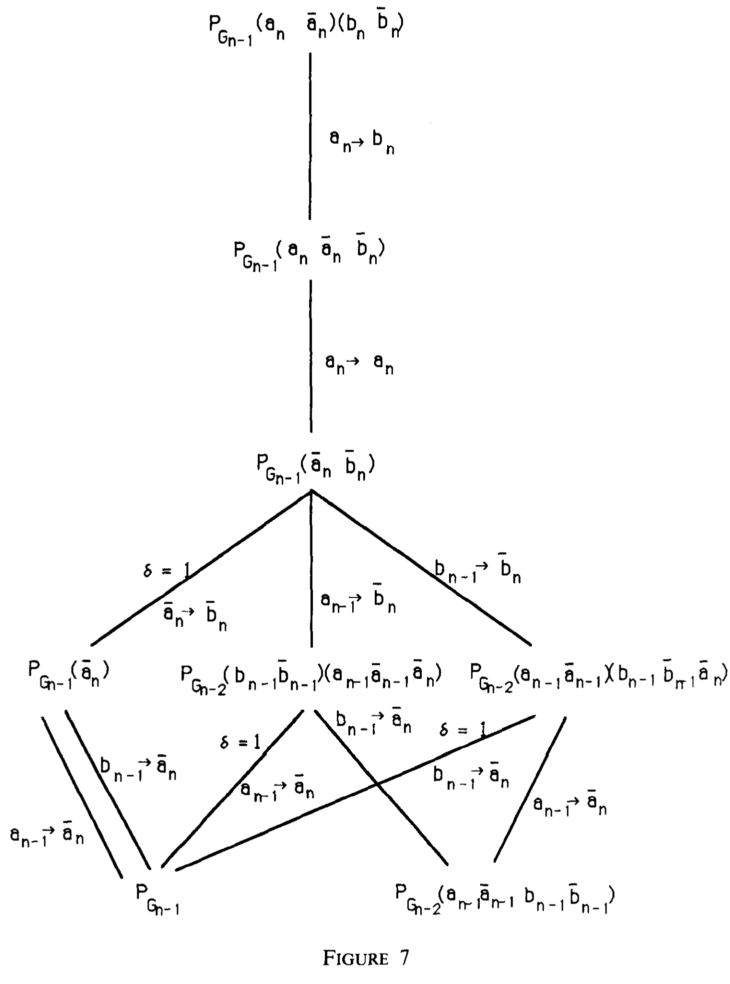

FIGURE 7

terms of the matrices $\{T^{(\varepsilon)}\}_{\varepsilon=0}^{\infty}$. For it follows from (2.1) that if $T_i^{(\varepsilon)}$ is the ith row of $T^{(\varepsilon)}$, then, for $n \geqslant 2$

$$r_{(k)}^{(n)} = \sum_{\varepsilon=0}^{h} T_i^{(\varepsilon)} \mathbf{r}^{(n-1)}(k-\varepsilon) = \sum_{\varepsilon=0}^{h} \sum_{j=1}^{d} T_{ij}^{(\varepsilon)} r_j^{(n-1)}(k-\varepsilon).$$

Set $Q_i(x) = \sum_{k=1}^{\infty} r_i^{(1)}(k) x^k$. It then follows from the above that

$$\begin{aligned} F_i(x, y) &= yQ_i(x) + \sum_{n=2}^{\infty} \sum_{k=1}^{\infty} \sum_{\varepsilon=0}^{h} \sum_{j=1}^{d} T_{ij}^{(\varepsilon)} r_j^{(n-1)}(k-\varepsilon) x^k y^n \\ &= yQ_i(x) + \sum_{j=1}^{d} \sum_{\varepsilon=0}^{h} T_{ij}^{(\varepsilon)} x^{\varepsilon} y \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} r_j^{(n)}(k) x^k y^n. \end{aligned}$$

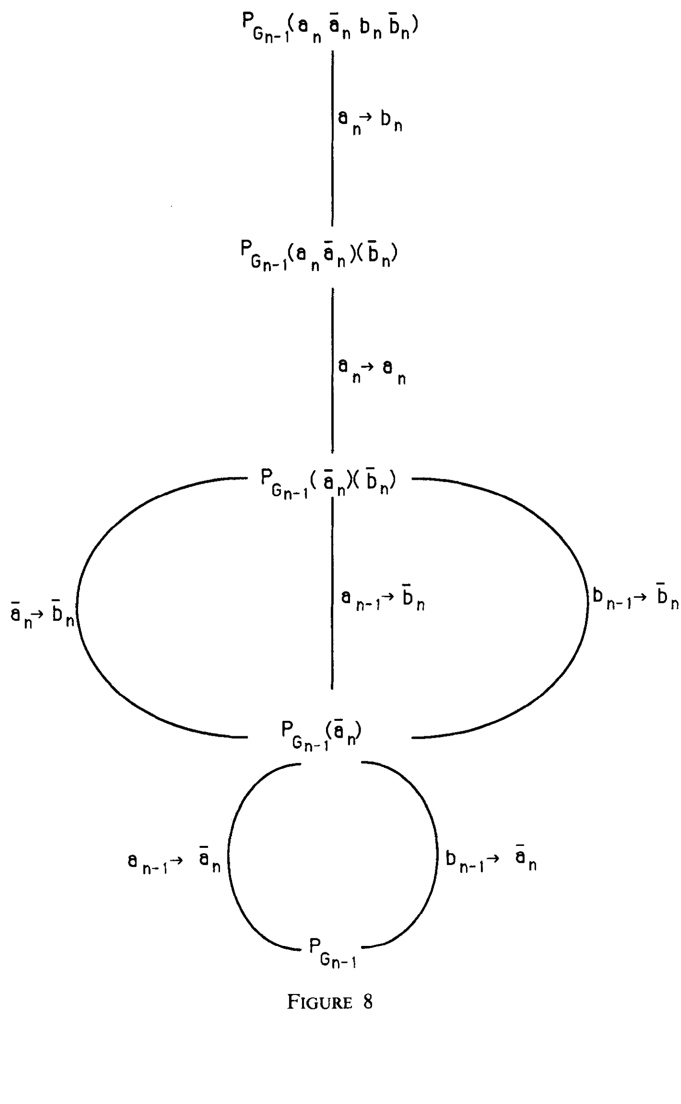

FIGURE 8

If we now define the polynomial matrix P via

$$P_{ij} = \sum_{\varepsilon=0}^{h} T_{ij}^{(\varepsilon)} x^{\varepsilon} y$$

and also the column vectors $\mathbf{F} = (F_1(x, y), \mathbf{F}_2(x, y), \ldots)$, $\mathbf{Q} = (Q_1(x), Q_2(x), \ldots)$ then the above equations can be written compactly as

$$\mathbf{F} = P\mathbf{F} + y\mathbf{Q}.$$

Since $P(0, 0)$ is the zero matrix it follows that $I - P$ is invertible and so

$$\mathbf{F} = y(I - P)^{-1}\mathbf{Q}. \tag{2.2}$$

The considerations which lead to (2.2) are now summarized.

THEOREM 2.6. *Let $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ be an H-linear family. Then there exists a positive integer h, a $d \times d$ matrix with entries in $\mathbb{Z}[x, y]$, and a d-vector* $\mathbf{Q}$ *with components in $\mathbb{Z}[x]$, such that $I - P$ is invertible and the first component of the d-vector*

$$\mathbf{F} = y(I - P)^{-1}\mathbf{Q}.$$

is the generating function for the region distributions of $\mathcal{G}$.

More specifically, if the said first component is

$$\sum_{k,n} a_{kn} x^k y^n,$$

then G_n has a_{kn} embeddings with k regions.

EXAMPLE 2.7. For the cobblestone path of Example 2.4 above we have

$$P = y\begin{pmatrix} 0 & 2 \\ 6 & 0 \end{pmatrix} + xy\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 4xy & 2y \\ 6y & 0 \end{pmatrix}.$$

To compute $\mathbf{Q}$ we note that

$$r_1^{(1)}(k) = \begin{cases} 1 & \text{if } k = 2 \\ 0 & \text{otherwise} \end{cases}$$

so that $Q_1(x) = x^2$. On the other hand $\alpha_2 = (a\ \bar{a}\ b\ \bar{b})$ and

$$(a\ \bar{a}\ b\ \bar{b})(a\ b)(\bar{a}\ \bar{b}) = (a\ \bar{b}\ b\ \bar{a}),$$

so that

$$r_2^{(1)}(k) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$Q_2(x) = x.$$

Thus,

$$\mathbf{F} = y\begin{pmatrix} 1-4xy & -2y \\ -6y & 1 \end{pmatrix}^{-1}\begin{pmatrix} x^2 \\ x \end{pmatrix}$$
$$= y(1-4xy-12y^2)^{-1}\begin{pmatrix} x^2+2xy \\ 2x^2y+x \end{pmatrix}.$$

Hence the number of embeddings of G_n that have k regions is the coefficient of $x^k y^n$ in the polynomial

$$(x^2y+2xy^2)\sum_{m=0}^{\infty} 4^m y^m (x+3y)^m,$$

or

$$2^{n+k-2}3^{(n-k-1)/2}\begin{pmatrix} \dfrac{n+k-3}{2} \\ \dfrac{n-k-1}{2} \end{pmatrix} + 2^{n+k-3}3^{(n-k+1)/2}\begin{pmatrix} \dfrac{n+k-3}{2} \\ \dfrac{n-k+1}{2} \end{pmatrix},$$

which agrees with the result of [4].

EXAMPLE 2.8. The diamond band of Example 2.5. Here

$$P = y\begin{pmatrix} 12 & 0 \\ 0 & 8 \end{pmatrix} + xy\begin{pmatrix} 0 & 4 \\ 16 & 0 \end{pmatrix} + x^2y\begin{pmatrix} 8 & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 12y+8x^2y & 4xy \\ 16xy & 8y \end{pmatrix}.$$

To compute $\mathbf{Q}$ we note that if $\Pi = \{\{\bar{a}, x, b\}, \{\bar{A}, \bar{x}, B\}, \{\bar{b}, \bar{B}\}, \{a, A\}\}$ then

$$r_{(\alpha_1,\Pi)}(k) = \begin{cases} 2 & \text{if } k=1 \\ 2 & \text{if } k=3 \\ 0 & \text{otherwise,} \end{cases}$$

$$r_{(\alpha_2,\Pi)}(k) = \begin{cases} 4 & \text{if } k=2 \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$Q_1(x) = 2x + 2x^3, \qquad Q_2(x) = 4x^2,$$

and

$$\mathbf{F} = y \begin{pmatrix} 1 - 12y - 8x^2y & -4xy \\ -16xy & 1 - 8y \end{pmatrix}^{-1} \begin{pmatrix} 2x + 2x^3 \\ 4x^2 \end{pmatrix}$$

$$= y(1 - 20y + 96y^2 - 8x^2y)^{-1} \begin{pmatrix} 2x + 2x^3 - 16xy \\ 4x^2 - 16x^2y \end{pmatrix}.$$

This yields

$$F_1 = 2xy(1 + x^2 - 8y) \sum_{m=0}^{\infty} (20y + 8x^2y - 96y^2)^m.$$

3. The Average Genus

We begin with a lemma about the growth rate of solutions to certain of simultaneous and nonhomogeneous difference equations. This is then used to show that modulo a certain periodicity in the constant term, the average genus of the member G_n of a linear family $\mathscr{G}$ is asymptotically linear in n.

Lemma 3.1. *Let T be a row stochastic $m \times m$ matrix and let $\mathbf{q}$, $\{\mathbf{v}^{(n)}\}_{n=1}^{\infty}$ be m-vectors such that*

$$\mathbf{v}^{(n+1)} = T\mathbf{v}^{(n)} + \mathbf{q} \qquad \textit{for} \quad n = 1, 2, 3, 4, \ldots .$$

If $v_j^{(n)}$ is the jth component of $\mathbf{v}^{(n)}$, then for each $j = 1, 2, \ldots, m$ there exist numbers a_j, b_j^n and integers n_j such that

$$v_j^{(n)} = a_j n + b_j^n + o(1)$$

and

$$b_j^n = b_j^{n'} \qquad \textit{whenever} \quad n \equiv n' \pmod{n_j}.$$

Proof. Let S be a nonsingular matrix that brings T to its rational form. In other words,

$$S^{-1}TS = \begin{pmatrix} B_1 & & \\ & B_2 & \\ & & \ddots \end{pmatrix},$$

where each B_i has the form

$$B_i = \begin{pmatrix} \lambda_i & & & & \\ 1 & \lambda_i & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \lambda_i \end{pmatrix}.$$

If we set $\mathbf{u}^{(n)} = S^{-1}\mathbf{v}^{(n)}$ and $\mathbf{p} = S^{-1}\mathbf{q}$, then

$$\mathbf{u}^{(n+1)} = S^{-1}TS\mathbf{u}^{(n)} + \mathbf{p}, \qquad n = 1, 2, 3, ... \tag{3.1}$$

Suppose now that the indices $\alpha+1, \alpha+2, ..., \alpha+\beta$ correspond to the irreducible block

$$B = \begin{pmatrix} \lambda & & & & & \\ 1 & \lambda & & & & \\ & 1 & \lambda & & & \\ & & 1 & \ddots & & \\ & & & \ddots & & \\ & & & & 1 & \lambda \end{pmatrix}$$

of $S^{-1}TS$. If $\mathbf{u}_j^{(n)}$ and $\mathbf{p}_j$ are the jth components of $\mathbf{u}^{(n)}$ and $\mathbf{p}$, respectively, then it follows from (3.1) that

$$\left.\begin{aligned} v_{\alpha+1}^{(n+1)} &= \lambda v_{\alpha+1}^{(n)} + p_{\alpha+1} \\ v_{\alpha+\mu}^{(n+1)} &= v_{\alpha+\mu-1}^{(n)} + \lambda v_{\alpha+\mu}^{(n)} + p_{\alpha+\mu} \end{aligned}\right\} \quad \begin{aligned} & n = 1, 2, 3, ... \\ & \mu = 2, 3, 4, ..., \beta. \end{aligned}$$

Consequently,

$$v_{\alpha+1}^{(n+1)} = \lambda^n v_{\alpha+1}^{(1)} + (1 + \lambda + \cdots + \lambda^{n-1})\, p_{\alpha+1}. \tag{3.2}$$

Since T is stochastic it has spectral radius 1. Moreover, it is known [6, Chap. XII] that if $|\lambda| = 1$ then $\beta = 1$ and λ is in fact a root of unity. Thus, when $|\lambda| = 1$ we have

$$v_{\alpha+1}^{(n+1)} = \begin{cases} v_{\alpha+1}^{(1)} + np_{\alpha+1} & \text{if } \lambda = 1 \\ \left(v_{\alpha+1}^{(1)} - \dfrac{p_{\alpha+1}}{1-\lambda}\right)\lambda^n + \dfrac{p_{\alpha+1}}{1-\lambda} & \text{otherwise.} \end{cases}$$

Finally, if $|\lambda| < 1$, then (3.2) becomes

$$v_{\alpha+1}^{(n+1)} = \lambda^n v_{\alpha+1}^{(1)} + \frac{1-\lambda^n}{1-\lambda}\, p_{\alpha+1}.$$

It now follows by induction on μ that for each $\mu = 1, 2, \ldots, \beta$ there exist a constant $c^{(\mu)}$ and a polynomial $P^{(\mu)}(x)$ of degree $< \mu$, both independent of n, such that

$$v_{\alpha+\mu}^{(n+1)} = \lambda^n P^{(\mu)}(n) + c^{(\mu)}, \qquad \mu = 1, 2, \ldots, \beta.$$

Since $|\lambda| < 1$, it follows that

$$v_{\alpha+\mu}^{(n)} = c^{(\mu)} + o(1).$$

The lemma now follows immediately from the fact that $\mathbf{v}^{(n)} = S\mathbf{u}^{(n)}$. Q.E.D.

THEOREM 3.2. *Let $\mathscr{G} = \{G_n\}_{n=1}^{\infty}$ be an H-linear family of graphs. Then there exist a constant A, an integer p, and a periodic sequence $\{B_n\}_{n=1}^{\infty}$ with $B_m = B_n$ whenever $m \equiv n$ (mod p), such that*

$$\mu_{G_n} = An + B_n + o(1).$$

Proof. Let t_n be the total number of embeddings of G_n. Then, since each member of

$$\mathscr{T}^{(h)}_{(P_{G_n}, \Pi_{G_n})}$$

has the same number of embeddings as G_{n-1},

$$t_n / t_{n-1} = \sum_{\varepsilon=0}^{h} \sum_{j=1}^{d} T_{ij}^{\varepsilon} \qquad \text{for each} \quad i = 1, 2, \ldots, d.$$

Hence, if $\boldsymbol{\mu}^{(n)}$ is the d-vector whose jth entry is

$$t_n^{-1} \sum_{k=1}^{\infty} k r_{(P_{G_n}^{\alpha_j}, \Pi_{G_n})}(k),$$

then

$$\begin{aligned}
\boldsymbol{\mu}^{(n)} &= t_n^{-1} \sum_{k=1}^{\infty} k \mathbf{r}^{(n)}(k) \\
&= t_n^{-1} \sum_{k=1}^{\infty} k \sum_{\varepsilon=0}^{h} T^{(\varepsilon)} \mathbf{r}^{(n-1)}(k-\varepsilon) \\
&= t_n^{-1} \sum_{\varepsilon=0}^{h} T^{(\varepsilon)} \sum_{k=1}^{\infty} (k-\varepsilon) \mathbf{r}^{(n-1)}(k-\varepsilon) \\
&\quad + t_n^{-1} \sum_{\varepsilon=0}^{h} T^{(\varepsilon)} \sum_{k=1}^{\infty} \varepsilon \mathbf{r}^{(n-1)}(k-\varepsilon) \\
&= t_n^{-1} \sum_{\varepsilon=0}^{h} T^{(\varepsilon)} \sum_{k=1}^{\infty} k \mathbf{r}^{(n-1)}(k) + t_n^{-1} \sum_{\varepsilon=0}^{h} T^{(\varepsilon)} \sum_{k=1}^{\infty} \varepsilon \mathbf{r}^{(n-1)}(k) \\
&= \left(\frac{t_{n-1}}{t_n} \sum_{\varepsilon=0}^{h} T^{(\varepsilon)} \right) \boldsymbol{\mu}^{(n-1)} + \left(\frac{t_{n-1}}{t_n} \sum_{\varepsilon=0}^{h} \varepsilon T^{(\varepsilon)} \right) \mathbf{1},
\end{aligned}$$

where $\mathbf{1} = (1, 1, ..., 1)$. By the definition of the $T^{(\varepsilon)}$, the matrix

$$T = \frac{t_{n-1}}{t_n} \sum_{\varepsilon=0}^{h} T^{(\varepsilon)}$$

is independent of n and is row stochastic. Setting

$$\mathbf{t} = \left(\frac{t_{n-1}}{t_n} \sum_{\varepsilon=0}^{h} \varepsilon T^{(\varepsilon)} \right) \mathbf{1},$$

the column vector $\mathbf{t}$ is also independent of n. Thus,

$$\boldsymbol{\mu}^{(n)} = T\boldsymbol{\mu}^{(n-1)} + \mathbf{t}. \tag{3.3}$$

Since μ_{G_n} is in fact the first component of $\boldsymbol{\mu}^{(n)}$, Lemma 3.1 implies the validity of the theorem. Q.E.D.

EXAMPLE 3.3. The cobblestone path. It follows from Example 2.4 that

$$T = \frac{1}{6} \left\{ \begin{pmatrix} 0 & 2 \\ 6 & 0 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \right\} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ 1 & 0 \end{pmatrix}$$

and

$$\mathbf{t} = \frac{1}{6} \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}.$$

Because of its small dimension it is actually easier to work with T rather than its rational form. Here (3.3) becomes

$$\boldsymbol{\mu}^{(n)} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ 1 & 0 \end{pmatrix} \boldsymbol{\mu}^{(n-1)} + \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}.$$

In other words,

$$\mu_1^{(n)} = \tfrac{2}{3}\mu_1^{(n-1)} + \tfrac{1}{3}\mu_2^{(n-1)} + \tfrac{2}{3}$$
$$\mu_2^{(n)} = \mu_1^{(n-1)}.$$

Or,

$$\mu_1^{(n)} = \tfrac{2}{3}\mu_1^{(n-1)} + \tfrac{1}{3}\mu_1^{(n-2)} + \tfrac{2}{3}.$$

Since $\mu_1^{(1)} = 2$ and $\mu_1^{(2)} = \frac{7}{3}$ it can be easily shown that

$$\mu_1^{(n)} = \frac{n}{2} + \frac{11}{8} - \frac{3}{8}\left(-\frac{1}{3} \right)^n = \frac{n}{2} + \frac{11}{8} + o(1).$$

EXAMPLE 3.4. The diamond band D_n. It follows from Example 2.5 that

$$T=\begin{pmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \qquad \mathbf{t}=\begin{pmatrix} \frac{5}{6} \\ \frac{2}{3} \end{pmatrix}.$$

The eigenvalues of T are 1 and $\frac{1}{6}$ with corresponding eigenvectors $(1, 1)$ and $(1, -4)$. Hence, if we set

$$S=\begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}$$

then

$$S^{-1}TS=\frac{1}{5}\begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{4}{6} & \frac{2}{6} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{6} \end{pmatrix}$$

and

$$\mathbf{p}=\frac{1}{5}\begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \frac{5}{6} \\ \frac{2}{3} \end{pmatrix}=\begin{pmatrix} \frac{4}{5} \\ \frac{1}{30} \end{pmatrix}.$$

It is easily verified by direct calculations that

$$\left.\begin{matrix} \mu_1^{(1)}=2 \\ \mu_2^{(1)}=2 \end{matrix}\right\} \qquad \text{or} \qquad \boldsymbol{\mu}^{(1)}=\begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

and so, if $\mathbf{v}^{(n)}=S^{-1}\boldsymbol{\mu}^{(n)}$

$$\mathbf{v}^{(1)}=\begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Thus we now have

$$\mathbf{v}^{(n)}=\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{6} \end{pmatrix}\mathbf{v}^{(n-1)}+\begin{pmatrix} \frac{4}{5} \\ \frac{1}{30} \end{pmatrix}$$

$$\mathbf{v}^{(1)}=\begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

which has

$$\mathbf{v}^{(n)}=\begin{pmatrix} \dfrac{4n}{5}+\dfrac{6}{5} \\ \dfrac{1}{25}\left(1-\dfrac{1}{6^{n-1}}\right) \end{pmatrix}$$

as its solution. But $\boldsymbol{\mu}^{(n)} = S\mathbf{v}^{(n)}$ and hence

$$\mu_{D_n} = \mu_1^{(n)} = \frac{4n}{5} + \frac{31}{25} - \frac{1}{25 \cdot 6^{n-1}}.$$

4. The Minimum Genus

The recursion matrix $T^{(\varepsilon)}$ can also be used to obtain information about the minimum (and maximum) genus of the H-linear graphs. Since a genus embedding of a graph is one that maximizes the number of the regions of the embeddings, we define the *maximum orbiticity* $\rho(P, \Pi)$ of an arbitrary permutation–partition pair as follows:

$$\rho(P, \Pi) = \max\{\|PQ\| \mid Q \in S(\Pi)\}.$$

Let $\mathscr{T}_G$ be the complete reduction diagram of the graph G and set $\rho(G) = \rho(P_G, \Pi_G)$. If L varies over all the maximal directed paths of $\mathscr{T}_G$ that start at its root, and if $\varepsilon(L)$ denotes the sum of the weights over all the edges of L, then it follows from Lemma 1.5 that

$$\rho(G) = \max_L \{\varepsilon(L)\}.$$

The recursion matrices $\{T_{ij}^{(\varepsilon)}\}$ associated with the H-linear family $\mathscr{G}$ are now used to define a new *weighted increment digraph* $I_{\mathscr{G}}$. The vertices of $I_{\mathscr{G}}$ are the copermutations $\alpha_1, \alpha_2, ..., \alpha_d$ and there is an arc from α_i to α_j if and only $T_{ij} > 0$. Furthermore, if a is an arc from α_i to α_j then we assign to it the weight

$$\omega(a) = \max\{\varepsilon \mid T_{ij}^{(\varepsilon)} > 0\}.$$

Example 4.1. The diamond band. Here, by Example 2.5

$$T^{(0)} = \begin{pmatrix} 12 & 0 \\ 0 & 8 \end{pmatrix} \qquad T^{(1)} = \begin{pmatrix} 0 & 4 \\ 16 & 0 \end{pmatrix} \qquad T^{(2)} = \begin{pmatrix} 8 & 0 \\ 0 & 0 \end{pmatrix},$$

and so $I_{\mathscr{G}}$ is the weighted graph of Fig. 9.

For each copermutation α_j let $\mathscr{W}_{j,n}$ denote the set of $\alpha_1 - \alpha_j$ directed walks of length n in $I_{\mathscr{G}}$, and let

$$\mathscr{W}_n = \bigcup_{j=1}^{d} \mathscr{W}_{j,n}.$$

The following lemma is an immediate consequence of Lemma 1.5.

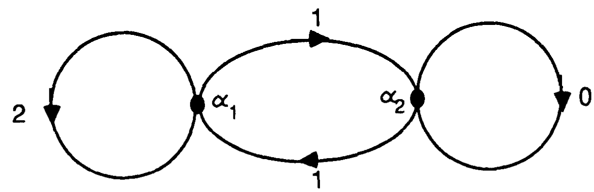

FIGURE 9

LEMMA 4.2. *Let $\mathscr{G}=\{G_n\}_{n=1}^{\infty}$ be an H-linear family and let $I_{\mathscr{G}}$ be its increment graph, Then*

$$\rho(G_n)=\max\{\omega(W)+\rho(\alpha_j, \Pi_H) \mid j=1, 2, ..., d, W\in \mathscr{W}_{j,n-1}\}.$$

A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be *eventually stable* if there exist positive integers n_0, n_1, and a constant c such that

$$a_{n+n_0}=a_n+c \qquad \text{for} \quad n\geqslant n_1.$$

In other words, a sequence is eventually stable if upon the removal of a sufficiently long initial segment the remainder is the union of a finite number of arithmetical progressions all of which have the same common difference.

THEOREM 4.3. *If $\mathscr{G}=\{G_n\}_{n=1}^{\infty}$ is an H-linear family then the sequence $\{\gamma(G_n)\}_{n=1}^{\infty}$ is eventually stable.*

Proof. Since the vertex and edge counts of $\{G_n\}_{n=1}^{\infty}$ form arithmetical progressions, it suffices to show that the sequence $\{\rho(G_n)\}_{n=1}^{\infty}$ is eventually stable. Note that if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are eventually stable, then so is the sequence

$$\{\max\{a_n, b_n\}\}_{n=1}^{\infty}. \tag{4.1}$$

Consequently it suffices to show that for each $j=1, 2, ..., d$ the sequence $\{\rho_{j,n}\}_{n=1}^{\infty}$, where

$$\rho_{j,n}=\max\{\omega(W)+\rho(\alpha_j, \Pi_H) \mid W\in \mathscr{W}_{j,n-1}\},$$

is eventually stable. We now regard j as fixed and simplify the rotation by replacing $\mathscr{W}_{j,n}$ with $\mathscr{W}_n$ in the sequel. It is clear that each directed walk W of $I_{\mathscr{G}}$ can be expressed (nonuniquely) as a formal sum

$$W=P_W+\sum_{i=1}^{q} m_i C_i,$$

where P_W is a directed $\alpha_1 - \alpha_j$ path, the C_i's are distinct edge disjoint directed cycles of $I_{\mathscr{G}}$, the m_i's are positive integers, and the path P_W shares at least one vertex with each of the C_i's. We shall refer to $\{P_W, C_1, C_2, ...\}$ as a *support* of W and write

$$\{P_W, C_1, C_2, ...\} \in \sup(W).$$

Since $I_{\mathscr{G}}$ is a finite digraph, the set $\mathscr{S}$ of all possible supports is also finite. If we set for every support $S \in \mathscr{S}$

$$\mathscr{W}_n(S) = \{W \in \mathscr{W}_n \mid S \in \sup(W)\},$$

then it is clear that $\mathscr{W}_n = \bigcup_{S \in \mathscr{S}} \mathscr{W}_n(S)$. It now follows from (4.1) that it suffices to prove the eventual stability of

$$\bar{\rho}_n = \max\{\omega(W) \mid W \in \mathscr{W}_n(S)\}$$

for each support S. Fix $S = \{P, C_1, C_2, ..., C_q\}$ and let p, c_1, c_2, ... denote the lengths of P, C_1, C_2, ..., respectively. In seeking to maximize $\omega(W)$, where W varies over all the walks in $\mathscr{W}_n(S)$ we are maximizing

$$\omega(P) + \sum_{i=1}^{q} m_i \omega(C_i), \tag{4.2}$$

where the m_i are subject to the constraint

$$p + \sum_{i=1}^{q} m_i c_i = n. \tag{4.3}$$

Thus (4.2, 4.3) can be restated as the integer program

$$\bar{\rho}_n = \max\{\mathbf{m} \cdot \boldsymbol{\omega} \mid \mathbf{m} \in \mathbb{Z}^{q+1}, \mathbf{m} \geqslant 0, \mathbf{m} \cdot \mathbf{c} = n\},$$

where

$$\mathbf{m} = (1, m_1, m_2, ..., m_q)$$
$$\boldsymbol{\omega} = (\omega(C_1), \omega(C_2), ..., \omega(C_q))$$
$$\mathbf{c} = (p, c_1, c_2, ..., c_q).$$

The eventual stability of $\{\bar{\rho}_n\}$ now follows from Lemma 4.4. ∎

LEMMA 4.4. *Let* $\mathbf{u}, \mathbf{v}$ *be m-vectors whose components are nonnegative integers. For each positive integer n set*

$$g(n) = \max\{\mathbf{y} \cdot \mathbf{u} \mid \mathbf{y} \cdot \mathbf{v} = n, \mathbf{y} \in \mathbb{Z}^m, \mathbf{y} \geqslant 0\}. \tag{4.4}$$

Then $\{g(n)\}_{n=1}^{\infty}$ *is eventually stable.*

Proof. We convert the given integer program to a linear program. In other words, let

$$g^* = \max\{\mathbf{y} \cdot \mathbf{u} \mid \mathbf{y} \cdot \mathbf{v} = 1, \mathbf{y} \in \mathbb{R}^m, \mathbf{y} \geqslant 0\}. \tag{4.5}$$

Since the maximum can be assumed to occur at the extrema of the associated polytope, and since all the constraints have integer coefficients, it follows that there is a vector $\mathbf{y}^{(0)} \in \mathbb{R}^m$ with nonnegative *rational* components, such that

$$g^* = \mathbf{y}^{(0)} \cdot \mathbf{u}.$$

Let d be an integer such that all the components of $d\mathbf{y}^{(0)}$ are nonnegative integers. Substituting $d\mathbf{y}^{(0)}$ into (4.4) we see that

$$\begin{aligned} g(d) \geqslant (d\mathbf{y}^{(0)}) \cdot \mathbf{u} &= d(\mathbf{y}^{(0)} \cdot \mathbf{u}) \\ &= dg^*, \end{aligned}$$

and hence

$$\frac{g(d)}{d} \geqslant g^*.$$

On the other hand, for any integer n, if $\mathbf{y}^{(n)}$ is a nonnegative vector of $\mathbf{Z}^m$ such that

$$g(n) = \mathbf{y}^{(n)} \cdot \mathbf{u} \qquad \text{and} \qquad \mathbf{y}^{(n)} \cdot \mathbf{v} = n,$$

then the vector $(1/n)\mathbf{y}^{(n)}$ satisfies the constraints of (4.5) and hence

$$g^* \geqslant \left(\frac{1}{n}\mathbf{y}^{(n)}\right) \cdot \mathbf{u} \geqslant \frac{1}{n}(\mathbf{y}^{(n)} \cdot \mathbf{u}) = \frac{g(n)}{n}.$$

Hence d is an integer such that

$$\frac{g(d)}{d} \geqslant \frac{g(n)}{n} \qquad \text{for all } n. \tag{4.6}$$

The remainder of the proof is a minor variation on the proof of [10, Theorem 11]. Since $g(a+b) \geqslant g(a) + g(b)$ it follows that $g(kd) \geqslant kg(d)$. It therefore follows from (4.6) that

$$g(kd) = kg(d). \tag{4.7}$$

Now, for arbitrary k and $0 \leqslant r < d$ define

$$D(k, r) = g(kd + r) - g(kd).$$

Then, by the superadditivity of g and by (4.7)

$$D(k, r) - D(k+1, r) = g(d) + g(kd+r) - g(kd+r+d) \leqslant 0. \qquad (4.8)$$

On the other hand,

$$\begin{aligned} g(kd+r) + g(d-r) &\leqslant g((k+1)d) = (k+1)\,g(d) \\ &= kg(d) + g(d) = g(kd) + g(d), \end{aligned}$$

and hence

$$D(k, r) \leqslant g(d) - g(d-r) \qquad \text{for all } k.$$

Thus,

$$D(k, r) \leqslant D(k+1, r) \leqslant g(d) - g(d-r) \qquad \text{for all } k.$$

Since $D(k, r)$ can assume only integer values it follows that for each r, $0 \leqslant r < d$, there exists an integer m_r such that $D(k, r) = D(k+1, r)$ whenever $k \geqslant m_r$. If $M = \max_r \{m_r\}$, then, for $k \geqslant M$,

$$D(k, r) = D(k+1, r) \qquad \text{for all} \quad 0 \leqslant r < d.$$

Set $n_1 = Md$. If $n \geqslant n_1$ then there exist integers k and r, satisfying $k \geqslant M$ and $0 \leqslant r < d$ such that $n = kd + r$. Consequently, $D(k, r) = D(k+1, r)$ and equality must hold in (4.8). Hence,

$$g(d) + g(kd+r) - g(kd+r+d) = 0$$

or

$$g(n) + g(d) = g(n+d). \qquad \text{Q.E.D.}$$

EXAMPLE 4.5. The diamond band. The increment graph $I_{\mathscr{G}}$ is given in Fig. 9. Here

$$\rho(\alpha_1, \Pi_H) = 3$$
$$\rho(\alpha_2, \Pi_H) = 2$$

and so

$$\rho(G_n) = 2(n-1) + 3 = 2n + 1.$$

Since G_n has $3n+1$ vertices and $5n$ edges, such an embedding of G_n with $2n+1$ regions is necessarily planar.

It is easily seen that a similar result holds for the maximum genus of G which corresponds to minimizing the number of regions. The details are omitted except to note that the auxiliary graph $I_{\mathscr{G}}$ needs to be replaced by a graph $J_{\mathscr{G}}$ each of whose arcs carries a weight that is equal to the *minimum* corresponding ε.

Conclusion

The author does not know of any H-linear families of graphs for which the integer p of Theorem 3.2 is different from 1. In other words, in all known cases there is no cycling in the constant term of the asymptotic limit of the everage genus. This is because in all known cases the recursion matrix T is primitive. However, I do not known if such is always the case. It is similarly quite possible that the genus sequence of Theorem 4.3 is immediately stable and so must be the union of a finite number of arithmetic progressions.

The method introduced in this paper can be modified so as to apply to other families of graphs as well. Thus the author has constructed a 24×24 recursion matrix P which describes the growth of the region distributions of such families as the ladders of [4], the circular ladders, and the Moebius ladders of [19]. In fact, this matrix applies to any family of graphs wherein each member is obtained from the previous one by a consistent application of the production which replaces a pair of parallel edges with the subgraph obtained by joining their midpoints with a new edge. While this matrix is too unwieldy to obtain the explicit formulas displayed in the above references, it does provide for a simple demonstration of the very surprising similarity between the distributions of the curcular and Moebius ladders which was noted first in [19] and also, with less accuracy, in [1]. Its size notwithstanding, this matrix also makes it immediate that all these ladders have an average genus that is asymptotically a linear function of their length. It is worthy of note that the genus sequence of the Moebius ladders is not immediately stable, thus making it seem unlikely that such an immediate stability would hold for the H-linear families.

The author thanks this colleagues John Bunce and Fred Van Vleck for their help with the matrix theoretic contents of Chapter III.

References

1. D. Archdeacon, Calculations on the average genus and genus distribution of graphs, *Congr. Numer.*, to appear.
2. D. Archdeacon, The orientable genus is nonadditive, *J. Graph Theory* **10** (1986), 385–402.

3. D. Archdeacon, The nonorientable genus is additive, *J. Graph Theory* **10** (1986), 363–384.
4. M. L. Furst, J. L. Gross, and R. Statman, Genus distributions of two classes of graphs, *J. Combin. Theory Ser. B* **46** (1989), 22–36.
5. J. L. Gross and M. L. Furst, Hierarchy for imbedding-distribution invariants of a graph, *J. Graph Theory* **11** (1987), 205–220.
6. F. R. Gantmakher, "The Theory of Matrices," Chelsea, New York, 1959.
7. J. L. Gross and T. W. Tucker, "Topological Graph Theory," Wiley, New York, 1987.
8. F. Harary, "Graph Theory," Addison–Wesley, Reading, MA, 1971.
9. C. H. Richardson, "An Introduction to the Calculus of Finite Differences," Van Nostrand, Princeton, NJ, 1963.
10. S. Stahl, *n*-Tuple Colorings and Associated Graphs, *J. Combin. Theory Ser. B* **20** (1976), 185–203.
11. S. Stahl, Permutation–partition pairs: A combinatorial generalization of graph embeddings, *Trans. Amer. Math. Soc.* **259** (1980), 129–145.
12. S. Stahl, Permutation–partition pairs. II. The genus of the amalgamation of graphs, *Trans. Amer. Math. Soc.* **27** (1982), 175–182.
13. S. Stahl, The average genus of classes of graph embeddings, *Congr. Numer.* **40** (1953), 375–388.
14. S. Stahl, Region distributions of graph embeddings and Stirling numbers, *Discrete Math.*, to appear.
15. S. Stahl, Permutation–partition pairs. IV. The average genus, in preparation.
16. S. Stahl, Region distributions of some small diameter graphs, *Discrete Math.*, to appear.
17. A. T. White, "Graphs, Groups and Surfaces," North-Holland, Amsterdam, 1984.
18. R. Rieper, Doctoral dissertation, Western Michigan University, 1987.
19. L. McGeoch, Doctoral dissertation, Carnegie–Mellon, 1987.