Oscillation and Nonoscillation in First Order Neutral Differential Equations

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Verifiable sufficient conditions are obtained respectively for the oscillation and nonoscillation of the neutral differential equation \( x'(t) - cx(t - \tau) + px(t - \sigma) = 0 \).

1. INTRODUCTION

The oscillation theory of delay differential equations has been extensively developed during the past few years. We refer to Arino et al. [1], Hunt and Yorke [10], Koplatadze and Canturija [13], Kusano [15], Ladas [16], Lakshmikantham et al. [19], Onose [21], Fukagai and Kusano [3], and the references therein for the literature concerned with the oscillation of delay differential equations. It is known that neutral delay differential equations (NDDEs for short) exhibit certain features which are not found in the case of delay differential equations (for example see Gromova and Zverkin [9], Gromova [8]). Only very recently there has been some interest in the oscillation theory of NDDEs and it appears that there are no known results about the existence of nonoscillatory solutions of first order NDDEs especially when the NDDE has one or more variable coefficients. Nonoscillation of linear second order NDDE has been considered by Jiong [11]. We refer to the works of Grammatikopoulos et al. [4-7], Jiong [12], Ladas and Sficas [17, 18], Sficas and Stavroulakis [22], Zahariev and Bainov [24], Zhang [25], and Kulenovic et al. [14] for some results related to the oscillation of NDDEs.

The purpose of this article is to derive sufficient conditions for oscillation and nonoscillation of first order NDDEs. The conditions we obtain are in

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many cases weaker than those available in the literature cited above; furthermore our conditions can be verified when the parameters are given. That is, wherever possible we derive conditions to guarantee the existence (or nonexistence) of real roots of the relevant characteristic equation while keeping the conditions weaker than those known. We remark that determining whether or not the characteristic equation associated with a NDDE has a real root is itself a problem. Thus we are concerned with the derivation of conditions which are easily verifiable. The authors are of the opinion that at this time there are no known sufficient conditions in the literature for the existence of nonoscillatory solutions of first order NDDEs and thus it appears that the results obtained in this article for nonoscillation of NDDEs are new.

2. Oscillation of NDDEs

We first consider the first order NDDE with constant coefficients

\[ \dot{x}(t) - c\dot{x}(t - \tau) + px(t - \sigma) = 0 \quad \left( \dot{x} = \frac{dx}{dt} \right). \]  

Our first result is the following:

**Theorem 2.1.** Assume

(i) \( 0 < c < 1 \).

(ii) \( \tau \geq 0, \sigma > 0, \) and \( p > 0 \).

(iii) \( p\sigma e > 1 - c\{1 + (\tau p/(1 - c))\} \).  

Then every solution of (2.1) is oscillatory.

**Proof.** The characteristic equation of (2.1) is

\[ f(\lambda) = \lambda - c\lambda e^{-\lambda \tau} + pe^{-\lambda \sigma} = 0. \]  

To prove the result, it suffices to prove that (2.3) has no real roots under the assumptions of the theorem [12]. One can note that any real root of (2.3) cannot be positive and since \( f(0) = p > 0, \lambda = 0 \) is not a root. Thus any real root of (2.3) can only be negative if it is possible. Let us set \( \lambda = -\mu \) for convenience and show that

\[ f(-\mu) = g(\mu) = 1 - ce^{\mu \tau} - (pe^{\mu \sigma}/\mu) = 0 \]  

(2.4)
has no positive roots when (2.2) holds. We define \( f_1, f_2 \) so that

\[
f_1(\mu) = 1 - ce^{\mu}, \quad f_2(\mu) = \frac{pe^{\mu}}{\mu}.
\] (2.5)

It is sufficient to show that \( f_2(\mu) > f_1(\mu) \) for \( \mu > 0 \). We note that \( f_2 \) has a global minimum at \( 1/\sigma \) and the minimum value is \( pe\sigma \). The strategy of our proof is to show the existence of a suitable curve lying between the graphs of the functions \( f_2 \) and \( f_1 \). One such curve is the graph of

\[
f(\mu) = (1 - c) - ct\mu; \quad \mu > 0.
\] (2.6)

It is easy to see,

\[
f(\mu) - f_1(\mu) = (1 - c) - ct\mu - (1 - ce^{\mu})
= c[e^{\mu} - \mu - 1] > 0 \quad \text{for} \quad \mu > 0
\]

and hence \( f(\mu) > f_1(\mu) \) for \( \mu > 0 \). From (2.6) and (2.5),

\[
f_2(\mu) - f(\mu) = \frac{pe^{\mu}}{\mu} + ct\mu - (1 - c).
\] (2.7)

Consider the value of \((f_2 - f)\) at \( \mu = (1/\sigma\alpha) \) for \( 1 \leq \alpha < \infty \);

\[
[f_2(\mu) - f(\mu)]_{\mu = (1/\sigma\alpha)} = \alpha\sigma pe^{(1/\alpha)} + (ct/\alpha\sigma) - (1 - c)
> p\sigma\alpha - (1 - c).
\] (2.8)

Thus if \( \alpha > (1 - c)/\sigma p \) then \( f_2(\mu) - f(\mu) > 0 \) for \( \mu \in (0, 1/\alpha\sigma) \) and hence for all \( \mu \in (0, p/(1 - c)) \), \([f_2(\mu) - f(\mu)] > 0\).

Let us now consider \( \mu \geq (p/1 - c) \) and note

\[
[f_2(\mu) - f(\mu)]_{\mu \geq p/(1 - c)} \geq pe\sigma + ct \frac{p}{1 - c} - (1 - c).
\] (2.9)

But by our assumption (2.2), \( pe\sigma + ct(\pi/(1 - c)) - (1 - c) > 0 \) showing that \( f_2(\mu) - f(\mu) > 0 \) for \( \mu \geq p/(1 - c) \). Thus we have shown that

\[f_2(\mu) > f_1(\mu) \quad \text{for all} \quad \mu > 0.
\]

It follows that (2.3) has no real roots and hence all solutions of (2.1) are oscillatory and the proof is complete.

We remark that the condition (2.2) is better than the corresponding con-
ditions obtained by Zhang [25] and Ladas and Sficas [17]. For instance in the example
\[ \dot{x}(t) - \frac{1}{2} \dot{x}(y - 1) + \frac{1}{4e} x(t - 2) = 0, \quad (2.10) \]
\( p \sigma = 1 - c = \frac{1}{2} \) and the results of Zhang [25] and Ladas and Sficas [17] do not apply for (2.10). But the condition (2.2) can be applied to (2.10) to conclude that all solutions of (2.10) are oscillatory.

**Corollary 2.1.** Assume that the constants \( c, \tau, \sigma_1, \ldots, \sigma_m, p_1, p_2, \ldots, p_m \) satisfy
\[ 0 < c < 1; \quad 0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_m; \quad p_i > 0, \quad i = 1, 2, \ldots, m \]
and
\[ \left( \sum_{i=1}^{m} p_i \sigma_i \right) e > 1 - c \left\{ 1 + \frac{\tau}{1 - c} \sum_{i=1}^{m} p_i \right\}. \quad (2.11) \]
Then all nontrivial solutions of
\[ \dot{x}(t) - c \dot{x}(t - \tau) + \sum_{i=1}^{m} p_i x(t - \sigma_i) = 0 \quad (2.12) \]
are oscillatory.

**Proof.** Since the details of the proof are similar to those of the above Theorem 2.1 we omit the proof to avoid repetitive arguments.

In a recent article Ladas and Sficas [17] have proved the following result: "If \( p, \sigma, \tau \) are positive constants and \( Q \) is \( \tau \)-periodic such that \( Q \in C([t_0, \infty), \mathbb{R}), \sigma > \tau, p < 1, \) and
\[ \lim_{t \to \infty} \int_{t-(\sigma-\tau)}^{t} Q(s) ds > (1-p)/c \quad (2.13) \]
then every solution of
\[ \dot{x}(t) - p \dot{x}(t - \tau) + Q(t) x(t - \sigma) = 0; \quad t > t_0 \]
is oscillatory."

The following result provides an alternative and somewhat weaker condition for all solutions of
\[ \dot{x}(t) - c \dot{x}(t - \tau) + p(t) x(t - \sigma) = 0 \quad (2.14) \]
to be oscillatory.
THEOREM 2.2. Assume the following:

(i) \( c, \tau, \sigma \) are positive constants, \( 0 < c < 1, \tau > 0, \sigma > 0; \sigma \geq \tau. \)

(ii) \( p \in C(\mathbb{R}, \mathbb{R}_+), p(t + \tau) = p(t), t \in \mathbb{R}, \mathbb{R}_+ = [0, \infty). \)

(iii) \( P_0 > 1/e[1 - (4c/P_0^2)] > 0 \) where \( P_0 = \int_{-\tau}^{\tau} p(s) \, ds. \) (2.15)

Then all nontrivial solutions of (2.14) are oscillatory.

Proof. Suppose the conclusion does not hold. Then there exists a non-oscillatory solution \( x(t) \) which we assume to be eventually positive; that is, there exists a \( T > 0 \) such that \( x(t) > 0 \) for \( t \geq T \). We have from (2.14),

\[
\frac{d}{dt} [x(t) - cx(t - \tau)] \leq 0 \quad \text{for} \quad t > T + \sigma - T_1
\]

which implies one of the following two alternatives:

(i) \( x(t) - cx(t - \tau) < 0 \) for \( t > T_1. \)

(ii) \( x(t) - cx(t - \tau) > 0 \) for \( t > T_1. \)

We first show that (i) is not possible. If case (i) holds, we have for some constant \( \delta > 0, \)

\[
x(t) - cx(t - \tau) \leq -\delta \quad \text{for} \quad t > T_1
\]

and hence

\[
x(t) \leq -\delta + cx(t - \tau) \\
\leq -\delta + c[ -\delta + cx(t - 2\tau)] \\
\leq -\delta[c + c^2 + \cdots + c^n] + c^{n+1}x(t - (n+1)\tau).
\]

If we let

\[
\|\varphi(t)\| = \sup_{t \in [T_1 - \tau, T_1]} |x(t)|
\]

then for \( t \geq T_1 \)

\[
x(t) \leq -\delta(c + c^2 + \cdots + c^n) + c^{n+1} \|\varphi(t)\|. \quad (2.16)
\]

Since \( 0 < c < 1, \) (2.16) implies that \( x(t) \) is negative when \( n \) is large enough and this contradiction shows that \( x(t) - cx(t - \tau) < 0 \) for \( t \geq T \) is not possible.
Let us now suppose \( x(t) - cx(t - \tau) > 0 \) for \( t \geq T \) and define

\[
w(t) = \frac{x(t - \tau) - cx(t - 2\tau)}{x(t) - cx(t - \tau)} > 1. \tag{2.17}
\]

Dividing both sides of (2.14) by \( x(t) - cx(t - \tau) \) and integrating

\[
\ln w(t) = \int_{t-\tau}^{t} \frac{p(s) x(s - \sigma)}{x(s) - cx(s - \tau)} \, ds
\]

\[
= \int_{t-\tau}^{t} \frac{p(s)[x(s - \sigma) - cx(s - \sigma - \tau) + cx(s - \sigma - \tau)]}{x(s) - cx(s - \tau)} \, ds
\]

\[
\geq \int_{t-\tau}^{t} \frac{p(s) cx(s - \sigma - \tau)}{x(s) - cx(s - \tau)} \, ds. \tag{2.18}
\]

Using the periodicity of \( p \) in (2.18),

\[
\ln w(t) \geq \int_{t-\tau}^{t} p(s) w(s) \, ds - c \int_{t-\tau}^{t} \frac{x(s - \tau) - cx(s - 2\tau)}{x(s) - cx(s - \tau)} \, ds
\]

\[
= \int_{t-\tau}^{t} p(s) w(s) \, ds - c \int_{t-\tau}^{t} w(s) \frac{d}{ds} \ln [x(s - \tau) - cx(2 - 2\tau)] \, ds. \tag{2.19}
\]

Let \( t^* \) be a number such that \( t - \tau < t^* < t \) and

\[
\int_{t-\tau}^{t^*} p(s) \, ds = \frac{P_0}{2}, \quad \int_{t^*}^{t} p(s) \, ds = \frac{P_0}{2}.
\]

We now show that \( w(t) \) is bounded above. On integrating (2.14) over \((t^*, t)\),

\[
x(t) - cx(t - \tau) - [x(t^*) - cx(t^* - \tau)] + \int_{t^*}^{t} p(s) x(s - \sigma) \, ds = 0
\]

which implies

\[
x(t^*) - cx(t^* - \tau) \geq \int_{t^*}^{t} p(s) x(s - \sigma) \, ds
\]

\[
\geq \int_{t^*}^{t} p(s)[x(s - \sigma) - cx(s - \sigma - \tau)] \, ds
\]

\[
\geq [x(t - \sigma) - cx(t - \sigma - \tau)] \int_{t^*}^{t} p(s) \, ds
\]

\[
= [x(t - \sigma) - cx(t - \sigma - \tau)](P_0/2). \tag{2.20}
\]
Integrating (2.14) over \((t-\tau, t^*)\),

\[
x(t^*) - cx(t^* - \tau) - \left[ x(t-\tau) - cx(t-2\tau) \right] + \int_{t-\tau}^{t^*} p(s) x(s-\sigma) \, ds = 0
\]

which implies

\[
x(t-\tau) - cx(t-2\tau) \geq \int_{t-\tau}^{t^*} p(s) \left[ x(s-\sigma) - cx(s-\sigma-\tau) \right] \, ds
\geq \left[ x(t^* - \sigma) - cx(t^* - \tau - \sigma) \right] (P_0/2).
\]

(2.21)

Since \(x(t) - cx(t - \tau)\) is decreasing, we can combine (2.20) and (2.21) so as to have

\[
x(t^*) - cx(t^* - \tau) \geq \left[ x(t-\tau) - cx(t-2\tau) \right] (P_0/2)
\]

\[
\geq \left[ x(t^* - \sigma) - cx(t^* - \tau - \sigma) \right] (P_0^2/2)
\]

\[
\geq \left[ x(t^* - \tau) - cx(t^* - 2\tau) \right] (P_0/2)^2.
\]

Thus

\[
w(t^*) = \frac{x(t^* - \tau) - cx(t^* - 2\tau)}{x(t^*) - cx(t^* - \tau)} \leq \frac{4}{P_0^2}
\]

(2.22)

for any \(t^* \geq T\). We let

\[
\lim \inf_{t \to \infty} w(t) = \overline{\gamma}
\]

(2.23)

and note that \(\overline{\gamma} < \infty\). It follows from (2.19) that

\[
\ln \overline{\gamma} \geq P_0 T + \lim \inf_{t \to \infty} \left[ -c \int_{t-\tau}^{t} w(s) \frac{d}{ds} \left\{ \ln \left[ x(s-\tau) - cx(s-2\tau) \right] \right\} \right] ds.
\]

(2.24)

We note

\[
\lim \inf_{t \to \infty} \left[ -c \int_{t-\tau}^{t} w(s) \frac{d}{ds} \left\{ \ln \left[ x(s-\tau) - cx(s-2\tau) \right] \right\} \right] ds
\]

\[
= \lim \sup_{t \to \infty} \left[ c \int_{t-\tau}^{t} w(s) \frac{d}{ds} \left\{ \ln \left[ x(s - \tau) \right] \right\} ds \right] \] dx(2.25)
and find that

$$\lim_{t \to \infty} \sup \left[ e \int_{t-\tau}^{t} w(s) \frac{d}{ds} \{\ln[x(s-\tau) - cx(s-2\tau)]\} \right] ds$$

$$\leq \frac{4c}{P_0^2} \lim_{t \to \infty} \sup \ln([w(t-\tau)]^{-1})$$

$$= \frac{4c}{P_0^2} \lim_{t \to \infty} \sup [-\ln w(t-\tau)]$$

$$= -\frac{4c}{P_0^2} \lim_{t \to \infty} \inf \ln[w(t-\tau)] = -\frac{4c \ln[\overline{T}]}{P_0^2}. \quad (2.26)$$

From (2.24), (2.25), (2.26),

$$\ln \overline{T} \geq \overline{T}_0 + \frac{4c \ln[\overline{T}]}{P_0^2}$$

and hence

$$\left[ 1 - \frac{4c}{P_0^2} \right] \frac{\ln \overline{T}}{\overline{T}} \geq P_0$$

leading to

$$\left[ 1 - \frac{4c}{P_0^2} \right] \frac{1}{e} \geq P_0. \quad (2.27)$$

But (2.27) contradicts (2.15) and this completes the proof.

**Corollary 2.2.** If \( p(t) \equiv p_0 > 0 \) and

$$p_0 \tau > \frac{1}{e} \left[ 1 - \frac{4c}{(p_0 \tau)^2} \right] \quad (2.28)$$

then every solution of

$$\dot{x}(t) - cx(t-\tau) + p_0 x(t-\sigma) = 0$$

is oscillatory.

**Remark.** We note that the existence of the limit in (2.13) has been assumed in [17]; such an assumption is not required in Theorem 2.2.
THEOREM 2.3. Assume the following:

(i) \( c, \tau, \sigma \) are nonnegative constants, \( 0 < c < 1, \tau > 0, \sigma > 0 \).

(ii) \( p \in C(\mathbb{R}, \mathbb{R}_+) \), \( p(t) \equiv p_0 > 0, \ t \in \mathbb{R} \)

(iii) \( p_0 \sigma e \geq 1 - c(1 + (\tau p_0 / (1 - c))) \).

Then every solution of

\[
\dot{x}(t) - c\dot{x}(t - \tau) + p(t)x(t - \sigma) = 0 \quad (2.30)
\]

is oscillatory.

Proof. We shall show that the existence of a nonoscillatory solution of \( (2.30) \) leads to a contradiction. Suppose \( \varphi \) is a nonoscillatory solution of \( (2.30) \); we suppose that \( \varphi(t) > 0 \) for all \( t \geq T \) for some \( T > 0 \). (If \( \varphi(t) < 0 \) eventually the procedure is similar.) It is known that nonoscillatory solutions of \( (2.30) \) tend to zero as \( t \to \infty \) due to \( p(t) \geq p_0 > 0 \) (see Ladas and Sficas [17]). Thus we have from \( (2.30) \),

\[
\varphi(t) = c\varphi(t - \tau) + \int_t^\infty p(s)\varphi(s - \sigma)\,ds; \quad t \geq T + \tau = t_0
\]

\[
\geq c\varphi(t - \tau) + p_0\int_t^\infty \varphi(s - \sigma)\,ds; \quad t > t_0. \quad (2.31)
\]

It is not difficult to see from \( (2.31) \) that

\[
\varphi(t) \geq c\varphi(t - \tau) \Rightarrow \varphi(t) \geq \alpha e^{-\mu t} \quad (2.32)
\]

for all large \( t \) where

\[
\mu = -\frac{(\ln c)}{\tau}; \quad \alpha = \varphi(t_0) e^{(\mu t_0/\tau)}. \quad (2.33)
\]

Define a sequence \( \{y_n(t)\} \) as

\[
y_0(t) \equiv \varphi(t) \quad (2.34)
\]

\[
y_{n+1}(t) \equiv cy_n(t - \tau) + p_0\int_t^\infty y_n(s - \sigma)\,ds; \quad t \geq t_0.
\]

It follows from \( (2.34) \) that

\[
y_{n+1}(t) \leq y_n(t) \leq \cdots \leq y_0(t); \quad t \geq t_0. \quad (2.35)
\]

Furthermore we have from \( (2.34) \),

\[
y_0(t) \geq \alpha e^{-\mu t}
\]
and also one can derive using (2.33) that
\[ y_1(t) \geq ae^{-\mu} \Rightarrow y_{n+1}(t) \geq ae^{-\mu}, \quad n = 1, 2, \ldots \]

Thus we have from (2.35),
\[ ae^{-\mu} \leq y_{n+1}(t) \leq y_n(t) \leq \cdots \leq y_0(t); \quad t \geq t_0. \tag{2.36} \]

By the Lebesgue’s convergence theorem the pointwise limit of \( \{ y_n(t) \} \) exists and hence
\[ ae^{-\mu} \leq y^*(t) = cy^*(t - \tau) + p_0 \int_{t}^{\infty} y^*(s - \sigma) \, ds, \tag{2.37} \]

where
\[ y^*(t) = \lim_{n \to \infty} y_n(t). \]

\( y^*(t) \) is a nonoscillatory solution of the NDDE,
\[ \dot{x}(t) - c \dot{x}(t - \tau) + p_0 x(t - \sigma) = 0. \tag{2.38} \]

But by Theorem 2.1, (2.38) cannot have a nonoscillatory solution when (2.29) holds. This contradiction proves the result.

3. **Nonoscillation in NDDEs**

It is known that (2.1) can have a nonoscillatory solution when the associated characteristic equation has a real root. However, verifiable sufficient conditions in terms of the parameters of (2.1) for the characteristic equation to have real roots are not known. In the case of (2.14), the method of the characteristic equation is not applicable. In this section we derive sufficient conditions for the equations of the form (2.1) and (2.14) to have nonoscillatory solutions. We use the following lemma which combines both the Banach contraction mapping and the Schauder’s fixed point theorem (Nashed and Wong [20]).

**Lemma.** Let \( X \) be a Banach space; \( \Omega \) be a bounded closed convex subset of \( X \); \( A, B \) be maps of \( \Omega \) into \( X \) such that \( Ax + By \in \Omega \) for every pair \( x, y \in \Omega \). If \( A \) is a strict contraction, i.e., it satisfies the condition that for all \( x, y \in \Omega \),
\[ \| Ax - Ay \| \leq \gamma \| x - y \| \]
for some \( \gamma, 0 \leq \gamma < 1 \), and \( B \) is completely continuous, i.e., \( B \) is continuous and compact (maps bounded sets into compact sets), then the equation

\[
Ax + Bx = x
\]

has a solution in \( \Omega \).

**Theorem 3.1.** Assume that there exists a positive number \( \mu \) satisfying

\[
pe^{\mu t} + \frac{pe^{\mu s}}{\mu} \leq 1. \tag{3.1}
\]

Then (2.1) has a nonoscillatory solution which tends to zero as \( t \to \infty \).

**Proof.** Let \( C = C([T - \rho, \infty), \mathbb{R}) \) denote the Banach space of all bounded continuous functions defined on \( [T - \rho, \infty) \) with values in \( \mathbb{R} = (-\infty, \infty) \), where \( \rho = \max(a, \tau) \) and the norm in \( C \) is the sup norm. Let \( \Omega \) be the subset of \( C \) defined by

\[
\Omega = \left\{ x \in C \left| e^{-\mu_2 t} \leq x \leq e^{-\mu_1 t}; \mu_1 > \mu_2 > 0 \right. \right. \quad \text{on } [T - \rho, \infty); \quad \left. \left. cx(t - \tau) < Dx(t); \ c < D < 1 \right. \right. \quad \text{for } t \geq T, \tag{3.2}
\]

where \( \mu_2 \) satisfies (3.1) and \( \mu_1 > \mu_2 \). Define a map \( S: \Omega \to C \) as

\[
S(x)(t) = S_1(x)(t) + S_2(x)(t), \tag{3.3}
\]

where

\[
S_1(x)(t) = \begin{cases} cx(t - \tau) & \text{for } t \geq T \\ \int_t^\infty px(s - \sigma) \, ds & \text{for } t \in [T - \rho, T]. \end{cases}
\]

\[
S_2(x)(t) = \begin{cases} ca & \text{for } t \geq T \\ (1 - c)a & \text{for } t \in [T - \rho, T]. \end{cases}
\]

The constant \( a \) is selected so as to make \( S_1(x) \) and \( S_2(x) \) continuous at \( t = T \). It is easy to see that the integral in the definition of \( S_2 \) is defined whenever \( x \in \Omega \). It also follows from (3.2) that \( S_1 \) is a contraction (due to \( c < D < 1 \)) and that \( S_2 \) is completely continuous. The set \( \Omega \) is closed, convex, and bounded in \( C \). We show that for every pair \( x, y \in \Omega \), \( S_1(x) + S_2(y) \in \Omega \). For instance we have for any \( x, y \) in \( \Omega \),
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\[ S_1(x)(t) + S_2(y)(t) \leq ce^{-\mu_2(t-\tau)} + p \int_{t}^{\infty} e^{-\mu_2(s-\sigma)} \, ds \]
\[ = e^{-\mu_2 t} \left[ ce^{\mu_2 t} + \frac{pe^{\mu_2 \sigma}}{\mu_2} \right] \]
\[ \leq e^{-\mu_2 t} \quad \text{for} \quad t \geq T - \rho, \] (3.4)

where \( \mu_2 \) satisfies hypothesis (3.1). Also
\[ S_1(x)(t) + S_2(y)(t) \geq ce^{-\mu_1(t-\tau)} + p \int_{t}^{\infty} e^{-\mu_1(s-\sigma)} \, ds \]
\[ \geq ce^{-\mu_1(t-\tau)} = ce^{\mu_1 t}e^{-\mu_1 t} \]
\[ \geq e^{-\mu_1 t} \quad \text{for} \quad t \geq T - \rho \] (3.5)

provided \( \mu_1 \) is sufficiently large. Equations (3.4) and (3.5) imply that \( S_1(x) + S_2(y) \in \Omega \) for \( x, y \in \Omega \). We note
\[ cS(x)(t-\tau) = c \left[ cx(t-2\tau) + \int_{t-\tau}^{\infty} px(s-\sigma) \, ds \right] \]
\[ \leq c \left[ Dc(t-\tau) + \int_{t-\tau}^{\infty} px(s-\sigma) \, ds \right] \]
\[ < D \left[ cx(t-\tau) + \int_{t}^{\infty} px(s-\sigma) \, ds \right] = DS(x)(t) \quad \text{for} \quad t \geq T. \] (3.6)

Thus \( S(\Omega) \subset \Omega \). By the Lemma, the map \( S: \Omega \to C \) has a fixed point in \( \Omega \) which is a nonoscillatory solution of (2.1) and this completes the proof.

**Corollary 3.1.** Assume that either
\[ (i) \quad pae < 1 - ce^{(\tau/\sigma)} \] (3.7)
or
\[ (ii) \quad pte^{(\sigma/\tau)} < 1 - ce \] (3.8)
holds. Then (2.1) has a nonoscillatory solution which tends to zero.

**Proof.** The result follows from Theorem 3.1 for the choices of \( \mu = 1/\sigma \) and \( \mu = 1/\tau \), respectively.

**Remark 3.1.** It is found that the sufficient conditions (3.7) and (2.2) are different showing the existence of a gap between them. It is evident that both of these conditions (3.7) and (2.2) can be improved so as to narrow the gap in the conditions.
EXAMPLE 3.1. In the NDDE,
\[ \dot{x}(t) - \left( \frac{1}{2e} \right) \dot{x}(t - 1) + \left( \frac{1}{2e} \right) x(t - 1) = 0, \quad (3.9) \]
the condition (3.7) of Corollary 3.1 is satisfied since \( p\sigma e = \left( \frac{1}{2} \right) \), \( 1 - ce^{(\tau/\sigma)} = \frac{1}{2} \). In fact (3.9) has a nonoscillatory solution \( x(t) = e^{-t} \).

THEOREM 3.2. Let \( c, \tau, \sigma \) be nonnegative constants, \( 0 < c < 1, \tau \geq 0, \sigma > 0 \). Let \( p \in C(\mathbb{R}_+, \mathbb{R}_+) \) and \( p(t) \to p_0 > 0 \) as \( t \to \infty \). Suppose there exists a positive number \( \mu \) satisfying
\[ ce^{\mu \tau} + \frac{e^{\mu \sigma} p_0}{\mu} \leq 1. \quad (3.10) \]
Then
\[ \dot{x}(t) - c\dot{x}(t - \tau) + p(t) x(t - \sigma) = 0 \quad (3.11) \]
has a nonoscillatory solution.

Proof. Details of the proof are similar to those of Theorem 3.1 and hence we will be brief. Define a map \( S: \Omega \to C([T - \rho, \infty), \mathbb{R}) \), where \( \Omega \) is defined as in Theorem 3.1 for suitably selected large positive number \( T \). We define \( S \) as
\[ S(x)(t) = c x(t - \tau) + \int_{t-\tau}^{\infty} p(s) x(s - \sigma) \, ds \]
\[ = S_1(x)(t) + S_2(x)(t) \text{ (say).} \quad (3.12) \]
To show that \( S_1(x)(t) + S_2(y)(t) \in \Omega \) for \( x, y \in \Omega \) we have
\[ S_1(x)(t) + S_2(y)(t) \leq ce^{-\mu_2(t - \tau)} + \int_{t-\tau}^{\infty} p(s) e^{-\mu_2(s - \sigma)} \, ds \]
\[ = ce^{-\mu_2(t - \tau)} - \frac{e^{\mu_2} p_0 e^{-\mu_2 \sigma}}{\mu_2} \int_{t-\tau}^{\infty} p(s) \, ds (e^{-\mu_2 s}) \]
\[ = e^{-\mu_2 t} \left[ ce^{\mu_2 \sigma} + \frac{p_0 e^{\mu_2 \sigma}}{\mu_2} \right] \leq e^{-\mu_2 t} \quad (3.13) \]
for all \( t \geq T - \rho \), where \( T \) is sufficiently large (we have used a limiting form of the mean value theorem of integral calculus in the last step in the derivation of (3.13)). The other details of the proof are similar to those of Theorem 3.1 and hence we omit them.
COROLLARY 3.2. Assume that either

(i) \( p_0 \sigma e \leq 1 - ce^{\tau/\sigma} \) \hspace{1cm} (3.14)

or

(ii) \( p_0 \tau \sigma / \tau \leq 1 - ce \) \hspace{1cm} (3.15)

holds. Then (3.11) has a nonoscillatory solution which tends to zero as \( t \to \infty \).

This corollary follows from Theorem 3.2 for the choice \( \mu = (1/\sigma) \) and \( \mu = (1/\tau) \), respectively.

THEOREM 3.3. Assume that \( c, \tau, \sigma \) are as in Theorem 3.1 and that \( \rho \in C(\mathbb{R}_+, \mathbb{R}_+) \) satisfies \( p(t) \leq p_0 \). If there exists a positive constant \( \mu \) satisfying

\[
se^{\mu} + \frac{p_0 e^{\mu \sigma}}{\mu} \leq 1
\]

then (3.11) has a nonoscillatory solution.

Proof: Let \( \varphi(t) \) be a nonoscillatory solution of

\[
\dot{x}(t) - c\dot{x}(t - \tau) + p_0 x(t - \sigma) = 0
\]

which exists by hypotheses and Theorem 3.1. Define a sequence \( \{x_n(t)\} \) \( (n = 1, 2, 3, ...) \) for \( t \in [T-\rho, \infty) \) as

\[
x_0(t) = \varphi(t)
\]

\[
x_{n+1}(t) = c x_n(t - \tau) + \int_{t-n}^{\infty} p(s) x_n(s - \sigma) \, ds.
\]

Since \( \varphi \) is a nonoscillatory solution of (3.17), \( \varphi(t) \to 0 \) as \( t \to \infty \) and hence

\[
\varphi(t) = c\varphi(t - \tau) + \int_{t}^{\infty} p_0 \varphi(s - \sigma) \, ds, \quad t > 0
\]

\[
\geq c\varphi(t - \tau) + \int_{t}^{\infty} p(s) \varphi(s - \sigma) \, ds, \quad t > 0.
\]

One can now show as in the proof of Theorem 2.3 that the sequence \( \{x_n(t)\} \) has a pointwise limit for \( t > 0 \), say \( x^*(t) \), satisfying

\[
x^*(t) = c x^*(t - \tau) + \int_{t}^{\infty} p(s) x^*(s - \sigma) \, ds
\]
and

\[ x^*(t) \geq \alpha e^{-\mu t} \quad t > 0 \]

for some positive constants \( \alpha \) and \( \mu \). \( x^* \) satisfying (3.20) is a nonoscillatory solution of (3.11) and this completes the proof.

The proof of the following is similar to that of Theorem 3.1 and so we formulate it without proof.

**THEOREM 3.4.** Assume the following: \( 0 < c < 1; \quad \tau \geq 0; \quad 0 < \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma; \quad p_i > 0; \quad i = 1, 2, \ldots, m. \) Suppose there is a number \( \mu > 0 \) such that

\[
 ce^{\mu r} + \sum_{i=1}^{m} p_i \frac{e^{\mu \sigma_i}}{\mu} \leq 1. \tag{3.21}
\]

Then

\[
 x(t) - c \dot{x}(t - \tau) + \sum_{i=1}^{m} p_i x(t - \sigma_i) = 0 \tag{3.22}
\]

has a nonoscillatory solution.

**COROLLARY 3.3.** Assume that either

(i) \( ce + \sum_{i=1}^{m} P_i [\exp(\sigma_i/\tau)] \leq 1 \) or

(ii) \( ce^{\tau/\sigma} + \sum_{i=1}^{m} P_i \sigma [\exp(\sigma_i/\sigma)] \leq 1 \)

holds. Then (3.22) has a nonoscillatory solution which tends to zero as \( t \to \infty \).

We conclude with the note that in the case of a delay differential equation of the type

\[
 x(t) + px(t - \sigma) = 0, \tag{3.23}
\]

a necessary and sufficient condition for all nontrivial solutions of (3.23) to be oscillatory is \( p e \sigma > 1 \); it will be both interesting and worthwhile to find such a single necessary and sufficient condition for all nontrivial solutions of (2.1) to be oscillatory.

**REFERENCES**


10. B. R. Hunt and J. A. Yorke, When all solutions of \( x' = -\sum q_i(x(t - \tau_i(t))) \) oscillate, *J. Differential Equations* 53 (1984), 139–145.


