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On Composite Nonlinearities and the Decomposition Method

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Accurate, convergent, computable solutions using the decomposition method have been demonstrated in and papers for wide classes of nonlinear and/or stochastic differential, partial differential, or algebraic equations. It is shown specifically in this paper that composite nonlinearities of the form $Nx = N_0(N_1(N_2(\cdots(x)\cdots)$ appearing in such equations where the N_i are nonlinear operators can also be handled with the Adomian A_n polynomials. © 1986 Academic Press, Inc.

We begin with some formal definitions consistent with Adomian's notation in [1, 2]. Let N represent a nonlinear operator and Nx a nonlinear term in an equation to be solved by decomposition. Terms such as x^2, e^x, sin x, etc. are viewed as zeroth-order composite nonlinearities and will be written as Nx, or preferably as $N_0 u^0$, where $u^0 \equiv x$, and expanded in Adomian's A_n polynomials [1], now identified as A_n^0 to correspond to the N_0 nonlinear operator. Thus $N_0 u^0 = \sum_{n=0}^{\infty} A_n^0$.

A first-order composite nonlinearity is defined as $\tilde{N}_1 x = N_0(N_1 u^1)$ or as $N_0 N_1 u^1$, where $u^1 = x$ and $u_0 = N_1 u^1$ with $N_0 u^0 = \sum_{n=0}^{\infty} A_n^0$ and $N_1 u^1 = \sum_{n=0}^{\infty} A_n^1$.

A second-order composite nonlinearity is $\tilde{N}_2 x = N_0 N_1 N_2 x$, or $N_0(N_1(N_2 x)))$, where $N_0 u^0 = \sum_{n=0}^{\infty} A_n^0$, $u^0 = N_1 u^1 = \sum_{n=0}^{\infty} A_n^1$, $u^1 = N_2 u^2 = \sum_{n=0}^{\infty} A_n^2$, and $u^2 = x$. When the decomposition is carried out, $u^0 = \sum_{n=0}^{\infty} u_n^0$, $u^1 = \sum_{n=0}^{\infty} u_n^1$, $u^2 = \sum_{n=0}^{\infty} u_n^2$.

A third-order composite nonlinearity is written $\tilde{N}_3 x = N_0(N_1(N_2(N_3 x)))) = N_0 N_1 N_2 N_3 x$ with $N_0 u^0 = \sum_{n=0}^{\infty} A_n^0$, $N_1 u^1 = \sum_{n=0}^{\infty} A_n^1$, $N_2 u^2 = \sum_{n=0}^{\infty} A_n^2$, $N_3 u^3 = \sum_{n=0}^{\infty} A_n^3$, and $u^3 = x$. By decomposition, $u^0 = \sum_{n=0}^{\infty} u_n^0$, $u^1 = \sum_{n=0}^{\infty} u_n^1$, $u^2 = \sum_{n=0}^{\infty} u_n^2$, $u^3 = \sum_{n=0}^{\infty} u_n^3$ with $u^0 = N_1 u^1$, $u^1 = N_2 u^2$, $u^2 = N_3 u^3$, and $u^3 = x$.

In general, $N_\nu u^\nu = \sum_{n=0}^{\infty} A_n^\nu = u^{\nu-1}$ for $1 \leqslant \nu \leqslant m$ with $u^m = x$ and $u^\nu = \sum_{n=0}^{\infty} u_n^\nu$.

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An nth order composite nonlinearity will be written

$$\tilde{N}_m(x) = N_0(N_1(N_2(\cdots(N_{m-2}(N_{m-1}(N_m(x)))\cdots))) = N_0(u^0) = \sum A_n^0$$

$$N_1(u^1) = \sum A_n^1 = u^0$$

$$N_2(u^2) = \sum A_n^2 = u^1$$

$$\vdots$$

$$N_\nu(u^\nu) = \sum A_n^\nu = u^{\nu-1}$$

$$\vdots$$

$$N_{m-1}(u^{m-1}) = \sum A_n^{m-1} = u^{m-2}$$

$$N_m(u^m) = \sum A_n^m = u^{m-1} \text{ with } u^m \equiv x$$

so that the u's are the variables of substitution. Equivalently, $\tilde{N}_m(x) = N_0 \cdot N_1 \cdot N_2 \cdot \cdots \cdot N_{m-1} \cdot N_m(x)$, i.e., a composition of operators. The objective is to determine the A_n polynomials as functions of the x_n's, i.e., $A_n(x_0, x_1, ..., x_n) = Nx$.

For an mth-order composite nonlinearity, we get

$$\begin{aligned} A_n^0 = A_n^0(&A_0^1(A_0^2(A_0^3(\cdots(A_0^\nu(\cdots(A_0^m(x_0))\cdots))\cdots))),..., \\ &A_n^1(A_0^2(A_0^3(\cdots(A_0^\nu(\cdots A_0^m(x_0))\cdots))\cdots)),..., \\ &A_n^2(A_0^3(\cdots(A_0^\nu(\cdots(A_0^m(x_0))\cdots))\cdots),..., \\ &A_n^3(\cdots(A_0^\nu(\cdots(A_0^m(x_0))\cdots),..., \\ &A_n^\nu(\cdots(A_0^m(x_0),..., A_n^m(x_0,..., x_n))\cdots))\cdots)))). \end{aligned}$$

EXAMPLE. First order. $\tilde{N}_1 x = e^{-\sin(x/2)} = N_0(N_1 x)$. Let $N_0 u^0 = e^{-u^0} = \sum_{n=0}^{\infty} A_n^0(u_0^0, u_1^0, ..., u_n^0)$ and $N_1 u^1 = \sin(u^1/2)$, where $u^1 = x$ and $u^0 = \sum_{n=0}^{\infty} u_n^0 = N_1 x = \sin(x/2)$. Calculating the A_n^0 polynomials for the $N_0 u^0$ term [1, 2]:

$$A_0^0 = e^{-u_0^0}$$

$$A_1^0 = e^{-u_0^0}(-u_1^0)$$

$$A_2^0 = e^{-u_0^0}(-u_2^0 + (\tfrac{1}{2})(u_1^0)^2)$$

$$A_3^0 = e^{-u_0^0}(-u_3^0 + u_1^0 u_2^0 - (\tfrac{1}{6})(u_1^0)^3)$$

$$\vdots$$

(If we omit the identifier superscript, we are dealing with $Nu = e^{-u} = \sum_{n=0}^{\infty} A_n$, where $A_0 = e^{-u_0}$, $A_1 = e^{-u_0}(-u_1)$, etc.) Now calculating the A_n for $N_1 x$, i.e., A_n^1, we have

$$\begin{aligned} A_0^1 &= \sin(x_0/2) \\ A_1^1 &= (x_1/2)\cos(x_0/2) \\ A_2^1 &= (x_2/2)\cos(x_0/2) - (x_1^2/8)\sin(x_0/2) \\ A_3^1 &= (x_3/2)\cos(x_0/2) - (x_1 x_2/4)\sin(x_0/2) \\ &\quad - (x_1^3/48)\cos(x_0/2) \\ &\vdots \end{aligned}$$

Since $N_0 u^0 = \sum_{n=0}^{\infty} A_n^0$, and $u^0 = N_1 x = \sum_{n=0}^{\infty} A_n^1 = \sum_{n=0}^{\infty} u_n^0$

$$\begin{aligned} u_0^0 &= A_0^1 = \sin(x_0/2) \\ u_1^0 &= A_1^1 = (x_1/2)\cos(x_0/2) \\ u_2^0 &= A_2^1 = (x_2/2)\cos(x_0/2) - (x_1^2/8)\sin(x_0/2) \\ &\vdots \end{aligned}$$

Now $N_0 u^0 = e^{-u^0} = \sum_{n=0}^{\infty} A_n^0 = A_0^0 + A_1^0 + \cdots = e^{-u_0^0} - u_1^0 e^{-u_0^0} + \cdots$. Thus, now dropping the unnecessary superscript,

$$\begin{aligned} A_0 &= e^{-\sin(x_0/2)} \\ A_1 &= -(x_1/2)\cos(x_0/2)\, e^{-\sin(x_0/2)} \end{aligned}$$

Any algebraic, differential, or partial differential equation in Adomian's standard form which contains a nonlinear term $e^{-\sin x/2}$ is now solved by decomposition [1, 2].

EXAMPLE. Second order: $\tilde{N}_2 x = e^{-\sin^2(x/2)}$. Let $N_0 u^0 = e^{-u^0} = \sum_{n=0}^{\infty} A_n^0$, $N_1 u^1 = (u^1)^2 = \sum_{n=0}^{\infty} A_n^1 = u^0 = \sum_{n=0}^{\infty} u_n^0$, and $N_2 u^2 = N_2 x = \sin(x/2) = \sum^{\infty} A_n^2 = u^1 = \sum_{n=0}^{\infty} u_n^1$. The A_n^0 were specified in the previous example. The A_n^1 are given by

$$\begin{aligned} A_0^1 &= (u_0^1)^2 \\ A_1^1 &= 2u_0^1 u_1^1 \\ A_2^1 &= (u_1^1)^2 + 2u_0^1 u_2^1 \\ A_3^1 &= 2u_1^1 u_2^1 + 2u_0^1 u_3^1 \\ &\vdots \end{aligned}$$

and the A_n^2 are

$$A_0^2 = \sin(x_0/2)$$
$$A_1^2 = (x_1/2)\cos(x_0/2)$$
$$A_2^2 = (x_2/2)\cos(x_0/2) - (x_1^2/8)\sin(x_0/2)$$
$$A_3^2 = (x_3/2)\cos(x_0/2) - (x_1 x_2/4)\sin(x_0/2) - (x_1^3/48)\cos(x_0/2).$$

Now let us consider an equation using the first-order example previously considered with $\tilde{N}_1 x = e^{-\sin(x/2)}$. Thus consider the equation

$$x = (\pi/2) + e^{-\sin(x/2)}.$$

Letting $x = \sum_{n=0}^{\infty} x_n$ we have

$$\sum_{n=0}^{\infty} x_n = \pi/2 + N_1 x$$

where

$$x_0 = \pi/2 = 1.570796327$$
$$x_1 = e^{-\sin(x_0/2)} = e^{-\sin(\pi/4)} = 0.4930686914$$
$$x_2 = e^{-\sin(x_0/2)}(-1)\left(\frac{x_1}{2}\cos\frac{x_0}{2}\right) = -0.0859547458$$
$$x_3 = 0.0480557892$$
$$x_4 = -0.0293847366$$
$$\vdots$$

The sum ϕ_5 of only five terms x_0 to x_4 is 1.99658132 which is correct within about $\frac{1}{4}$ of 1%. (One can see that the next term should add about 0.01. If we guess $x = 2$ and calculate the right side, we have $1.570796327 + 0.431075951 = 2.00187228$.)

Now consider the second-order example with $x = k + \tilde{N}_2 x$, where $k = \pi/2$ and $\tilde{N}_2 x = e^{-\sin^2(x/2)} = N_0 N_1 N_2 x$. Using the A_n we have already written we now have

$$A_0^0 = e^{-u_0^0} = e^{-A_0^1} = e^{-(u_0^1)^2} = e^{-(A_0^2)^2} = e^{-\sin^2(x_0/2)}$$
$$A_1^0 = e^{-u_0^0}(-u_1^0) = e^{-\sin^2(x_0/2)}(-u_1^0)$$

where

$$u_1^0 = A_1^1 = 2u_0^1 u_1^1 = 2A_0^2 A_1^2$$
$$= 2\sin(x_0/2)(x_1/2)\cos(x_0/2)$$

$$A_1^0 = -x_1 \sin(x_0/2)\cos(x_0/2)\, e^{\quad \sin^2(x_0/2)}$$
$$A_2^0 = e^{-u_0^0}(-u_2^0 + (\tfrac{1}{2})(u_1^0)^2)$$
$$u_2^0 = A_2^1 = (u_1^1)^2 + 2u_0^1 u_2^1$$
$$u_1^0 = A_1^1 = 2u_0^1 u_1^1$$
$$u_1^1 = A_1^2 = (x_1/2)\cos(x_0/2)$$
$$u_0^1 = A_0^2 \sin(x_0/2)$$
$$A_2^0 = e^{-u_0^0}(-\{(u_1^1)^2 + 2u_0^1 u_2^1\} + (\tfrac{1}{2})(2u_0^1 u_1^1)^2)$$
$$= e^{-u_0^0}(-(u_1^1)^2 - 2u_0^1 u_2^1 + (\tfrac{1}{2})(2u_0^1 u_1^1)^2)$$

where

$$u_1^1 = A_1^2 = (x_1/2)\cos(x_0/2)$$
$$u_0^1 = \sin(x_0/2) = A_0^2$$
$$u_2^1 = A_2^2 = (x_2/2)\cos(x_0/2) - (x_1^2/8)\sin(x_0/2)$$
$$A_2^0 = e^{-\sin^2(x_0/2)}\{-(x_1^2/4)\cos^2(x_0/2)$$
$$-2(\sin(x_0/2))((x_2/2)\cos(x_0/2) - (x_1^2/8)\sin(x_0/2))$$
$$+2(\sin(x_0/2))^2\,((x_1/2)\cos(x_0/2))^2\}$$

Finally

$$A_2^0 = e^{-\sin^2(x_0/2)}\,\{(x_1^2/4)\sin^2(x_0/2) - (x_1^2/4)\cos^2(x_0/2)$$
$$-x_2\sin(x_0/2)\cos(x_0/2) + (x_1^2/2)\sin^2(x_0/2)\cos^2(x_0/2)\}$$
$$A_3^0 = e^{-u_0^0}(-u_3^0 + u_1^0 u_2^0 - (\tfrac{1}{6})(u_1^0)^3)$$
$$= e^{-A_0^1}(-A_3^1 + A_1^1 A_2^1 - ((\tfrac{1}{6})\, A_1^1)^3)$$
$$\vdots$$

so that $x = \sum_{n=0}^{\infty} x_n$, where

$$x_0 = \pi/2 = 1.570796327$$
$$x_1 = e^{-\sin^2(x_0/2)} = 0.999812126$$
$$x_2 = -x_1\sin(x_0/2)\cos(x_0/2)\, e^{-\sin^2(x_0/2)} = -0.0137009172$$

etc., using the above A_n, and it does not appear worthwhile to go further; so a three-term approximation $\phi_3 = 2.58430937$. Check with $x = \phi_3$, $e^{-\sin^2(\phi_3/2)} = 0.999491607$ so the right side is 2.57028793.

It is easy to see now from the referenced work [1, 2] and this paper that these results apply not only to algebraic equations but also to differential equations in the form $Ly + Ny = g(x)$, where Ny is a composite nonlinearity since we get $L^{-1}Ly = L^{-1}g(x) - L^{-1}Ny = L^{-1}g(x) - L^{-1}\sum_{n=0}^{\infty} A_n$. If $Ly = dy/dx$ and $y(0) = k$, for example, $y = \sum_{n=0}^{\infty} y_n = k + L^{-1}g - L^{-1}\sum_{n=0}^{\infty} A_n$, where $y_0 = k + L^{-1}g$, and $y_{n+1} = -L^{-1}A_n$ for $n \geqslant 0$ and the A_n are calculated by the methods discussed.

References

1. G. Adomian, "Stochastic Systems," Academic Press, New York, 1983.
2. G. Adomian, "Nonlinear Stochastic Operator Equations," Academic Press, Orlando/San Diego, to be published.