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# Convex hypersurfaces and $L^p$ estimates for Schrödinger equations

Quan Zheng,[a,b,*] Xiaohua Yao,[a,c] and Dashan Fan[a,d]

[a] Department of Mathematics, Huazhong University of Science and Technology, Wuhan 430074, People's Republic of China
[b] Center for Optimal Control and Discrete Mathematics, Huazhong Normal University, Wuhan 430079, People's Republic of China
[c] Department of Mathematics, Lanzhou University, Lanzhou 730000, People's Republic of China
[d] Department of Mathematics, University of Wisconsin-Milwaukee, Milwaukee, WI 53201, USA

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**Abstract**

This paper is concerned with Schrödinger equations whose principal operators are homogeneous elliptic. When the corresponding level hypersurface is convex, we show the $L^p$–$L^q$ estimate of the solution operator in the free case. This estimate, combined with the results of fractionally integrated groups, allows us to further obtain the $L^p$ estimate of solutions for the initial data belonging to a dense subset of $L^p$ in the case of integrable potentials.


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## 1. Introduction

In this paper, we take interest in $L^p$, $1 \leqslant p < \infty$, estimates of solutions for the following Schrödinger equation:

$$\frac{\partial u}{\partial t} = (iP(D) + V)u, \quad u(0, \cdot) = u_0 \in L^p(\mathbf{R}^n), \tag{$*$}$$

*Corresponding author.
*E-mail addresses:* qzheng@hust.edu.cn (Q. Zheng), yaoxiaohua@hust.edu.cn (X. Yao), fan@csd.uwm.edu (D. Fan).


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where $D = -i(\partial/\partial x_1, \dots, \partial/\partial x_n)$, $P : \mathbf{R}^n \to \mathbf{R}$ is a homogeneous elliptic polynomial of order $m$ ($m$ must be even, except $n = 1$), and $V$ is a suitable potential function. In the sequel, we may assume without loss of generality that $P(\xi) > 0$ for $\xi \neq 0$. Otherwise, we have $P(\xi) < 0$ for $\xi \neq 0$, for which the following hypersurface $\Sigma$ should be replaced by

$$\{\xi \in \mathbf{R}^n | \, P(\xi) = -1\}.$$

In order to obtain $L^p$ estimates of the solution of $(*)$, we will first treat $L^p$–$L^q$ estimates of $e^{itP(D)}$, which is the solution operator of $(*)$ with $V = 0$. To this end, we need to consider the compact hypersurface

$$\Sigma = \{\xi \in \mathbf{R}^n | \, P(\xi) = 1\}.$$

When the Gaussian curvature of $\Sigma$ is nonzero everywhere, it is known that $L^p$–$L^q$ estimates of $e^{itP(D)}$ $(t \neq 0)$ can be deduced from Miyachi [12]. In fact, Miyachi gave some remarks on these estimates in a more general case where $P$ is a positive and smooth homogeneous function, provided the nonvanishing Gaussian curvature on $\Sigma$. Also, dropping the homogeneity of $P$, Balabane and Emami-Rad [4] studied these estimates under a suitable nondegenerate condition. However, one can check that the nondegenerate condition is equivalent to the nonzero Gaussian curvature if $P$ is homogeneous.

As we know, the nonvanishing Gaussian curvature plays a crucial rule in estimating many oscillatory integrals [16]. This is the reason why one needs such a condition in [4,12]. However, there exist many hypersurfaces $\Sigma$ whose Gaussian curvatures may vanish at some points (although we have observed that if $m = 2$ then $\Sigma$ has nonzero Gaussian curvature everywhere under our assumptions on $P$). These examples are easily available, for instance, the hypersurfaces $\Sigma$ associated with polynomials $\xi_1^m + \cdots + \xi_n^m$ $(m = 4, 6, \dots)$ or $\xi_1^4 + 6\xi_1^2\xi_2^2 + \xi_2^4$.

On the other hand, an important subclass of hypersurfaces with vanishing Gaussian curvature at some points is the class of convex hypersurfaces of finite type [5]. The main purpose of this paper is to investigate the $L^p$ estimate of the solution of $(*)$ when $\Sigma$ is a convex hypersurface of finite type. Roughly speaking, this means that $P$ allows to be degenerate on a subset of $\mathbf{R}^n$.

This paper is organized as follows.

In Section 2, we study $L^p$–$L^q$ estimates of the solution operator $e^{itP(D)}$ $(t \neq 0)$ and the resolvent operator $(\lambda - iP(D))^{-1}$ $(\operatorname{Re} \lambda \neq 0)$ when $\Sigma$ is a convex hypersurface of finite type. The method used is quite different from those in the previous papers [4,12], due to the nature of the vanishing Gaussian curvature. Our proof depends heavily on a decay estimate for the kernels $\mathscr{F}^{-1}(e^{\pm iP})$, in which we need to use a powerful theorem in [5]. Since the proof is involved and very technical, we will present it in Section 3.

In Section 4, we show that the operator $iP(D) + V$ with suitable integrable potential $V$ generates an integrated group on $L^p(\mathbf{R}^n)$. As we know, the semigroup of operators is a useful abstract tool to treat Cauchy problems. However, the Cauchy problem $(*)$ in $L^p(\mathbf{R}^n)$ $(p \neq 2)$ cannot be treated by classical semigroups of operators (i.e. $C_0$-semigroups). In fact, the Schrödinger operator $iP(D)$ generates $C_0$-semigroups in $L^p(\boldsymbol{R}^n)$ if and only if $p=2$ (see [10,12]). Thus, several generalizations of $C_0$-semigroups, such as smooth distribution semigroups [3], integrated semigroups [1,9], and regularized semigroups [7,8] were introduced and applied to different general differential operators [9,18]. In our case, we use fractionally integrated groups to deal with the Cauchy problem $(*)$ in $L^p(\mathbf{R}^n)$, which will lead to better results than using smooth distribution semigroups (see [3]). Moreover, when $P$ is nondegenerate, we will show how our results present an improvement over Theorems 2′ and 6 in [4].

Throughout this paper, $\Sigma$ denotes the hypersurface $\{\xi \in \mathbf{R}^n \mid P(\xi)=1\}$. Assume that $P:\mathbf{R}^n \to [0,\infty)$ is always a homogeneous elliptic polynomial of order $m$ where $n \geqslant 2$, $m$ is even and $\geqslant 4$.

## 2. $L^p$-$L^q$ Estimates for Schrödinger equations without potentials

We start with the concept of finite type. $S$ denotes the smooth hypersurface $\{\xi \in \mathbf{R}^n \mid \phi(\xi)=0\}$, where $\phi \in C^\infty(\mathbf{R}^n)$ and $\nabla\phi(\xi) \neq 0$ for $\xi \in S$. We say that $S$ is of finite type if any one-dimensional tangent line has at most a finite order of contact with $S$. The precise definition is as follows.

$\mathbf{S}^{n-1}$ denotes the unit sphere in $\mathbf{R}^n$. Let

$$\nabla_\eta = \sum_{j=1}^{n} \eta_j \partial/\partial x_j \quad \text{for } \eta=(\eta_1,\dots,\eta_n) \in \mathbf{S}^{n-1},$$

which is the directional derivative in direction $\eta$, and let $\nabla_\eta^j$ be the $j$th power of this derivative.

**Definition 2.1.** Let $k$ be an integer. The smooth hypersurface $S$ is of type $k$ if there exists a constant $\delta > 0$ such that

$$\sum_{j=1}^{k} |\nabla_\eta^j \phi(\xi)| \geqslant \delta \quad \text{for } \xi \in S \text{ and } \eta \in \mathbf{S}^{n-1}.$$

Moreover, we say that $S$ is convex if

$$S \subset \{\eta \in \mathbf{R}^n \mid \langle \eta-\xi, \nabla\phi(\xi)\rangle \geqslant 0\} \quad \text{for } \xi \in S$$

or

$$S \subset \{\eta \in \mathbf{R}^n \mid \langle \eta-\xi, \nabla\phi(\xi)\rangle \leqslant 0\} \quad \text{for } \xi \in S.$$

It is clear that $k \geqslant 2$, and that if $S$ is of type $k$ it is also of type $k'(>k)$. For the hypersurface $\Sigma$ (i.e. $\{\xi \in \mathbf{R}^n \mid P(\xi)=1\}$), since

$$\langle \xi, \nabla P(\xi)\rangle = mP(\xi) = m \quad \text{for } \xi \in \Sigma,$$

it follows that $\nabla P(\xi) \neq 0$ for $\xi \in \Sigma$, and thus $\Sigma$ is smooth. Also, a simple computation leads to

$$\nabla_\eta^m (P(\xi) - 1) = m! P(\eta) \quad \text{for } \xi \in \Sigma \text{ and } \eta \in \mathbf{S}^{n-1}.$$

Hence, we have

**Proposition 2.2.** *$\Sigma$ is a smooth compact hypersurface of type less than or equal to m.*

A simple example of polynomials whose level hypersurface $\Sigma$ is of type $m$ is $\xi_1^m + \cdots + \xi_n^m$ $(m = 4, 6, \ldots)$. We notice that there exist polynomials $P$ whose level hypersurfaces $\Sigma$ are of type $k(<m)$. For example, when $P(\xi) = \xi_1^6 + 5\xi_1^2\xi_2^4 + \xi_2^6$, the corresponding hypersurface $\Sigma$ is of type 4, but $m = 6$.

We now turn to the Cauchy problem $(*)$ with $V = 0$. In this case, for every initial data $u_0 \in \mathcal{S}(\mathbf{R}^n)$ (the Schwartz space), the solution is given by

$$u(t, \cdot) = e^{itP(D)} u_0 := \mathcal{F}^{-1}(e^{itP}) * u_0,$$

where $\mathcal{F}$ (or $\hat{}$) denotes the Fourier transform, $\mathcal{F}^{-1}$ its inverse, and $\mathcal{F}^{-1}(e^{itP})$ is understood in the distributional sense. Therefore, to obtain $L^p$–$L^q$ estimates of $e^{itP(D)}$ $(t \neq 0)$, the key result is to show estimates of the kernels $\mathcal{F}^{-1}(e^{\pm iP})$.

In the sequel, denote by $p'$ the conjugate index of $p$, and $\|\cdot\|_{L^p-L^q}$ the norm in $\mathcal{L}(L^p, L^q)$ (the space of all bounded linear operators from $L^p$ to $L^q$). Let

$$h(m,n,k) = \frac{m-2}{2(m-1)} + \frac{(m-k)(n-1)}{k(m-1)} \quad \text{for } 2 \leqslant k \leqslant m,$$

$\tau = n/h(m,n,k)$, and $q(p) = q(m,n,k,p)$ where

$$\frac{1}{q(m,n,k,p)} = \frac{1}{\tau p} + \frac{1}{\tau' p'} \quad \text{for } 1 \leqslant p < 2.$$

We first remark that when $2 \leqslant k \leqslant m$,

$$\frac{2(m-1)}{m-2} \leqslant \tau \leqslant \frac{2n(m-1)}{m-2}.$$

Since $m \geqslant 4$, it follows that $\tau \in (2, 3n]$. Next, we remark that

$$\frac{1}{2} > \frac{1}{\tau p} + \frac{1}{\tau' p'} > \frac{1}{p'} \quad \text{for } 1 \leqslant p < 2$$

and thus $2<q(p)<p'$. Moreover, denote by $I_p$ $(1\leqslant p\leqslant 2)$ the following subset of $[2,\infty]$:

$$I_p=\begin{cases}(q(p),\infty] & \text{if } 1\leqslant p<\tau',\\ \left(q(p),\dfrac{p(2-\tau')}{p-\tau'}\right) & \text{if } \tau'\leqslant p<2,\\ \{2\} & \text{if } p=2.\end{cases}$$

**Theorem 2.3.** *Suppose $\Sigma$ is a convex hypersurface of type $k$. Then $\mathscr{F}^{-1}(e^{\pm iP})\in C^\infty(\mathbf{R}^n)$ and*

$$(\mathscr{F}^{-1}(e^{\pm iP}))(x)=O(|x|^{-h(m,n,k)})\quad \textit{as } |x|\to\infty.$$

The proof is lengthy and is given in the next section.

**Theorem 2.4.** *Suppose $\Sigma$ is a convex hypersurface of type $k$. If $p\in[1,2]$ and $q\in I_p$, then there exists a constant $C>0$ such that*

$$||e^{itP(D)}||_{L^p-L^q}\leqslant C|t|^{\frac{n}{m}\left(\frac{1}{q}-\frac{1}{p}\right)}\quad \text{for } t\neq 0.$$

**Proof.** By Theorem 2.3, $\mathscr{F}^{-1}(e^{\pm iP})\in L^s(\mathbf{R}^n)$ for $s>\tau$. Since $P$ is homogeneous, one has

$$\mathscr{F}^{-1}(e^{itP})(x)=|t|^{-n/m}\mathscr{F}^{-1}(e^{itP/|t|})(|t|^{-1/m}x)\quad \text{for } t\neq 0 \text{ and } x\in\mathbf{R}^n,$$

and thus

$$||\mathscr{F}^{-1}(e^{itP})||_{L^s}=|t|^{-n/ms'}||\mathscr{F}^{-1}(e^{itP/|t|})||_{L^s}\leqslant C|t|^{-n/ms'}\quad \text{for } t\neq 0,$$

where the constant $C$ is independent of $t$. The remainder of the proof will be divided into several steps.

*Step* 1. When $1\leqslant p<\tau'$ and $\frac{\tau' p}{\tau'-p}<q\leqslant\infty$, it follows from Young's inequality that

$$||e^{itP(D)}||_{L^p-L^q}\leqslant||\mathscr{F}^{-1}(e^{itP})||_{L^s}\leqslant C|t|^{\frac{n}{m}\left(\frac{1}{q}-\frac{1}{p}\right)}\quad \text{for } t\neq 0,$$

where $\frac{1}{s}=1+\frac{1}{q}-\frac{1}{p}$, which implies $s>\tau$.

*Step* 2. Since $P(D)$ is self-adjoint in $L^2(\mathbf{R}^n)$, $||e^{itP(D)}||_{L^2-L^2}=1$ for $t\geqslant 0$ by Stone's theorem. When $1\leqslant p<2$ and $q(p)<q\leqslant p'$, we deduce from the Riesz–Thorin

interpolation theorem that

$$||e^{itP(D)}||_{L^p-L^q} \leqslant ||e^{itP(D)}||^{1-2/p'}_{L^1-L^s} ||e^{itP(D)}||^{2/p'}_{L^2-L^2} \leqslant C|t|^{\frac{n}{m}\left(\frac{1}{q}-\frac{1}{p}\right)} \quad \text{for } t \neq 0,$$

where $s = \frac{q(p'-2)}{p'-q} > \tau$.

*Step* 3. When $1 \leqslant p < \tau'$ and $q(p) < q \leqslant \infty$, we notice $q(p) < \frac{\tau' p}{\tau' - p}$. Since $L^{s_2}(\mathbf{R}^n) \subset L^{s_1}(\mathbf{R}^n) + L^{s_3}(\mathbf{R}^n)$ for $1 \leqslant s_1 \leqslant s_2 \leqslant s_3 \leqslant \infty$, the desired estimate is a direct consequence of the conclusions in Steps 1 and 2.

*Step* 4. When $\tau' \leqslant p < 2$ and $q(p) < q < \frac{p(2-\tau')}{p-\tau'}$, we put $\lambda = \frac{2(p-p_0)}{p(2-p_0)}$, where $p_0 \in [1, \tau')$ such that $q < \frac{2}{\lambda} < \frac{p(2-\tau')}{p-\tau'}$. A simple computation leads to

$$\frac{\lambda}{2} < \frac{1}{q} < \frac{1}{q(p)} = \frac{1-\lambda}{q(p_0)} + \frac{\lambda}{2}.$$

Consequently, there exists a unique $q_0 > q(p_0)$ such that $\frac{1}{q} = \frac{1-\lambda}{q_0} + \frac{\lambda}{2}$. Also, $\frac{1}{p} = \frac{1-\lambda}{p_0} + \frac{\lambda}{2}$. The desired estimate now can be deduced from the Riesz–Thorin interpolation theorem and the conclusion in Step 1. □

The subsequent theorem deals with $L^p$–$L^q$ estimates of the resolvent of $iP(D)$.

**Theorem 2.5.** *Suppose $\Sigma$ is a convex hypersurface of type $k$. If $p \in [1,2]$, $q \in I_p$, and $\frac{1}{p} - \frac{1}{q} < \frac{m}{n}$, then there exists a constant $C > 0$ such that*

$$||(\lambda - iP(D))^{-1}||_{L^p-L^q} \leqslant C|\mathrm{Re}\,\lambda|^{\frac{n}{m}\left(\frac{1}{p}-\frac{1}{q}\right)-1} \quad \textit{for } \mathrm{Re}\,\lambda \neq 0.$$

**Proof.** For $\mathrm{Re}\,\lambda > 0$ and $f \in \mathscr{S}(\mathbf{R}^n)$, one has

$$\begin{aligned}(\lambda - iP(D))^{-1} f &= \mathscr{F}^{-1}((\lambda - iP)^{-1}\hat{f}) \\ &= \int_0^\infty e^{-\lambda t} \mathscr{F}^{-1}(e^{itP}\hat{f})\, dt \\ &= \int_0^\infty e^{-\lambda t} e^{itP(D)} f\, dt.\end{aligned}$$

It follows therefore from Theorem 2.4 that

$$\begin{aligned}||(\lambda - iP(D))^{-1}||_{L^p-L^q} &\leqslant C \int_0^\infty e^{-(\mathrm{Re}\,\lambda)t} t^{\frac{n}{m}\left(\frac{1}{q}-\frac{1}{p}\right)}\, dt \\ &= C|\mathrm{Re}\ \lambda|^{\frac{n}{m}\left(\frac{1}{p}-\frac{1}{q}\right)-1}.\end{aligned}$$

For Re $\lambda<0$ and $f\in\mathscr{S}(\mathbf{R}^n)$, one has

$$(\lambda - iP(D))^{-1}f = -(-\lambda + iP(D))^{-1}f = \int_0^\infty e^{\lambda t}e^{-itP(D)}f\,dt,$$

and thus the desired estimate follows from Theorem 2.4. $\square$

Since $p'\in I_p$ for $p\in[1,2]$, we have

**Corollary 2.6.** *Suppose $\Sigma$ is a convex hypersurface of type $k$. If $p\in[1,2]$ then*

$$\|e^{itP(D)}\|_{L^p-L^{p'}} \leqslant C|t|^{\frac{n}{m}\left(1-\frac{2}{p}\right)} \quad \textit{for } t\neq 0.$$

*If, in addition, $p>\frac{2n}{n+m}$, then*

$$\|(\lambda - iP(D))^{-1}\|_{L^p-L^{p'}} \leqslant C|\text{Re }\lambda|^{-\frac{n}{m}\left(\frac{2}{p}-1\right)-1} \quad \textit{for } \text{Re }\lambda\neq 0.$$

## 3. The proof of Theorem 2.3

By our assumptions on $P$, $\phi := P^{1/m}$ is a positive and smooth homogeneous function of degree 1, and $\Sigma = \{\xi\in\mathbf{R}^n |\, \phi(\xi) = 1\}$. Let $\varphi\in C^\infty(\mathbf{R})$ such that $\text{supp}\,\varphi\subset[1,\infty)$ and $\varphi(t) = 1$ for $t\geqslant 2$. Obviously, in order to estimate $\mathscr{F}^{-1}(e^{iP})$ (similarly for $\mathscr{F}^{-1}(e^{-iP})$), it suffices to estimate $\mathscr{F}^{-1}((\varphi\circ\phi)e^{iP})$ (cf. [4, p. 363]). Consider the integral

$$\begin{aligned} K_\varepsilon(x) &:= \int_{\mathbf{R}^n} e^{-\varepsilon\phi(y)+iP(y)+i\langle x,y\rangle}\varphi(\phi(y))\,dy \\ &= \int_0^\infty e^{-\varepsilon t+it^m}t^{n-1}\varphi(t)\left(\int_\Sigma \frac{e^{itr\langle\eta,\xi\rangle}}{|\nabla\phi(\xi)|}\,d\sigma(\xi)\right)dt \quad \text{for } \varepsilon>0, \end{aligned}$$

where $r = |x|$, $x = r\eta$, and $d\sigma$ is the induced surface measure on $\Sigma$. We will show that

$$\mathscr{F}^{-1}((\varphi\circ\phi)e^{iP})(x) = (2\pi)^{-n}\lim_{\varepsilon\to 0} K_\varepsilon(x)$$

uniformly for $x$ in compact subsets of $\mathbf{R}^n$, and $K_\varepsilon(x)$ decays as $|x|^{-h(m,n,k)}$. From this we have $\mathscr{F}^{-1}((\varphi\circ\phi)e^{iP})\in C(\mathbf{R}^n)$ since it is clear that $K_\varepsilon\in C(\mathbf{R}^n)$.

Denote by $\Pi$ the Gaussian map

$$\xi\in\Sigma\mapsto\frac{\nabla\phi(\xi)}{|\nabla\phi(\xi)|}\in\mathbf{S}^{n-1}.$$

Since $\Sigma$ is a compact convex hypersurface, $\Pi$ is a homeomorphism from $\Sigma$ to $\mathbf{S}^{n-1}$. Given $\eta \in \mathbf{S}^{n-1}$, let $\xi_\pm = \Pi^{-1}(\pm\eta)$. Then

$$\langle \eta, \xi_\pm \rangle = \pm \left\langle \frac{\nabla\phi(\xi_\pm)}{|\nabla\phi(\xi_\pm)|}, \xi_\pm \right\rangle = \pm \frac{\phi(\xi_\pm)}{|\nabla\phi(\xi_\pm)|} = \pm \frac{1}{|\nabla\phi(\xi_\pm)|},$$

Noting that $\pm\eta$ is the outward unit normal to $\Sigma$ at $\xi_\pm$, by Theorem B in [5] (also cf. [6]) we have

$$\int_\Sigma \frac{e^{i\lambda\langle \eta,\xi\rangle}}{|\nabla\phi(\xi)|}\, d\sigma(\xi) = e^{i\lambda\langle \eta,\xi_+\rangle} H_+(\lambda) + e^{i\lambda\langle \eta,\xi_-\rangle} H_-(\lambda) + H_\infty(\lambda) \quad \text{for } \lambda > 0.$$

Here $H_\pm \in C^\infty((0,\infty))$,

$$|H_\pm^{(j)}(\lambda)| \leqslant C_j \lambda^{-j-(n-1)/k} \quad \text{for } j \in \mathbf{N}_0,$$

and

$$|H_\infty(\lambda)| \leqslant C_j \lambda^{-j} \quad \text{for } j \in \mathbf{N},$$

where $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$ and constants $C_j$ depend only on the hypersurface $\Sigma$. Hence,

$$\begin{aligned} K_\varepsilon(x) = & \int_0^\infty e^{-\varepsilon t + it^m + itr\langle \eta,\xi_+\rangle} t^{n-1} \varphi(t) H_+(tr)\, dt \\ & + \int_0^\infty e^{-\varepsilon t + it^m + itr\langle \eta,\xi_-\rangle} t^{n-1} \varphi(t) H_-(tr)\, dt \\ & + \int_0^\infty e^{-\varepsilon t + it^m} t^{n-1} \varphi(t) H_\infty(tr)\, dt \\ := & J_1^\varepsilon + J_2^\varepsilon + J_3^\varepsilon. \end{aligned}$$

In the remainder of this section, for the sake of convenience, we will denote by $C$ a generic constant independent of $r$, $t$ and $\varepsilon$.

We consider first the integral $J_3^\varepsilon$. It is obvious that

$$|J_3^\varepsilon| \leqslant C \int_1^\infty t^{n-1} (tr)^{-(n+1)}\, dt \leqslant Cr^{-(n+1)}.$$

Since $r = |x|$, it follows from the dominated convergence theorem that $J_3^\varepsilon$ $(\varepsilon \to 0)$ converges uniformly for $x$ in compact subsets of $\mathbf{R}^n \backslash \{0\}$, and decays as $|x|^{-h(m,n,k)}$, where we notice that $h(m,n,k) \leqslant n$.

Next, consider the integral $J_1^\varepsilon$. Let

$$\begin{cases} u(t) = -\varepsilon t + it^m + itr\langle \eta, \xi_+ \rangle, \\ v(t) = t^{n-1}\varphi(t) H_+(tr) \end{cases}$$

for $t>0$. Since $u'(t)\neq 0$ for $t>0$, we can define $D_{\#}f=-gf'$ and $D_{*}f=(gf)'$ for $f\in C^1((0,\infty))$, where $g=-1/u'$. By induction on $j$ we find that

$$g^{(j)}(t)=\sum_{l=l_0}^{j} a_l t^{l(m-1)-j} g(t)^{l+1} \quad \text{for } j\in \mathbf{N}_0,$$

where constants $a_l$ depend only on $l$ and $m$, and $l_0\in\mathbf{N}_0$ such that $l_0\geqslant j/(m-1)$. Since there exists a constant $c>1$ such that $c^{-1}\leqslant|\nabla\phi(\xi)|\leqslant c$ for $\xi\in\Sigma$, we have

$$|g(t)|=1/|u'(t)|\leqslant 1/r\langle\eta,\xi_+\rangle=|\nabla\phi(\xi_+)|/r\leqslant c/r \quad \text{for } t>0.$$

Also, $|g(t)|\leqslant\frac{1}{m}t^{1-m}$ for $t>0$. Hence,

$$|g^{(j)}(t)|\leqslant Cr^{-1}t^{-j} \quad \text{for } j\in\mathbf{N}_0.$$

On the other hand, one sees

$$\left|\frac{d^j}{dt^j}(H_+(tr))\right|\leqslant Ct^{-j}(tr)^{-(n-1)/k} \quad \text{for } j\in\mathbf{N}_0,$$

and thus by Leibniz's formula

$$|v^{(j)}(t)|\leqslant Cr^{-(n-1)/k}t^{-j+n-1-(n-1)/k} \quad \text{for } j\in\mathbf{N}_0. \tag{3.1}$$

Since it is not hard to show

$$D_*^j v=\sum_{\alpha} a_\alpha g^{(\alpha_1)}\cdots g^{(\alpha_j)}v^{(\alpha_{j+1})} \quad \text{for } j\in\mathbf{N},$$

where the sum runs over all $\alpha=(\alpha_1,\dots,\alpha_{j+1})\in\mathbf{N}_0^{j+1}$ such that $|\alpha|=j$ and $0\leqslant\alpha_1\leqslant\cdots\leqslant\alpha_j$, it follows that

$$|(D_*^j v)(t)|\leqslant Cr^{-j-(n-1)/k}t^{-j+n-1-(n-1)/k} \quad \text{for } j\in\mathbf{N}_0, \tag{3.2}$$

where we used the notation $D_*^0 v=v$. Noting that $D_{\#}^n e^u=e^u$ we have

$$J_1^\varepsilon=\int_0^\infty (D_{\#}^n e^u)(t)v(t)\,dt=\int_0^\infty e^{u(t)}(D_*^n v)(t)\,dt.$$

Consequently,

$$|J_1^\varepsilon|\leqslant Cr^{-n-(n-1)/k}\int_1^\infty t^{-1-(n-1)/k}\,dt\leqslant Cr^{-n-(n-1)/k}.$$

Thus, the dominated convergence theorem yields that $J_1^\varepsilon$ converges uniformly for $x$ in compact subsets of $\mathbf{R}^n\backslash\{0\}$, and decays as $|x|^{-h(m,n,k)}$.

We now consider the integral $J_2^\varepsilon$. In this case, we put

$$\begin{cases} u(t) = -\varepsilon t + it^m - it\bar{r}, \\ v(t) = t^{n-1}\varphi(t)H_-(tr) \end{cases}$$

for $t>0$, where

$$\bar{r} := -r\langle \eta, \xi_- \rangle = r/|\nabla\phi(\xi_-)|.$$

Since $t_0 := (\bar{r}/m)^{1/(m-1)}$ is the unique critical point of the oscillatory integral $J_2^\varepsilon$, we write

$$\begin{aligned} J_2^\varepsilon &= \left\{ \int_{2t_0}^\infty + \int_{t_0/2}^{2t_0} + \int_0^{t_0/2} \right\} e^{u(t)} v(t)\, dt \\ &:= J_{2,1}^\varepsilon + J_{2,2}^\varepsilon + J_{2,3}^\varepsilon. \end{aligned}$$

From integration by parts one gets

$$J_{2,1}^\varepsilon = -\frac{e^{u(2t_0)}}{u'(2t_0)} \sum_{j=0}^{n-1} (D_*^j v)(2t_0) + \int_{2t_0}^\infty e^{u(t)} (D_*^n v)(t)\, dt.$$

Since

$$|u'(t)| \geqslant mt^{m-1} - \bar{r} \geqslant (2^{m-1}-1)\bar{r} \geqslant (2^{m-1}-1)r/c \quad \text{for } t \geqslant 2t_0$$

and since

$$|u'(t)| \geqslant mt^{m-1} - \bar{r} \geqslant m(1-2^{1-m})t^{m-1} \quad \text{for } t \geqslant 2t_0,$$

the estimate (3.2) still holds for $t \geqslant 2t_0$. Hence,

$$\begin{aligned} |J_{2,1}^\varepsilon| &\leqslant Cr^{-1} \sum_{j=0}^{n-1} r^{-j-(n-1)/k} (2t_0)^{-j+n-1-(n-1)/k} \\ &\quad + C \int_{2t_0}^\infty r^{-n-(n-1)/k} t^{-1-(n-1)/k}\, dt \\ &\leqslant Cr^{(n-m-m(n-1)/k)/(m-1)} \sum_{j=0}^{n-1} r^{-jm/(m-1)}. \end{aligned}$$

But

$$(n-m-m(n-1)/k)/(m-1) \leqslant -h(m,n,k),$$

$J^{\varepsilon}_{2,1}$ converges uniformly for $x$ in compact subsets of $\mathbf{R}^n\setminus\{0\}$, and decays as $|x|^{-h(m,n,k)}$. Since $|u'(t)|\geqslant Ct^{m-1}$ and $\geqslant Cr$ for $0<t\leqslant t_0/2$, a slight modification of the above method leads to the same conclusion for $J^{\varepsilon}_{2,3}$. We omit the details.

In order to deal with $J^{\varepsilon}_{2,2}$, set $w(t)=t^m-t\bar{r}$ and $v(t)$ is defined as in $J^{\varepsilon}_{2,1}$. Obviously,

$$\lim_{\varepsilon\to 0} J^{\varepsilon}_{2,2} = J_{2,2} := \int_{t_0/2}^{2t_0} e^{iw(t)}v(t)\,dt$$

uniformly for $x$ in compact subsets of $\mathbf{R}^n\setminus\{0\}$. It remains to show that $J_{2,2}$ decays as $|x|^{-h(m,n,k)}$. Let $y=(t-t_0)/t_0$, which maps $[t_0/2, 2t_0]$ onto $[-1/2, 1]$, and let $\lambda = m(m-1)t_0^m$. Then

$$\begin{aligned}\Phi(y) &:= \frac{1}{\lambda}(w(t_0(y+1))-w(t_0))\\ &= \frac{1}{m(m-1)}((y+1)^m - my - 1)\\ &= \frac{1}{m(m-1)}\sum_{l=2}^{m}\binom{m}{l}y^l \quad \text{for } y\in[-1/2,1].\end{aligned}$$

Consequently,

$$\Phi''(y) = (y+1)^{m-2} \geqslant 2^{2-m} \quad \text{for } y\in[-1/2,1]$$

and

$$J_{2,2} = t_0 e^{iw(t_0)} \int_{-1/2}^{1} e^{i\lambda\Phi(y)} v(t_0(y+1))\ dy.$$

It follows thus from van der Corput's theorem (cf. [16, p. 334]) that

$$\begin{aligned}|J_{2,2}| &\leqslant C\lambda^{-1/2}t_0\left(|v(2t_0)| + \int_{-1/2}^{1} |t_0 v'(t_0(y+1))|\,dy\right)\\ &\leqslant Ct_0^{1-m/2}\left(|v(2t_0)| + \sup_{t_0/2\leqslant t\leqslant 2t_0} |t_0 v'(t)|\right).\end{aligned}$$

Since (3.1), in which $H_+$ is replaced by $H_-$, still holds,

$$|J_{2,2}| \leqslant Cr^{-(n-1)/k} t_0^{n-m/2-(n-1)/k} \leqslant Cr^{-h(m,n,k)},$$

as desired.

Finally, we only need to show that $K_\varepsilon(x)$ converges uniformly for $x$ in some neighborhood of the origin. Let $U$ be the ball $\{x\in\mathbf{R}^n|\ |x|\leqslant m/2M\}$, where $M$ is a constant such that $|\xi| \leqslant M$ for $\xi \in \Sigma$. For a given $x \in U$, let

$$u(t) = -\varepsilon t + it^m + it\langle x, \xi\rangle \quad \text{for } t \geqslant 1.$$

Then

$$K_\varepsilon(x) = \int_\Sigma \left( \int_1^\infty e^{u(t)} t^{n-1} \varphi(t)\, dt \right) \frac{d\sigma(\xi)}{|\nabla \phi(\xi)|}.$$

Since

$$|u'(t)| \geqslant |mt^{m-1} + \langle x, \xi\rangle| \geqslant mt^{m-1} - M|x| \geqslant (m/2)t^{m-1} \geqslant m/2,$$

as in the case of the estimate of $J_1^\varepsilon$ we obtain

$$\left| \int_1^\infty e^{u(t)} t^{n-1} \varphi(t)\, dt \right| \leqslant C,$$

and therefore the claim follows.

Moreover, an analogous method as above leads to

$$D^\alpha \mathscr{F}^{-1}(e^{\pm iP}) = \mathscr{F}^{-1}(\xi^\alpha e^{\pm iP(\xi)}) \in C(\mathbf{R}^n) \quad \text{for } \alpha \in \mathbf{N}_0^n,$$

i.e. $\mathscr{F}^{-1}(e^{\pm iP}) \in C^\infty(\mathbf{R}^n)$ and

$$D^\alpha(\mathscr{F}^{-1}(e^{\pm iP}))(x) = O(|x|^{-h(m,n,k)+|\alpha|/(m-1)}) \ (|x| \to \infty) \quad \text{for } \alpha \in \mathbf{N}_0^n.$$

## 4. $L^p$ Estimates for Schrödinger equations

It was Balabane and Emami-Rad [4] who first applied smooth distribution semigroups to higher order Schrödinger equations, and showed the $L^p$ estimate of solutions. Arendt and Kellermann [2] showed that a smooth distribution semigroup is equivalent to an integrally integrated semigroup. However, it is known that the fractionally integrated semigroup is a generalization of the integrally integrated semigroup and is a more suitable tool for elliptic differential operators in $L^p(\mathbf{R}^n)$ (cf. [9,17]). We start with its definition.

**Definition 4.1.** Let $A$ be a linear operator on a Banach space $X$ and $\beta \geqslant 0$. Then a strongly continuous family $T : [0, \infty) \to \mathscr{L}(X)$ is called a $\beta$-times integrated semigroup on $X$ with generator $A$ if there exist constants $C, \omega \geqslant 0$ such that $||T(t)|| \leqslant Ce^{\omega t}$ for $t \geqslant 0$, $(\omega, \infty) \subset \rho(A)$ (the resolvent set of $A$), and

$$(\lambda - A)^{-1}x = \lambda^\beta \int_0^\infty e^{-\lambda t} T(t)x\, dt \quad \text{for } \lambda > \omega \text{ and } x \in X.$$

If $A$ and $-A$ both are generators of $\beta$-times integrated semigroups on $X$, we say $A$ is the generator of a $\beta$-times integrated group on $X$.

Here is a sufficient condition for an operator to be the generator of integrated semigroups, which is due to Hieber [9, p. 30] (see [2] for a special case).

**Lemma 4.2.** *Let $A$ be a linear operator on a Banach space $X$. Suppose there exist constants $C, \omega \geqslant 0$ and $\gamma \geqslant -1$ such that for $\operatorname{Re} \lambda > \omega$, $\lambda \in \rho(A)$ and $\|(\lambda - A)^{-1}\| \leqslant C|\lambda|^{\gamma}$. Then $A$ generates a $\beta$-times integrated semigroup on $X$, where $\beta > \gamma + 1$.*

Assume that the operator $P(D)$ has maximal domain in the distributional sense on $L^p(\mathbf{R}^n)$ $(1 \leqslant p < \infty)$, and thus it is closed and densely defined. It is known that $D(P(D)) = W^{m,p}(\mathbf{R}^n)$ for $1 < p < \infty$. The following Lemma 4.3(a) can be found in [9,17], and Lemma 4.3(b) follows from Lemma 4.3(a) and Definition 4.1, immediately.

**Lemma 4.3.** *Let $1 \leqslant p < \infty$ and $\beta > n_p := n|\frac{1}{2} - \frac{1}{p}|$.*

(a) *$iP(D)$ generates a $\beta$-times integrated group $T(t)$ $(t \in \mathbf{R})$ on $L^p(\mathbf{R}^n)$, and*

$$\|T(t)\|_{L^p - L^p} \leqslant C|t|^{\beta} \quad \textit{for } t \in \mathbf{R}.$$

(b) $\{\lambda \in \mathbf{C} | \operatorname{Re} \lambda \neq 0\} \subset \rho(iP(D))$ *on $L^p(\mathbf{R}^n)$, and*

$$\|(\lambda - iP(D))^{-1}\|_{L^p - L^p} \leqslant C|\lambda|^{\beta}/|\operatorname{Re} \lambda|^{\beta+1} \quad \textit{for } \operatorname{Re} \lambda \neq 0.$$

Let $V$ be a measurable function defined on $\mathbf{R}^n$. We consider $V$ as a multiplication operator on $L^p(\mathbf{R}^n)$ with $D(V) := \{f \in L^p(\mathbf{R}^n) | Vf \in L^p(\mathbf{R}^n)\}$. The domain of $iP(D) + V$ is $D(P(D)) \cap D(V)$. Denote by $I'_p$ $(1 \leqslant p \leqslant 2)$, the following subset of $[1, \infty]$:

$$I'_p = \begin{cases} \left[p, \dfrac{\tau' p}{2-p}\right) & \text{if } 1 \leqslant p < \tau', \\ \left(\dfrac{p(2-\tau')}{2-p}, \dfrac{\tau' p}{2-p}\right) & \text{if } \tau' \leqslant p < 2, \\ \{\infty\} & \text{if } p = 2, \end{cases}$$

where $\tau = n/h(m,n,k)$ and $h(m,n,k)$ is defined as in Section 2.

**Theorem 4.4.** *Let $V = V_1 + V_2$ with $V_j \in L^{s_j}(\mathbf{R}^n)$ for some $s_j \in (\frac{n}{m}, \infty]$ $(j = 1, 2)$.*

(a) *If $s_j \in I'_p$ for some $p \in [1, 2]$, then $iP(D) + V$ generates a $\beta$-times integrated group on $L^p(\mathbf{R}^n)$, where $\beta > n_p + 1$.*

(b) *If $s_j \in I'_{p'}$ for some $p \in (2, \infty)$, then an extension of $iP(D) + V$, i.e. $(-iP(D) + \bar{V})^*$ generates a $\beta$-*times *integrated group on $L^p(\mathbf{R}^n)$, where $\beta > n_p + 1$.*

**Proof.** Since $iP(D) + V$ and $-(iP(D) + V)$ satisfy the same assumptions, it suffices to show that $iP(D) + V$ generates a $\beta$-times integrated semigroup on $L^p(\mathbf{R}^n)$.

We consider first the case $1 \leqslant p \leqslant 2$. Let $\frac{1}{q_j} = \frac{1}{p} - \frac{1}{s_j}$ $(j = 1, 2)$. Then $s_j \in I'_p$ implies $q_j \in I_p$ ($I_p$ is defined in Section 2). So we obtain by Theorem 2.5 and Hölder's inequality that

$$\begin{aligned} ||V(\lambda - iP(D))^{-1}||_{L^p - L^p} &\leqslant \sum_{j=1,2} ||V||_{L^{q_j} - L^p} ||(\lambda - iP(D))^{-1}||_{L^p - L^{q_j}} \\ &\leqslant C \sum_{j=1,2} ||V||_{L^{s_j}} |\mathrm{Re}\,\lambda|^{\frac{n}{ms_j} - 1}. \end{aligned}$$

In view of $\frac{n}{ms_j} - 1 < 0$, there exists $\omega \geqslant 1$ such that

$$||V(\lambda - iP(D))^{-1}||_{L^p - L^p} \leqslant 1/2 \quad \text{for } \mathrm{Re}\,\lambda > \omega.$$

Consequently, $\lambda \in \rho(iP(D) + V)$ and

$$(\lambda - iP(D) - V)^{-1} = (\lambda - iP(D))^{-1} \sum_{j=0}^{\infty} (V(\lambda - iP(D))^{-1})^j.$$

This implies by Lemma 4.3(b) that

$$\begin{aligned} ||(\lambda - iP(D) - V)^{-1}||_{L^p - L^p} &\leqslant 2||(\lambda - iP(D))^{-1}||_{L^p - L^p} \\ &\leqslant C|\lambda|^{n_p + \varepsilon} \quad \text{for } \mathrm{Re}\,\lambda > \omega, \end{aligned}$$

where $\varepsilon \in (0, \beta - n_{p-1})$. It follows now from Lemma 4.2 that $iP(D) + V$ generates a $\beta$-times integrated semigroup on $L^p(\mathbf{R}^n)$.

Next, we consider the case $2 < p < \infty$. From the proof of (a) one sees that $-iP(D) + \bar{V}$ is densely defined on $L^{p'}(\mathbf{R}^n)$, and thus $(-iP(D) + \bar{V})^*$ exists and is densely defined on $L^p(\mathbf{R}^n)$. It is easy to check $iP(D) + V \subset (-iP(D) + \bar{V})^*$. Also, an adjointness argument implies

$$||(\lambda - (-iP(D) + \bar{V})^*)^{-1}||_{L^p - L^p} = ||(\bar{\lambda} - (-iP(D) + \bar{V}))^{-1}||_{L^{p'} - L^{p'}}.$$

Since $s_j \in I'_{p'}$ and $n_{p'} = n_p$, this leads to the same estimate as in the case $1 < p < 2$. It follows therefore from Lemma 4.2 that $(-iP(D) + \bar{V})^*$ generates a $\beta$-times integrated semigroup on $L^p(\mathbf{R}^n)$. $\square$

When $2<p<\infty$, we rewrite $I'_{p'}$ as

$$I'_{p'} = \begin{cases} \left(\dfrac{p(2-\tau')}{p-2}, \dfrac{\tau' p}{p-2}\right) & \text{if } 2<p\leqslant\tau, \\ \left[\dfrac{p}{p-1}, \dfrac{\tau' p}{p-2}\right) & \text{if } \tau<p<\infty. \end{cases}$$

From this and the subsequent proposition one sees that it is not always true that $iP(D)+V=(-iP(D)+\bar{V})^*$ in Theorem 4.3(b), so is such situation in [4, Theorem 6]. This means that the operator $iP(D)+V$ in [4, Theorem 6] should be replaced by $(-iP(D)+\bar{V})^*$ for $p>2$.

**Proposition 4.5.** *Let* $1<p<\infty$ *and* $\frac{n}{m}<s<\infty$.

(a) *If* $V\in L^s(\mathbf{R}^n)$ *and* $s\geqslant p$, *then* $W^{m,p}(\mathbf{R}^n)\subset D(V)$ *and* $V$ *is a compact operator from* $W^{m,p}(\mathbf{R}^n)$ *to* $L^p(\mathbf{R}^n)$.

(b) *If* $V\in L^s(\mathbf{R}^n)$ *and* $s\geqslant\max\{p,p'\}$, *then* $iP(D)+V=(-iP(D)+\bar{V})^*$ *on* $L^p(\mathbf{R}^n)$.

(c) *If* $1\leqslant s<p$, *then there exists* $V\in L^s(\mathbf{R}^n)$ *such that* $W^{m,p}(\mathbf{R}^n)\cap D(V)=\{0\}$.

**Proof.** (a) is a direct consequence of Theorem 10.2 in [15, p. 147], in which condition (5) is satisfied. Since $\rho(P(D))\neq\emptyset$ on $L^{p'}(\mathbf{R}^n)$, (b) follows from (a) and Theorem 6.1 in [15, p. 94]. To show (c), we will modify slightly an example in [14, p. 60]. Let $f(x)=|x|^{-n/p}$ for $|x|\leqslant 1$ and $=0$ for $|x|>1$. Define

$$V(x)=\sum_{j=1}^{\infty} 2^{-j} f(x-\alpha_j) \quad \text{for } x\in\mathbf{R}^n,$$

where $\{\alpha_j\}_{j=1}^{\infty}=\mathbf{Q}^n$ ($\mathbf{Q}$ denotes the set of all rational numbers). Then

$$||V||_{L^s}\leqslant\sum_{j=1}^{\infty} 2^{-j}||f(x-\alpha_j)||_{L^s}=||f||_{L^s}<\infty.$$

If $0\neq g\in W^{m,p}(\mathbf{R}^n)\cap D(V)$, one has $g\in C(\mathbf{R}^n)$ by Sobolev's embedding theorem. Thus, $|g(x)|\geqslant c>0$ in some open subset $\Omega\subset\mathbf{R}^n$. Taking $\alpha_j\in\Omega$ yields

$$\int_{\mathbf{R}^n} |V(x)g(x)|^p \, dx \geqslant 2^{-np} c^p \int_{|x-\alpha_j|\leqslant\delta} |x-\alpha_j|^{-n} \, dx = \infty \quad \text{for small } \delta>0,$$

which contradicts $g\in D(V)$. $\square$

Set $I'_p = I'_{p'} \cap [p, \infty)$ for $p \in (2, \infty)$. Then $I'_p = \emptyset$ for $p \geqslant 2 + \tau'$ and

$$I'_p = \begin{cases} \left(\dfrac{p(2-\tau')}{p-2}, \dfrac{\tau' p}{p-2}\right) & \text{if } 2 < p \leqslant 4 - \tau', \\ \left[p, \dfrac{\tau' p}{p-2}\right) & \text{if } 4 - \tau' < p < 2 + \tau'. \end{cases}$$

Combining Theorem 4.4 and Proposition 4.5(b) leads to

**Theorem 4.6.** *Let* $1 \leqslant p < 2 + \tau'$ *and* $V = V_1 + V_2$ *with* $V_j \in L^{s_j}(\mathbf{R}^n)$ *for some* $s_j \in I'_p \cap \left(\frac{n}{m}, \infty\right]$ $(j = 1, 2)$. *Then* $iP(D) + V$ *generates a* $\beta$*-times integrated group on* $L^p(\mathbf{R}^n)$, *where* $\beta > n_p + 1$.

Corresponding to Corollary 2.6 we have

**Corollary 4.7.** *Let* $1 \leqslant p \leqslant 3$, $n_p < m/2$ *and* $V \in L^{\frac{p}{|p-2|}}(\mathbf{R}^n)$. *Then* $iP(D) + V$ *generates a* $\beta$*-times integrated group on* $L^p(\mathbf{R}^n)$, *where* $\beta > n_p + 1$.

In order to give $L^p$ estimates of the solution for Schrödinger equations we need Straub's fractional powers (cf. [13]). If a densely defined operator $A$ is the generator of a $\beta$-times integrated group $T(t)$ $(t \in \mathbf{R})$ satisfying $||T(t)|| \leqslant Ce^{\omega|t|}$ $(t \in \mathbf{R})$, then for $\delta, \varepsilon > 0$ the fractional powers $(\omega + \delta \pm A)^{\beta+\varepsilon}$ are well-defined and their domains are independent of $\delta > 0$. We note that $D((\omega + \delta + A)^{\beta+\varepsilon}) \cap D((\omega + \delta - A)^{\beta+\varepsilon})$ for small $\varepsilon > 0$ contains the dense subspace $D(A^{[\beta]+1})$. The following result is a consequence of Theorem 4.6 and Theorem 1.1 in [13].

**Theorem 4.8.** *Suppose* $p$, $V$ *and* $\beta$ *satisfy the assumptions of* Theorem 4.6. *Then, there exist constants* $C, \omega > 0$ *such that for every data* $u_0 \in D((\omega + iP(D) + V)^\beta) \cap D((\omega - iP(D) - V)^\beta)$, *the Cauchy problem* $(*)$ *has a unique solution* $u \in C(\mathbf{R}, L^p(\mathbf{R}^n))$ *and*

$$||u(t, \cdot)||_{L^p} \leqslant Ce^{\omega|t|}||(\omega \pm iP(D) \pm V)^\beta u_0||_{L^p} \quad \textit{for } t \in \mathbf{R},$$

*where we choose* $+$ (*resp.* $-$) *if* $t \geqslant 0$ (*resp.* $< 0$).

When $P$ is nondegenerate (i.e. $\det\left(\frac{\partial^2 P(\xi)}{\partial \xi_i \partial \xi_j}\right)_{n \times n} \neq 0$ for $\xi \in \mathbf{R}^n \backslash \{0\}$), the Gaussian curvature of $\Sigma$ is nonzero everywhere (cf. [11]). In this case $k = 2$, and thus $h(m, n, k) = \frac{n(m-2)}{2(m-1)}$. In order to compare our results with those in [4] for homogeneous polynomial $P$, we denote by $I_p(\tau)$ (resp. $I'_p(\tau)$) the set $I_p$ (resp. $I'_p$) defined in Section 2 (resp. 4). As a direct consequence of Theorems 2.4 and 4.4(a) we have

**Corollary 4.9.** *Suppose $P$ is nondegenerate. Let $1 \leqslant p \leqslant 2$ and $\tau_0 = \frac{2(m-1)}{m-2}$. If $q \in I_p(\tau_0)$ (resp. $s_j \in I'_p(\tau_0) \cap (\frac{n}{m}, \infty])$, then the conclusion of* Theorem 2.4 *(resp.* 4.4*(a)) is true.*

We first note that for homogeneous polynomial $P$, the hypothesis (H2) in [4] is equivalent to that $P$ is nondegenerate. Thus, Corollary 4.9 improves the corresponding Theorem 2′ and 6 in [4] in several respects:

(1) For fixed $p \in [1, 2)$, the interval $I_p(\tau_0)$ is replaced by smaller $(q(\tau_1, p), p']$ in Theorem 2′, where $\frac{1}{q(\tau,p)} = \frac{1}{\tau p} + \frac{1}{\tau' p'}$ and $\tau_1 = \frac{2n(m-1)}{mn-2n-3m+2}$. In fact, it is clear that $q(\tau, p)$ $(\tau > 2)$ is strictly increasing. Since $p' \in I_p(\tau_0)$ and $2 < \tau_0 < \tau_1$, $I_p(\tau_0)$ contains properly $(q(\tau_1, p), p']$. Similarly, the interval $I'_p(\tau_0)$ is replaced by smaller $[\frac{p}{2-p}, \frac{\tau'_1 p}{2-p})$ in Theorem 6.

(2) The hypothesis (H3′), i.e. $n > 3 + \frac{4}{m-2}$ is required in Theorems 2′ and 6, but not in Corollary 4.9. For example, when $n = 2, 3$ and $m \geqslant 4$, one has $n \leqslant 3 + \frac{4}{m-2}$, and thus Theorems 2′ and 6 cannot deal with such a case. However, in this case, $I'_p(\tau_0) \cap (\frac{n}{m}, \infty] = I'_p(\tau_0)$, which means that for every $p \in [1, 2]$, we can choose $q$ and $s_j$'s values such that the conclusion of Corollary 4.9 holds.

(3) We first note that $p \neq 1$ in Theorems 2′ and 6, but it is admitted that $p = 1$ in Corollary 4.9. Furthermore, it is required in Theorem 6 that $p > \frac{2c}{c+1}$, where $c$ is an integer with $c > \frac{n}{m-1}$. However, the restriction of $p$ in Corollary 4.9 is only caused by $I'_p(\tau_0) \cap (\frac{n}{m}, \infty] \neq \emptyset$, which is equivalent to $p > \frac{2n}{n+2m-2}$. It is easy to see that this is naturally an improvement of the corresponding condition in Theorem 6.

(4) The conclusion in Theorem 6 (see its remark for homogeneous $P$) is that $iP(D) + V$ generates a smooth distribution group on $L^p(\mathbf{R}^n)$ of order $\beta$, which is equivalent to a $\beta$-times integrated group on $L^p(\mathbf{R}^n)$, where $\beta$ is an integer with $\beta > n_p + 2$. Our conclusion in Corollary 4.9 however admits that $\beta$ is a real number with $\beta > n_p + 1$.

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