Journal of Combinatorial Theory, Series A 116 (2009) 1326–1343

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Indecomposable permutations, hypermaps and labeled Dyck paths

Robert Cori

Labri Université Bordeaux 1, 351 Cours de la Liberation, Talence cedex, France

ARTICLE INFO

Article history:
Received 4 June 2008
Available online 21 May 2009

Keywords:
Combinatorial maps
Permutations
Dyck paths
Bijective proofs
Enumeration
Stirling numbers

ABSTRACT

Hypermaps were introduced as an algebraic tool for the representation of embeddings of graphs on an orientable surface. Recently a bijection was given between hypermaps and indecomposable permutations; this sheds new light on the subject by connecting a hypermap to a simpler object. In this paper, a bijection between indecomposable permutations and labeled Dyck paths is proposed, from which a few enumerative results concerning hypermaps and maps follow. We obtain for instance an inductive formula for the number of hypermaps with n darts, p vertices and q hyperedges; the latter is also the number of indecomposable permutations of $\mathcal{S}_n$ with p cycles and q left-to-right maxima. The distribution of these parameters among all permutations is also considered.

0. Introduction

Permutations and maps on surfaces have an old common history. Heffter [16] was probably the first who mentioned the fact that any embedding of a graph on an orientable surface could be represented by a pair consisting of a permutation and a fixed point free involution; J.R. Edmonds [13] and J. Youngs [29] gave in the early sixties a more precise presentation of this idea by showing how to compute the faces of an embedding using the cycles of the product of the permutation and the fixed point free involution, giving a purely combinatorial definition of the genus. A. Jacques [17] proved that this could be generalized to any pair of permutations (called a hypermap in [4]), hence relaxing the condition that the second one should be a fixed point free involution. He defined the genus of a pair of permutations by a formula involving the number of their cycles and of that of their product.

W.T. Tutte [26] generalized these constructions by introducing a combinatorial object consisting of three fixed point free involutions in order to represent embeddings in a nonorientable surface.

E-mail address: robert.cori@labri.fr.

0097-3165/$ – see front matter
doi:10.1016/j.jcta.2009.02.008

The combinatorial representation allows one to obtain results on automorphisms of maps and hypermaps, for instance A. Machí [20] obtained a combinatorial version of the Riemann–Hurwitz formula for hypermaps. A coding theory of rooted maps by words [4] had also some extent for explaining the very elegant formulas found by W.T. Tutte [25] for the enumeration of maps. In the same years Jones and Singerman [18] settled some important algebraic properties of maps. Recently G. Gonthier (see [15]), used hypermaps in giving a formal proof of the 4 color theorem. A survey of the combinatorial and algebraic properties of maps and hypermaps is given in [5].

In 2004 P. Ossona de Mendez and P. Rosenstiehl [7] proved an important combinatorial result: they constructed a bijection between (rooted) hypermaps and indecomposable permutations (also called connected or irreducible). Indecomposable permutations are a central object in combinatorics (see for instance [24, Problem 5.13]), they were considered in different contexts, and probably for the first time by A. Lentin [19] while solving equations in the free monoid. They were also considered for instance as a basis of a Hopf algebra defined by Malvenuto and Reutenauer (see [1] or [12]), and in the enumeration of a certain kind of Feynman diagrams [6].

In this paper we present the Ossona–Rosenstiehl result in simpler terms and focus on the main property of the bijection (which is not explicitly stated in [7]): the number of cycles and of left-to-right maxima of the indecomposable permutation are equal to the number of vertices and hyperedges of the rooted hypermap associated with it.

This property has some nice consequences for the enumeration: it allows one to give a formula for the number of rooted hypermaps on n darts, or with n darts and p vertices. The property shows that the number of indecomposable permutations of $\mathcal{S}_n$ with p cycles and q left-to-right maxima is symmetric in p, q. By a straightforward argument this result can be generalized to all permutations, answering a conjecture of Guo-Niu Han and D. Foata. We introduce a simple bijection between some labeled Dyck paths and permutations which allows us to obtain a formula for the polynomials enumerating indecomposable permutations by the number of cycles and left-to-right maxima (hence of hypermaps by vertices and hyperedges).

The paper is organized as follows: in Section 1, we give a few elementary results on indecomposable permutations focusing mainly on the parameters left-to-right maxima and cycles. Section 2 is devoted to hypermaps and the bijection of P. Ossona de Mendez and P. Rosenstiehl. All the details of the proof of correctness are in [7]; we give here the key points and some examples in order to facilitate the reading of their paper. The main result of the present paper and consequences of this bijection are given at the end of this section.

In Section 3 we recall some notions about Dyck paths and their labeling. We describe a bijection between them and permutations and show the main properties of this bijection. In Section 4 we introduce a family of polynomials enumerating permutations and derive a formula for the generating function of the permutations with respect to the number of left-to-right maxima and cycles.

In the last section we restrict the hypermaps to be maps and permutations to be fixed point free involutions, obtaining in a simpler way some old enumeration formulas for them (see [2,28]).

1. Indecomposable permutations

In this section we give some notation and recall some basic results on permutations. Then we shall focus on indecomposable permutations, often also called connected permutations.

1.1. Definition and counting formula

Permutations are the central object of this paper. We shall express them in two ways either as *sequences*, or as *sets of cycles*.

The set of all permutations (i.e. the symmetric group) on $\{1, 2, \ldots, n\}$ will be denoted by $\mathcal{S}_n$. The notation of a permutation as a sequence is:

$$\alpha = a_1, a_2, \ldots, a_n.$$

In this setting, a_i is the image of i by α, also denoted $\alpha(i)$.

Definition 1. A permutation $\theta = a_1, a_2, \ldots, a_n$ is *decomposable*, if there exists $p < n$ such that for all i, $1 \leqslant i \leqslant p$:

$$1 \leqslant a_i \leqslant p,$$

it is called *indecomposable* otherwise.

Hence a permutation $\theta = a_1, a_2, \ldots, a_n$ is indecomposable if for any $p < n$ the left factor subsequence $a_1, a_2, \ldots, a_p$ contains at least one $a_j > p$. Equivalently, θ is indecomposable if for $p < n$, there is no initial interval $[1, \ldots, p]$ fixed by θ, or no union of a subset of the set of cycles equal to $[1, \ldots, p]$. For instance, $3, 1, 2, 5, 4$ is a decomposable permutation of S_5, while any permutation $\alpha = a_1, a_2, \ldots, a_n$ for which $a_1 = n$ is indecomposable.

Let c_n be the number of indecomposable permutations of $\mathcal{S}_n$. The following formula is well known, and is obtained by noting that any decomposable permutation can be written as the concatenation of an indecomposable permutation of length $p < n$ and a permutation of length $n - p$ on $\{p + 1, \ldots, n\}$:

$$c_n = n! - \sum_{p=1}^{n-1} c_p (n-p)!.$$

From this formula we obtain the first values of the number of indecomposable permutations which are:

$$1, 1, 3, 13, 71, 461, 3447, \ldots.$$

1.2. Left-to-right maxima and cycles

Let $\alpha = a_1, a_2, \ldots, a_n$ be a permutation, i is the index of a left-to-right maximum if $a_j < a_i$ for all $1 \leqslant j < i$.

For any α, 1 is the index of a left-to-right maximum, and k such that $a_k = n$ also is, hence the number of left-to-right maxima of a permutation α is equal to 1 if and only if $a_1 = n$.

1.2.1. Bijection

The following algorithm describes a bijection from the set of permutations having k cycles to the set of permutations having k left-to-right maxima. It is often called the *First fundamental transform* and is extensively used for the determination of permutation statistics (see [14]).

To obtain the transform β from α, write the cycles $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ of the permutation α, such that the first element of each cycle Γ_i is the maximum among the elements of Γ_i. Then reorder the Γ_i in such a way that the first elements of the cycles appear in increasing order, and finally, delete the parenthesis around the cycles obtaining β as a sequence.

For instance, let $\alpha = 4, 7, 2, 1, 3, 6, 5, 9, 8$, then we write

$$\alpha = (1, 4)(2, 7, 5, 3)(6)(8, 9)$$

putting the maximum at the beginning of each cycle and reordering the cycles gives:

$$\alpha = (4, 1)(6)(7, 5, 3, 2)(9, 8)$$

hence

$$\beta = 4, 1, 6, 7, 5, 3, 2, 9, 8.$$

Proposition 1. *The permutation α is decomposable if and only if its first fundamental transform β is.*

Proof. If α is decomposable, then for some $p < n$ the subset $\{1, 2, \ldots, p\}$ is the union of cycles of α. Hence the left factor $b_1, b_2, \ldots, b_p$ of β, is a permutation of $\{1, 2, \ldots, p\}$ proving the decomposability of β. The proof of the converse is obtained by similar arguments. $\square$

Corollary. *The number of indecomposable permutations with k cycles is equal to the number of indecomposable permutations with k left-to-right maxima.*

Remark. It is customary for permutations to consider in an obvious manner left-to-right minima, right-to-left maxima and right-to-left minima. Consider for a permutation $a_1, a_2, \dots, a_n$, the reverse $a_n, a_{n-1}, \dots, a_1$ and the complement $n+1-a_1, n+1-a_2, \dots, n+1-a_n$; these operations show that the statistics for these four parameters are equal. However this is not true for indecomposable permutations. For instance, the numbers of indecomposable permutations of $\mathcal{S}_4$ with $k = 1, 2, 3, 4$ left-to-right minima are 0, 7, 5, 1, respectively, and those with the same number of left-to-right maxima are $6, 6, 1, 0$. Nevertheless the operation $\alpha \to \alpha^{-1}$ transforms an indecomposable permutation into an indecomposable permutation, showing that the number of indecomposable permutations of $\mathcal{S}_n$ with k left-to-right maxima (resp. minima) is equal to the number of indecomposable permutations of $\mathcal{S}_n$ with k right-to-left minima (resp. maxima).

1.2.2. Enumeration

It is well known that the number of permutations $s_{n,k}$ of $\mathcal{S}_n$ having k cycles is equal to the coefficient of x^k in the polynomial:

$$A_n(x) = x(x+1)(x+2)\cdots(x+n-1).$$

These numbers are the unsigned Stirling numbers of the first kind.

Proposition 2. *The number $c_{n,k}$ of indecomposable permutations of S_n $(n > 1)$ with k cycles (or with k left-to-right maxima) is given by each one of the following formulas:*

$$c_{n,k} = s_{n,k} - \sum_{p=1}^{n-1} \sum_{i=1}^{\min(k,p)} c_{p,i} s_{n-p,k-i}, \qquad c_{n,k} = \sum_{p=1}^{n-1} \sum_{i=1}^{\min(k,p)} p c_{p,i} s_{n-p-1,k-i}.$$

Proof. The first formula follows from the observation that a decomposable permutation of $\mathcal{S}_n$ with k cycles is the concatenation of an indecomposable permutation of $\mathcal{S}_p$ with i cycles and a permutation of $\mathcal{S}_{n-p}$ with $k-i$ cycles.

For the second, observe that the deletion of n from its cycle in an indecomposable permutation α of $\mathcal{S}_n$ with k cycles gives a (possibly decomposable) permutation with k cycles; since if n was alone in its cycle then α would have been decomposable. Conversely, let β be any permutation with k cycles written as the concatenation of an indecomposable permutation θ on $\{1, 2, \dots, p\}$ (with $p \leqslant n$) and a permutation β' on $\{p+1, \dots, n-1\}$ having respectively i and $k-i$ cycles. When inserting n in any position of the cycles of θ one gets an indecomposable permutation with k cycles. The formula follows from the fact that there are exactly p places where n can be inserted, since inserting n in a cycle of β' gives a decomposable permutation. □

Let $C_n(x) = \sum_{k=1}^{n-1} c_{n,k} x^k$; the equalities become

$$C_n(x) = A_n(x) - \sum_{p=1}^{n-1} A_{n-p}(x) C_p(x), \qquad C_n(x) = \sum_{p=1}^{n-1} p A_{n-1-p}(x) C_p(x).$$

Indecomposable Stirling numbers. The first values of the numbers $c_{n,k}$ of indecomposable permutations of $\mathcal{S}_n$ with k cycles, for $2 \leqslant n \leqslant 7$, are given in the table below; these numbers might be called indecomposable Stirling numbers of the first kind since they count indecomposable permutations by their number of cycles.

1					
2	1				
6	6	1			
24	34	12	1		
120	210	110	20	1	
720	1452	974	270	30	1

2. Hypermaps

In this section we recall some elementary facts about hypermaps, state the main result of P. Ossona de Mendez and P. Rosenstiehl and give a simplified proof of it.

2.1. Definition

Let B be a finite set, the elements of it being called *darts*. In the sequel we will take $B = \{1, 2, \ldots, n\}$.

Definition 2. A *hypermap* is given by a pair of permutations (σ, α), acting on B such that the group they generate is transitive on B.

The transitivity condition can be translated in simple combinatorial terms, remarking that it is equivalent to the connectivity of the graph $G_{\sigma,\alpha}$ with vertex set B and edge set:

$$E = \bigcup_{b\in B} \{b, \alpha(b)\} \bigcup_{b\in B} \{b, \sigma(b)\}.$$

The cycles of σ are called the *vertices* of the hypermap while the cycles of α are called its *hyperedges*.

An example of hypermap with 3 vertices and three hyperedges is given by

$$\sigma = (1, 2, 3)(4, 5, 6)(7, 8, 9), \qquad \alpha = (1, 6, 7)(2, 5, 8)(3, 4, 9).$$

Hypermaps have been introduced, as a generalization of combinatorial maps, for the representation of embeddings of hypergraphs in surfaces, showing that the cycles of $\alpha^{-1}\sigma$ can be considered as representing a kind of faces, and defining a genus in a formula like Euler's for maps. In this paper we will not consider hypermaps as a topological embedding but as the very simple object consisting of a pair of permutations generating a transitive subgroup of $\mathcal{S}_n$.

2.2. Labeled, unlabeled and rooted hypermaps

In enumerative combinatorics it is customary to consider labeled objects and unlabeled ones. Since in the above definition of hypermaps we consider the elements of B as distinguishable numbers, they should be called *labeled hypermaps*.

As an example, the number of labeled hypermaps with 3 darts is 26, since among the 36 pairs of permutations on $\{1, 2, 3\}$ there are 10 which do not generate a transitive group. These are given by (where ε is the identity and $\tau_{i,j}$ the transposition exchanging i and j):

- $\sigma = \varepsilon$ and $\alpha = \tau_{i,j}$ $i \neq j \in \{1, 2, 3\}$ or $\alpha = \epsilon$ (4 pairs)
- $\sigma = \tau_{i,j}$ and $\alpha = \varepsilon$ or $\alpha = \sigma$ (2 pairs for each of the 3 transpositions).

Two hypermaps (σ, α) and (σ', α') are *isomorphic* if there exists a permutation ϕ such that:

$$\phi^{-1}\alpha\phi = \alpha', \qquad \phi^{-1}\sigma\phi = \sigma'.$$

The set of *unlabeled hypermaps* is the quotient of the set of labeled ones by the isomorphism relation. For instance the number of unlabeled hypermaps with 3 darts is 7. Representatives of the 7 isomorphism classes are given below:

	H_1	H_2	H_3	H_4	H_5	H_6	H_7
σ	$(1,2,3)$	$(1,2,3)$	$(1,2,3)$	$(1,2,3)$	$(1,2)(3)$	$(1)(2,3)$	$(1)(2)(3)$
α	$(1,2,3)$	$(1,3,2)$	$(1,2)(3)$	$(1)(2)(3)$	$(1,2,3)$	$(1,3)(2)$	$(1,2,3)$

In general, the enumeration of unlabeled objects is difficult and the formulas one obtains are complicated. Thus, intermediate objects are introduced: the *rooted* ones, this is done by selecting one element, *the root*, in the object, and considering isomorphisms which fix the root. For hypermaps we select here n as the root, two labeled hypermaps (σ, α) and (σ', α') being *isomorphic* as *rooted hypermaps* if there exists ϕ such that:

$$\phi^{-1}\alpha\phi = \alpha', \qquad \phi^{-1}\sigma\phi = \sigma', \qquad \phi(n) = n.$$

Such a ϕ will be called a *rooted isomorphism*. There are 13 different rooted hypermaps with 3 darts, to the 7 above we have to add these below, which are isomorphic to one of the previous ones but for which the isomorphism ϕ is not a rooted isomorphism.

	H_8	H_9	H_{10}	H_{11}	H_{12}	H_{13}
σ	$(1,2,3)$	$(1,2,3)$	$(1)(2,3)$	$(1)(2,3)$	$(1,2)(3)$	$(1)(2,3)$
α	$(1,3)(2)$	$(1)(2,3)$	$(1,3,2)$	$(1,2,3)$	$(1,3)(2)$	$(1,2)(3)$

In the sequel we will denote by h_n the number of rooted hypermaps with n darts.

Proposition 3. *The number of labeled hypermaps with n darts is equal to*

$$(n-1)!h_n.$$

Proof. Since there are $(n-1)!$ permutations ϕ such that $\phi(n) = n$, we only have to prove that for a hypermap (σ, α) and a ϕ such that $\phi(n) = n$ if

$$\phi^{-1}\alpha\phi = \alpha, \qquad \phi^{-1}\sigma\phi = \sigma$$

then ϕ is the identity. But this follows from the fact that for such an isomorphism ϕ, $\phi(a) = a$ implies $\phi(\sigma(a)) = \sigma(a)$, $\phi(\alpha(a)) = \alpha(a)$ and from the transitivity of the group generated by σ and α. □

2.3. Bijection

The following algorithm is a slightly modified version of that of P. Ossona de Mendez and P. Rosenstiehl in [7]:

2.3.1. Algorithm OMR

Let $\theta = a_1, a_1, a_2, \ldots, a_{n+1}$ be an indecomposable permutation. A pair of permutations (σ, α) is associated with θ through the following algorithm:

- Determine the indexes of the left-to-right maxima of θ (recall that these are the indexes $i_1, i_2, \ldots, i_k$ satisfying $j < i_p \Rightarrow a_j < a_{i_p}$). Note that $i_1 = 1$ and $a_{i_k} = n + 1$.
- Let σ_1 be the permutation split into cycles as:

$$\sigma_1 = (1, 2, \ldots, i_2 - 1)(i_2, i_2 + 1, \ldots, i_3 - 1) \cdots (i_k, \ldots, n + 1).$$

- The permutations α and σ are obtained from θ and σ_1, respectively, by deleting $n + 1$ from their cycles (observe that the lengths of these cycles are not less than 2).

We will denote by $\Psi(\theta)$ the pair of permutations (σ, α) obtained from θ by means of the algorithm OMR.

Example. Consider the indecomposable permutation

$$\theta = 6, 5, 7, 4, 2, 10, 3, 8, 9, 1$$

then the indexes of the left-to-right maxima are $1, 3, 6$ giving

$$\sigma_1 = (1,2)(3,4,5)(6,7,8,9,10).$$

Since $\theta = (1,6,10)(2,5)(3,7)(4)(8)(9)$ we have

$$\sigma = (1,2)(3,4,5)(6,7,8,9), \qquad \alpha = (1,6)(2,5)(3,7)(4)(8)(9).$$

Theorem 1. *The above algorithm yields a bijection Ψ between the set of indecomposable permutations on $\mathcal{S}_{n+1}$ and the set of rooted hypermaps with darts $1, 2, \ldots, n$. Moreover for $(\sigma, \alpha) = \Psi(\theta)$, α and θ have the same number of cycles and the number of cycles of σ is equal to the number of left-to-right maxima of θ.*

The key point in the proof of this theorem is the following characterization of the smallest elements of the cycles of the permutation σ given by the algorithm:

Lemma 1. *A hypermap (σ, α) is such that there exists an indecomposable permutation θ satisfying*

$$\Psi(\theta) = (\sigma, \alpha)$$

if and only if:

- *The permutation σ has cycles consisting of consecutive integers in increasing order*:

$$\sigma = (1, 2, \ldots, i_2 - 1)(i_2, i_2 + 1, \ldots, i_3 - 1) \cdots (i_k, i_k + 1, \ldots, n).$$

- *The set of right-to-left minima of α^{-1} is the union of $\{i_1, i_2, \ldots, i_{k-1}\}$ and a (possibly empty) subset of the interval $[i_k, i_k + 1, \ldots, n]$.*

Proof. Suppose that (σ, α) satisfy the conditions above, consider the notation of $\alpha = a_1, a_2, \ldots, a_n$ as a sequence and let

$$\theta = a_1, a_2, \ldots a_{i_k - 1}, n + 1, a_{i_k + 1}, \ldots, a_n, a_{i_k}.$$

Then $\theta(i_k) = n + 1$ and $\theta(n + 1) = \alpha(i_k)$, hence the indexes of the left-to-right maxima of θ are $i_1, i_2, \ldots, i_k$ giving $\Psi(\theta) = (\sigma, \alpha)$.

Conversely let θ be an indecomposable permutation and let $= \Psi(\theta) = (\sigma, \alpha)$, then:

(1) By the definition of Ψ the cycle of σ containing n is $(i_k, \ldots, n)$.
(2) For any permutation θ, the indexes $i_1, i_2, \ldots, i_k$ of the left-to-right maxima are exactly the right-to-left minima of θ^{-1}. Deleting $n + 1$ from its cycle in θ in order to obtain α has the following effect on the sequence $b_1, b_2, \ldots, b_{n+1}$ representing θ^{-1}: $b_i = n + 1$ is replaced by b_{n+1}. Clearly, $i_1, i_2, \ldots, i_{k-1}$ are still right-to-left minima in the sequence obtained by this transformation since they are smaller than b_{n+1}. $\square$

2.3.2. Proof of Theorem 1

(1) The pair of permutations (σ, α) defines a hypermap.

Consider a cycle $(i_p, i_p + 1, \ldots, i_{p+1} - 1)$ of σ with $p < k$, then $\alpha(i_p) = \theta(i_p)$ is a left-to-right maximum of θ and the next one is $\alpha(i_{p+1})$, hence $\theta(i) < \theta(i_p)$ for $i_p < i < i_{p+1}$ and for $i < i_p$. Since θ is indecomposable this implies $\theta(i_p) = \alpha(i_p) \geqslant i_{p+1}$. We have thus observed that for any cycle Γ_p of σ, which does not contain n, there is an element (namely i_p) such that $\alpha(i_p)$ is in another cycle of σ whose smallest element is greater than the smallest element of Γ_p; this observation clearly implies the transitivity of the group generated by (σ, α).

(2) Let θ and θ' two different indecomposable permutations then $\Psi(\theta)$ and $\Psi(\theta')$ are nonisomorphic as rooted hypermaps.

- Suppose that there exists a rooted isomorphism ϕ between $\Psi(\theta) = (\sigma, \alpha)$ and $\Psi(\theta') = (\sigma', \alpha')$, then the cycles of σ and σ' containing n have the same length, since $\phi(n) = n$ and $\sigma' = \phi^{-1}\sigma\phi$ implies:

$$\sigma^i(n) = n \quad \Leftrightarrow \quad \sigma'^i(n) = n.$$

- Hence i_k and $i'_{k'}$ the smallest elements of the cycles of σ and σ' containing n are equal, and $\phi(i) = i$ for all $i_k \leqslant i \leqslant n$.
- By Lemma 1, i_{k-1} is equal to $\alpha^{-1}(j)$ for the maximal j in $[i_k, \ldots, n]$ such that $\alpha^{-1}(j) \notin [i_k, \ldots, n]$. But since $\phi(i) = i$ for $i \in [i_k, \ldots, n]$ we have

$$\alpha^{-1}(\ell) \notin [i_k, \ldots, n] \quad \Leftrightarrow \quad \alpha'^{-1}(\ell) \notin [i_k, \ldots, n].$$

Hence $\alpha'^{-1}(j) \notin [i_k, \ldots, n]$, and $i'_{k-1} = \alpha'^{-1}(j)$. Moreover the cycles of σ and σ' containing respectively i_{k-1} and $i'_{k'-1}$ have the same length, giving $i_{k-1} = i'_{k'-1}$ and $\phi(i) = i$ for all $i_{k-1} \leqslant i \leqslant n$.
- By repeating the above argument for all the i_p we conclude that ϕ is the identity.

(3) For any hypermap (σ, α) there exists an indecomposable permutation θ such that (σ, α) and $\Psi(\theta)$ are isomorphic as rooted hypermaps.

It suffices to show that there exists an isomorphism ϕ such that the hypermap $(\sigma', \alpha') = (\phi^{-1}\sigma\phi, \phi^{-1}\alpha\phi)$ satisfies the conditions of Lemma 1. We have to find among all the conjugates of σ one in which the cycles consist of consecutive integers and such that the smallest elements in each cycle are the right-to-left minima of the conjugate of α^{-1}. For that we write down the cycles $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ of σ in a specific order, then we will write σ' (having the same numbers of cycles of each length as σ and with cycles consisting of consecutive numbers in increasing order) above σ in such a way that cycles of the same length correspond. The automorphism ϕ is then obtained by the classical construction for conjugates of a permutation (see for instance [23, Chapter 3]).

Since $\phi(n) = n$, $\Gamma_k = (z_{j_1}, z_{j_2}, \ldots, z_{j_k})$ should be the cycle of σ containing n, it has to be written so that $z_{j_k} = n$. In order to find which cycle is Γ_{k-1} we use Lemma 1: the first element of this cycle should correspond to a right-to-left minima of α'^{-1}, hence this element is the first among $\alpha^{-1}(z_{j_k}), \alpha^{-1}(z_{j_{k-1}}), \ldots, \alpha^{-1}(z_{j_1})$ which is not in Γ_k, such an element existing by the transitivity of the group generated by $\{\sigma, \alpha\}$. We continue by computing the image under α^{-1} of the elements already written down, taken from right to left; when an element $\alpha^{-1}(z_i) = u_1$, not written down is obtained, the whole cycle of σ containing u_1 is written with u_1 at the beginning. The algorithm terminates when all $\{1, 2, \ldots, n\}$ are obtained, and this termination is also a consequence of the transitivity of the group generated by $\{\sigma, \alpha\}$.

To end we write σ' above σ in such a way that the elements $1, 2, \ldots, n$ appear in that order with the lengths of cycles corresponding to those of $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$, and the isomorphism ϕ is determined. Then α' is obtained by $\alpha' = \phi^{-1}\alpha\phi$.

2.3.3. Example

We give an example of a hypermap $H = (\sigma, \alpha)$ and the computation of the hypermap $H' = (\sigma', \alpha')$ such that H and H' are isomorphic as rooted hypermaps and there exists θ satisfying $\Psi(\theta') = (\sigma', \alpha')$. We take for this example, the hypermap obtained by exchanging vertices and hyperedges in the hypermap considered above:

$$\sigma = (1,6)(2,5)(3,7)(4)(8)(9), \qquad \alpha = (1,2)(3,4,5)(6,7,8,9).$$

We begin the list of cycles by $\Gamma_6 = (9)$, then since $\alpha^{-1}(9) = 8$ after two steps the list consists of

$$(8)(9).$$

Now since $\alpha^{-1}(8) = 7$, the list grows

$$(7,3)(8)(9)$$

then $\alpha^{-1}(3) = 5$ gives:

$$(5, 2)(7, 3)(8)(9).$$

Then, since $\alpha^{-1}(7) = 6$:

$$(6, 1)(5, 2)(7, 3)(8)(9).$$

We end by $\alpha^{-1}(2) = 1, \alpha^{-1}(5) = 4$ and obtain finally

$$(4)(6, 1)(5, 2)(7, 3)(8)(9).$$

Aligning with

$$(1)(2, 3)(4, 5)(6, 7)(8)(9)$$

we obtain $\phi = 4, 6, 1, 5, 2, 7, 3, 8, 9$; then we have

$$\sigma' = \phi^{-1}\sigma\phi = (1)(2, 3)(4, 5)(6, 7)(8)(9), \qquad \alpha' = \phi^{-1}\alpha\phi = (3, 5)(7, 1, 4)(2, 6, 8, 9).$$

To obtain θ' we remark that the last cycle of σ is of length 1, hence the position of 10 in the sequence representing θ should be one place before the end, giving from $\alpha' = 4, 6, 5, 7, 3, 8, 1, 9, 2$:

$$\theta' = 4, 6, 5, 7, 3, 8, 1, 9, 10, 2.$$

2.4. Main results

Corollary 1. *The number of rooted hypermaps with n darts is equal to c_{n+1}, the number of those with n darts and k vertices is $c_{n+1,k}$.*

We also obtain another proof of a result of J.D. Dixon [10].

Corollary 2. *The probability p_n that a pair of permutations randomly chosen among the permutations in $\mathcal{S}_n$ generates a transitive group is*:

$$p_n = 1 - \frac{1}{n} - \frac{1}{n^2} - \frac{4}{n^3} - \frac{23}{n^4} - \frac{171}{n^5} - \frac{11542}{n^6} - \frac{16241}{n^7} - \frac{194973}{n^8} + O\left(\frac{1}{n^9}\right).$$

Proof. We have seen that the number of labeled hypermaps with n darts is $(n-1)!c_{n+1}$; hence the probability t_n is

$$\frac{(n-1)!c_{n+1}}{n!n!} = \frac{c_{n+1}}{nn!}.$$

In [3] L. Comtet proves that the number of indecomposable permutations c_n of $\mathcal{S}_n$ satisfies

$$\frac{c_n}{n!} = 1 - \frac{2}{n} - \frac{1}{(n)_2} - \frac{1}{(n)_3} - \frac{19}{(n)_4} - \frac{110}{(n)_5} - \frac{745}{(n)_6} - \frac{5752}{(n)_7} - \frac{49775}{(n)_8} + O\left(\frac{1}{476994}\right)$$

where $(n)_k = n(n-1)\cdots(n-k+1)$. Replacing n by $n+1$ gives the result. $\square$

The following theorem answers positively a conjecture of Guo-Niu Han and D. Foata.[1]

Theorem 2. *The number of permutations of $\mathcal{S}_n$ with p cycles and q left-to-right maxima is equal to the number of permutations of $\mathcal{S}_n$ with q cycles and p left-to-right maxima.*

[1] Personal communication.

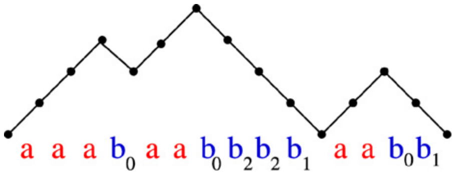

Fig. 1. A labeled Dyck path.

Proof. We define a bijection Φ between these two subsets. Let θ be an indecomposable permutation with p cycles and q left-to-right maxima, we define $\Phi(\theta)$ as $\Psi^{-1}(\alpha', \sigma')$ where $\Psi(\theta) = (\sigma, \alpha)$, and (α', σ') is the unique hypermap isomorphic to (α, σ) as a rooted hypermap and such that there exists an indecomposable permutation θ' satisfying $\Psi(\theta') = (\alpha', \sigma')$.

Clearly, θ' has the same numbers of cycles and left-to-right maxima as θ. Then Φ is a bijection among indecomposable permutations having the desired property.

Now a decomposable permutation β can be written as the concatenation of k indecomposable ones:

$$\beta = \theta_1\theta_2 \cdots \theta_k.$$

Define $\Phi(\beta)$ by

$$\Phi(\beta) = \Phi(\theta_1)\Phi(\theta_2) \cdots \Phi(\theta_k)$$

with an obvious convention on the numbering of the elements on which the θ_i act. Clearly $\Phi(\beta)$ has also as many cycles as β has left-to-right maxima and as many left-to-right maxima as β has cycles, completing the proof. $\square$

3. A bijection with labeled Dyck paths

The bijection described below will allow us to obtain a formula for the number of indecomposable permutations with a given number of cycles and of left-to-right maxima.

3.1. Dyck paths and labeled Dyck paths

A Dyck path can be defined as a word w on the alphabet $\{a, b\}$ where the number of occurrences of the letter a (denoted by $|w|_a$) is equal to the number of occurrences of the letter b, and such that any left factor contains no more occurrences of the letter b than those of a. We write:

$$|w|_a = |w|_b \quad \text{and} \quad \forall w = w'w'' \quad |w'|_a \geqslant |w'|_b.$$

A Dyck path is *primitive* if it is not the concatenation of two Dyck paths or equivalently if it is equal to $aw'b$, where w' is a Dyck path. Such a path is usually drawn as a sequence of segments in the cartesian plane starting at the point $(0, 0)$ and going from the point (x, y) to $(x + 1, y + 1)$ for each letter a and from the point (x, y) to $(x + 1, y - 1)$ for each letter b. The conditions on the occurrences of the letters translates in the fact that the path ends on the x axis and never crosses it. The path *aaabaabbbbaabb* is drawn in Fig. 1.

Definition 3. A *labeling* of a Dyck path w consists in assigning integer labels to the occurrences of b, obtaining a word f on the alphabet $\{a, b_0, b_1, \ldots, b_i, \ldots\}$ satisfying the following conditions:

- The occurrence b_i in f is preceded by an a if and only if $i = 0$.
- For each occurrence of b_i in f ($f = gb_ih$) with $i > 0$, $i \leqslant |g|_a - |g|_b$ (where $|g|_b$ denotes $\sum_{i \geqslant 0} |g|_{b_i}$).

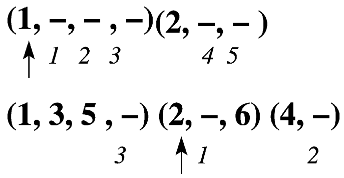

Fig. 2. Partial representations.

3.2. From permutations to labeled Dyck paths

We associate a labeled Dyck path f with a permutation α by the algorithm described below. This algorithm uses a *partial representation* of a permutation defined as follows

Definition 4. The partial representation of the permutation α at step i consists of the sequence of cycles of α containing elements less than i and presented in increasing order of their smallest element. In this representation only the elements less than i appear and the symbol $-$ is written to replace all the other elements of these cycles. The positions of the symbol $-$ are called free positions.

For example, the partial representations of the permutation

$$\alpha = (1,3,5,9)(2,7,6)(4,8)$$

at steps 3 and 7 are:

$$(1,-,-,-)(2,-,-) \quad \text{and} \quad (1,3,5,-)(2,-,6)(4,-).$$

In these representations we define the *pivot* as the smallest element less than i such that $\alpha(i) > i$ and we number the free positions from 1 to p beginning immediately at the right of the pivot and proceeding in cyclic order.

In the above example the pivots are respectively 1 and 2, and the numbering of free positions is given in Fig. 2, where arrows indicate the pivots.

Algorithm $\boldsymbol{\Delta}$. This algorithm builds a labeled Dyck word w from a permutation α of $\mathcal{S}_n$. It uses partial representations of α and starts with an empty word w.

For each i in $\{1,2,\dots,n\}$ **do**

- **If** i is the smallest element in a cycle of length k, **then** open a new cycle in the partial representation of α, with i as the first element and $k-1$ free positions in it. Append the word $a^k b_0$ to w.
- **Else** Determine the pivot of the partial representation and the numbering of the free positions; let j be the numbering of the free position where i has to be inserted, append b_j to w.

We will denote $\Delta(\alpha) = w$ the labeled Dyck path associated with α by the above algorithm.

3.2.1. Example

For the permutation α considered above, the determination of the word $w = \Delta(\alpha)$ is illustrated in Fig. 3, where at each step of the algorithm the pivot is represented by an arrow pointing to it.

This gives $w = w_1 w_2 \cdots w_9$ where: $w_1 = aaaab_0$, $w_2 = aaab_0$, $w_3 = b_1$, $w_4 = aab_0$, $w_5 = b_4$, $w_6 = b_2$, $w_7 = b_1$, $w_8 = b_1$, $w_9 = b_1$. Hence

$$\Delta(\alpha) = aaaab_0aaab_0b_1aab_0b_4b_2b_1b_1b_1.$$

Note that the word w is a Dyck path since each cycle of the permutation is considered for the first time by its smallest element, creating an excess of $k-1$ for the number of occurrences of the a's

(1, –, – , –)

(1, –, – , –) (2, –, –)

(1, 3, – , –) (2, –, –)

(1, 3, – , –) (2, –, –) (4, –)

(1, 3, 5 , –) (2, –, –) (4, –)

(1, 3, 5 , –) (2, –, 6) (4, –)

(1, 3, 5 , –) (2, 7, 6) (4, –)

(1, 3, 5 , –) (2, 7, 6) (4, 8)

(1, 3, 5, 9) (2, 7, 6) (4, 8)

Fig. 3. Associating a labeled Dyck path to a permutation.

with respect to those of the b's. A letter b is written for each other element of the cycle, so that there cannot be an excess of the number of occurrences of b's compared to that of a's.

Δ is a bijection since from a labeled Dyck path w one can obtain the permutation α from w in an obvious way reading the word w from left to right and building the partial representation of α each time a letter b_0 is read.

Inverse algorithm. Let w be a labeled Dyck path:

- If the ith occurrence of b in w is b_0 let k be the number of occurrences of a immediately before it, then open a new cycle of length k with i as first element and $k-1$ free positions in it.
- If the ith occurrence of a b is labeled b_p, with $p > 0$, then insert i in the pth free position of the opened cycles starting from the pivot.

3.3. Characterization

Theorem 3. *Let α be a permutation and $w = \Delta(\alpha)$, then:*

- *The number of cycles in α is equal to the number of occurrences of b_0 in w.*
- *The permutation α is indecomposable if and only if the word w is primitive.*
- *If the permutation α is indecomposable, then the number of left-to-right maxima of α is equal to the number of occurrences of b_1 in w.*

Proof. • Since each smallest element in a cycle inserts a b_0 in w the number of cycles is equal to the number of occurrences of b_0.

• The permutation is decomposable if and only if there is no free place at a step before the very end of the algorithm. But this exactly means that there is a left factor of w which is a Dyck path.

• The smallest element i of a cycle of length greater than 1 of a permutation α cannot be a left-to-right maximum of α since $i < a_i$. If a fixed point j of α is a left-to-right maximum then $j = a_j$ and $a_i < j$ for $i < j$ implies that $\{a_1, \ldots, a_{j-1}\} = \{1, 2, \ldots, j-1\}$ showing that α is decomposable. Moreover an occurrence of b_1 corresponds to an element written immediately after the pivot, at some step of the algorithm; this element is the image of the pivot which is the smallest element already considered with a free position after it; then it gives rise to a left-to-right maximum. □

Corollary 3. *The number of indecomposable permutations of $\mathcal{S}_n$ with p cycles and q left-to-right maxima, hence the number of hypermaps with n darts, p vertices and q hyperedges, is equal to the number of primitive labeled Dyck paths of length $2n$ with p occurrences of b_0 and q occurrences of b_1.*

3.4. Permutations by numbers of left-to-right maxima and right-to-left minima

In [22], E. Roblet and X.G. Viennot define a bijection similar to Δ, between permutations and another kind of labeled Dyck paths. Their labeling is on the alphabet $\{a, b_1, b_2, \ldots, b_i, \ldots\}$ and such that if w' is a labeled Dyck word, in their definition, then:

- Each occurrence of b_i in w' (such that $w' = gb_ih$) satisfies $1 \leqslant i \leqslant |g|_a - |g|_b$ moreover $i = 1$ if g ends with an a.

Let us denote by $w' = \Delta'(\alpha)$ the labeled Dyck path obtained from the permutation α by the bijection of E. Roblet and X.G. Viennot. The main feature of the bijection (see [22, Theorem 5]) is that the number of right-to-left minima of α is equal to the number of factors ab_1 in w' (often called the peaks of the Dyck path). Moreover the number of left-to-right maxima in α is equal to the number of occurrences of b_k ($w' = gb_kh$) such that $k = |g|_a - |g|_b$. As for the bijection Δ, α is indecomposable if and only if w' is a primitive Dyck word.

Note that an occurrence of b_1 preceded by an a in w' may correspond to both a left-to-right maximum and right-to-left minimum of α if $|g|_a - |g|_b = 1$, but this cannot happen if α is indecomposable.

Corollary 4. *The number of permutations of $\mathcal{S}_n$ with p left-to-right maxima and q cycles is equal to the number of permutations of $\mathcal{S}_n$ with p left-to-right maxima and q right-to-left minima.*

Proof. We first show that this is true for indecomposable permutations. Let θ be an indecomposable permutation and let $w = \Delta'(\theta)$, then replace each occurrence of b_1 in w preceded by an a by b_0 and each occurrence of b_i ($w = gb_kh$) not preceded by an a by b_j, where $j = |g|_a - |g|_b + 1 - i$. Denote w' the word such obtained, it satisfies the conditions of Definition 3, then $\theta' = \Delta^{-1}(w')$ is well defined. θ' has as many left-to-right maxima as θ and as many cycles as θ has right-to-left minima, hence this gives the result for indecomposable permutations.

In order to complete the proof for all permutations it suffices to use the same argument as in the end of Theorem 2. □

Remark. Note that this corollary gives another proof of the symmetry of the statistics for the number of cycles, and number of left-to-right maxima. Indeed, this symmetry is clear for the parameters left-to-right maxima and right-to-left minima since they are exchanged by the inverse operation on permutations.

4. Bivariate polynomials associated to Dyck paths

In this section we introduce two polynomials whose construction is very similar to an operation introduced by P. Deleham (see [9]) which he called the Δ-operator.

4.1. Polynomial associated to a given Dyck path

A factor ab in a Dyck path is often called a *peak*. We go back to the labeled Dyck paths considered in Section 3, and associate a polynomial $L(w)$ in two variables x, y with a Dyck path w as follows:

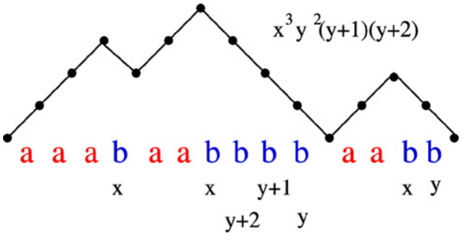

Fig. 4. The polynomial associated with the Dyck path *aabaabbbbaabb*.

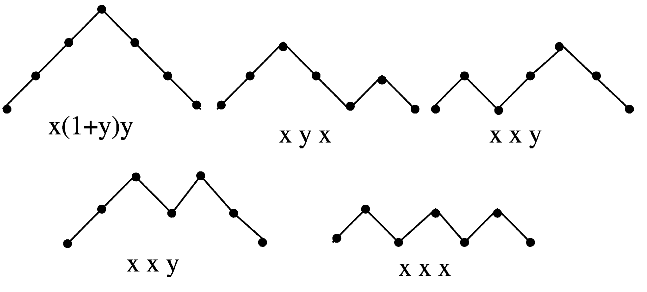

Fig. 5. The computation of polynomial L_3.

Definition 5. Let w be a Dyck path of length $2n$ and let $w'_1b, w'_2b, \dots, w'_nb$ be the left factors of w ending with an occurrence of b, that is $w = w'_ibw''_i$.

The polynomial $L(w)$ is the product of binomials u_i associated with each w'_i by the following rule:

- If w'_i ends with a, then $u_i = x$,
- else $u_i = y + h_i$, where h_i is given by $h_i = |w'_ib|_a - |w'_ib|_b$.

Remark that the number of possible values of i in b_i for a labeling of w is $h_i + 1$, hence $L(1,1)$ is exactly the number of possible labelings of w.

An example of a polynomial associated with a Dyck path is given in Fig. 4.

Proposition 4. *The polynomial $L(w) = x^p \sum_{i=1}^{n} a_i y^i$ associated with a Dyck path w is such that p is the number of factors ab in w, and a_i is the number of labelings of w such that i occurrences of b are labeled b_1.*

Proof. The first part is immediate since each $u_i = x$ corresponds to a factor ab in w. For the second part, note that the possible labelings b_i of an occurrence of b not preceded by an a are such that $1 \leqslant i \leqslant |w'b|_a - |w'|_b = h_i + 1$. Since we have in this case $u_i = y + h_i$ then y in u_i may be interpreted as the labeling b_1 for that occurrence, the other labelings corresponding to the integer h_i. $\square$

4.2. Sum of the polynomials for all paths of a given length

Let D_n be the set of Dyck paths of length $2n$ and D'_n the set of primitive Dyck paths of length $2n$; clearly $D'_n = aD_{n-1}b$. We consider the polynomials $L_n(x,y) = \sum_{w \in D_n} L(w)$ and $L'_n(x,y) = \sum_{w \in D'_n} L(w)$. An example of the polynomials associated with the five Dyck paths of length 6 and allowing us to compute $L_3 = x^3 + 3x^2y + xy^2 + xy$ is given in Fig. 5.

Proposition 5. *The polynomials L_n and L'_n satisfy the following relations for $n > 1$:*

$$L'_n(x, y) = yL_{n-1}(x, y+1),$$
$$L_n(x, y) = L'_n(x, y) + \sum_{p=1}^{n-1} L'_p(x, y)L_{n-p}(x, y).$$

Proof. For the first relation note that the value of $L(awb)$ is obtained from $L(w)$ by replacing $y+1$ for y and multiplying by y. To prove the second relation observe that a Dyck path of length $2n$ is either primitive or the concatenation of a primitive Dyck path of length $2p$ where $1 \leqslant p < n$ and another one (not necessarily primitive) of length $2n - 2p$. $\square$

The first few values of these polynomials are:

$$L_1 = x, \qquad L'_1 = x, \qquad L'_2 = xy, \qquad L_2 = xy + x^2,$$
$$L'_3 = xy^2 + x^2y + xy, \qquad L_3 = x^3 + xy^2 + 3x^2y + xy,$$
$$L'_4 = xy^3 + 3xy^2 + 3y^2x^2 + 2xy + 3yx^2 + yx^3,$$
$$L_4 = xy^3 + 3xy^2 + 6y^2x^2 + 2xy + 5yx^2 + 6yx^3 + x^4.$$

Corollary 5. *The coefficient of $x^p y^q$ in the polynomial L'_n is the number of indecomposable permutations of $\mathcal{S}_n$ with p cycles and q left-to-right maxima. Moreover, the polynomials L'_n are symmetric in x, y: for all $n > 1$ we have*

$$L'_n(x, y) = L'_n(y, x).$$

Proof. The first part is a direct consequence of Theorem 3 and Proposition 5. The symmetry of the polynomials follows from the fact that the number of indecomposable permutations of $\mathcal{S}_n$ with p cycles and q left-to-right maxima is equal to the number of indecomposable permutations of $\mathcal{S}_n$ having q cycles and p left-to-right maxima. $\square$

Corollary 6. *The number of permutations of $\mathcal{S}_n$ having p cycles and q left-to-right maxima is the coefficient of $z^n x^p y^q$ in the power series:*

$$\frac{1}{1 - zL'_1 - z^2L'_2 - z^3L'_3 \cdots}.$$

5. Indecomposable fixed points free involutions and maps

A *map* is a pair of permutations σ, α where α is a fixed point free involution. Maps may be considered as a subset of the set of hypermaps, but conversely a hypermap may be considered as a bipartite map [27]. Maps are an important combinatorial and algebraic tool for dealing with embeddings of graphs in surfaces (see [13,26]) they are sometimes called *rotation system* as in [21].

5.1. Fixed point free involutions

A fixed point free involution is permutation where all cycles have length 2, the number of fixed point free involutions of $\mathcal{S}_{2m}$ is the double factorial:

$$(2m-1)!! = (2m-1)(2m-3)\cdots(3)(1) = \frac{(2m)!}{m!2^m}.$$

As expected, an indecomposable fixed point free involution is a fixed point free involution which is indecomposable as a permutation. The number i_m of indecomposable fixed point free involutions of $\mathcal{S}_{2m}$

satisfies the following inductive relation, which is very similar to that satisfied by the indecomposable permutations.

Proposition 6.

$$i_m = (2m-1)!! - \sum_{p=1}^{m-1} i_p (2m-2p-1)!!.$$

Proof. A decomposable fixed point free involution of $\mathcal{S}_{2m}$ can be written as the concatenation of an indecomposable fixed point free involution of $\mathcal{S}_{2p}$ $(1 < p < m)$ and a fixed point free involution of $\mathcal{S}_{2m-2p}$. $\square$

5.2. Bijection

The bijection between rooted hypermaps and indecomposable permutations specializes for maps and fixed point free involutions as follows.

Proposition 7. *There exist a bijection Ψ' between the set of indecomposable fixed point free involutions on $1, 2, \dots, 2m+2$ and the set of rooted maps on $1, 2, \dots, 2m$. Moreover for $(\sigma, \alpha) = \Psi'(\theta)$, the number of cycles of σ is equal to the number of left-to-right maxima of θ.*

Proof. The bijection Ψ associates with an indecomposable fixed point free involution θ of $\mathcal{S}_{2m+2}$ a hypermap (σ', α') such that α' has m cycles of length 2 and 1 cycle of length 1. The element in this cycle is $j = \theta(2m+2)$, clearly this j is not a left-to-right maxima of θ. Consider the pair of permutations (σ, α) obtained by deleting j from its cycle in σ' and deleting the cycle of length 1 in α, then renumbering the darts by $\phi(i) = i - 1$ for $i > j$. Then (σ, α) is a map. $\square$

Note that this construction was described in detail by P. Ossona de Mendez and P. Rosenstiehl (see [8]).

Corollary 7. *The number of rooted maps with m edges is equal to i_{m+1}, and the number of maps with n vertices and m edges is equal to the number of indecomposable fixed point free involutions with n left-to-right maxima.*

5.3. Involutions and Dyck paths

Theorem 3 specializes for fixed-point free involutions as follows:

Proposition 8. *Let θ be a fixed point free involution of $\mathcal{S}_{2m}$ and w the labeled Dyck path associated with it then*:

- *The length of w is $4m$ and there are m factors of aab_0 in w.*
- *The fixed point free involution θ is indecomposable if and only if the word w is primitive.*
- *If the permutation θ is indecomposable, then the number of left-to-right maxima is equal to the number of occurrences of b_1 in w.*

Replacing in w each factor aab_0 by a gives a new kind of labeled Dyck path w' in which there is no occurrence of b_0 and in which for any occurrence $w' = ub_iv$ of b we have

$$|u|_a - |u|_b \geqslant i.$$

This suggests associating the polynomial $M(w)$ with a Dyck path w by:

Definition 6. Let w be a Dyck path of length $2n$ and let $w'_1b, w'_2b, \ldots, w'_nb$ be the left factors of w ending with an occurrence of b, that is $w = w'_ibw''_i$. The polynomial $M(w)$ is the product of binomials v_i associated with each occurrence of b by the following rule: Let $h_i = |w'_ib|_a - |w'_ib|_b$ $v_i = y + h_i$.

5.4. An enumeration formula

We define the polynomials $M_m(y)$ and $M'_m(y)$ by $M_m(y) = \sum_{w \in D_m} M(w)$ and $M'_m(y) = \sum_{w \in D'_m} M(w)$.

Proposition 9. *The polynomials $M_m(y)$ satisfy the following equations*:

$$M_1(y) = M'_1(y) = y$$

and for $m > 1$:

$$M'_m(y) = yM_{m-1}(1+y),$$
$$M_m(y) = M'_m(y) + \sum_{p=1}^{n-1} M'_p(y)M_{m-p}(y).$$

Moreover, in $M'_m(y)$ the coefficient of y^n is the number of rooted maps with $m-1$ edges and n vertices.

The first values of M'_m are:

$$M'_2 = y^2 + y, \qquad M'_3 = 2y^3 + 5y^2 + 3y, \qquad M'_4 = 5y^4 + 22y^3 + 32y^2 + 15y.$$

This gives a simpler proof of a formula given by D. Arques and J.-F. Béraud [2] (see also [28] for the computation of the first values).

Corollary 8. *Let $U(z, y)$ be the formal power series enumerating rooted maps by numbers of edges and vertices,*

$$U(z, y) = \sum_{m,p>0} \mu_{m,p} z^m y^p = \sum_{m \geqslant 1} z^m M'_m(y),$$

where $\mu_{m,p}$ is the number of rooted maps with m edges and p vertices. Then

$$U(z, y) = y + zU(z, y)U(z, y+1).$$

Proof. By Proposition 9 we have

$$\frac{M'_{m+1}(y)}{y} = M_m(y+1) = M'_m(y+1) + \sum_{p=1}^{n-1} M'_p(y+1)M_{m-p}(y+1),$$
$$\frac{M'_{m+1}(y)}{y} = M'_m(y+1) + \sum_{p=1}^{n-1} M'_p(y+1)\frac{M'_{m+1-p}(y)}{y}.$$

Hence:

$$M'_{m+1}(y) = yM'_m(y+1) + \sum_{p=1}^{n-1} M'_p(y+1)M'_{m+1-p}(y).$$

Multiplying these equalities by z^{m+1} for all $m > 1$ and adding gives the result. $\square$

Note that similar results to those of this section were obtained recently by D. Drake for Hermite polynomials related to weighted involutions which he calls matchings (see [11]).

Acknowledgments

The author wishes to thank warmly Domnique Foata, Antonio Machí, Gilles Schaeffer and Xavier Viennot for fruitful discussions and comments on earlier versions of this paper.

References

[1] M. Aguiar, F. Sottile, Structure of the Malvenuto–Reutenauer Hopf algebra of permutations, Adv. Math. 191 (2004) 225–275.
[2] D. Arques, J.-F. Béraud, Rooted maps on orientable surfaces, Riccati's equation and continued fractions, Discrete Math. 215 (2000) 1–12.
[3] L. Comtet, Sur les coefficients de l'inverse de la série formelle $\sum n!t^n$, C. R. Acad. Sci. Paris Ser. A 275 (1972) 569–572.
[4] R. Cori, Un code pour les graphes planaires ses applications, Astérisque 27 (1975) 1–169.
[5] R. Cori, A. Machí, Maps hypermaps and their automorphisms, a survey, Expo. Math. 10 (1992) 403–467.
[6] P. Cvitanovic, B. Lautrup, R.B. Pearson, Number and weights of Feynman diagrams, Phys. Rev. D 18 (1978) 1939–1949.
[7] P. Ossona de Mendez, P. Rosenstiehl, Transitivity and connectivity of permutations, Combinatorica 24 (2004) 487–502.
[8] P. Ossona de Mendez, P. Rosenstiehl, Encoding pointed maps by double occurrence words, KAM-DIMATIA Ser. 752 (2005) 487–502.
[9] P. Deleham, Delta operator, in: On-Line Encyclopedia of Integer Sequences, A084938, 2003.
[10] J.D. Dixon, Asymptotics of generating the symmetric and alternating groups, Electron. J. Combin. 12 (2005) R56.
[11] D. Drake, The combinatorics of associated Hermite polynomials, European J. Combin. 30 (2009) 1005–1021.
[12] G. Duchamp, F. Hivert, J.-Y. Thibon, Noncommutative symmetric functions VI: Free quasi-symmetric functions and related algebras, Internat. J. Algebra Comput. 12 (2002) 671–717.
[13] J.R. Edmonds, A combinatorial representation for polyhedral surfaces, Notices Amer. Math. Soc. 7 (1960) 646.
[14] D. Foata, M.P. Schützenberger, Théorie géométrique des polyômes Eulériens, Lecture Notes in Math., Springer, Paris, 1970.
[15] G. Gonthier, A computer checked proof of the four colour theorem, http://research.microsoft.com/~gonthier/4colproof.pdf, 2005.
[16] L. Heffter, Über das Problem der Nachbargebiete, Math. Ann. 38 (1891) 477–508.
[17] A. Jacques, Sur le genre d'une paire de substitutions, C. R. Acad. Sci. Paris 267 (1968) 625–627.
[18] G. Jones, D. Singerman, Theory of maps on orientable surfaces, Proc. London Math. Soc. 31 (1978) 211–256.
[19] A. Lentin, Contribution à une théorie des equations dans les monoïdes libres, PhD thesis, Faculté des Sciences, Université de Paris, 1969.
[20] A. Machí, The Riemann–Hurwitz formula for the centralizer of a pair of permutations, Arch. Mat. 42 (1984) 280–288.
[21] B. Mohar, C. Thomassen, Graphs on surfaces, John Hopkins Univ. Press, 2001.
[22] E. Roblet, X.G. Viennot, Théorie combinatoire des T-fractions approximants de Padé en deux points, Discrete Math. 153 (1996) 271–288.
[23] Joseph J. Rotman, An Introduction to the Theory of Groups, Springer, 1999.
[24] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge Univ. Press, 1999.
[25] W.T. Tutte, A census of planar maps, Canad. J. Math. 15 (1963) 249–271.
[26] W.T. Tutte, What is a map?, in: New Directions in the Theory of Graphs, Academic Press, New York, 1973, pp. 309–325.
[27] T.R.S. Walsh, Hypermaps versus bipartite maps, J. Combin. Theory Ser. B 18 (1975) 155–163.
[28] T.R.S. Walsh, A.B. Lehman, Counting rooted maps by genus I, J. Combin. Theory Ser. 13 (1972) 192–218.
[29] J. Youngs, Minimal embeddings the genus of a graph, J. Math. Mech. 12 (1963) 303–315.