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Embeddings in Free Modules and Artinian Rings*

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1. INTRODUCTION

It is well known that if R is a ring such that each right R-module embeds in a free module, then R is a quasi-Frobenius (QF) ring. However, if the embedding is restricted to finitely generated right R-modules, then the situation is much more difficult to handle. A ring R is called right FGF whenever every finitely generated right R-module embeds in a free module and the so-called FGF problem which asks whether a right FGF ring is QF remains open (see [5] for a discussion of this question and [7, 8, 10, 12, 13, 16] for more recent results).

In [9] Menal used the counting arguments developed by Osofsky in [11] to prove that if R is a ring such that each cyclic right R-module embeds in a free module (R is then called a right CF ring) and the injective envelope $E(R_R)$ is projective, then R is already a QF ring. Thus, given a ring R, there exists a cardinal c with the property, that if every c-generated right R-module embeds in a free module, then R is QF. However, Menal

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conjectured that in general it is not enough to take $c = \aleph_0$, i.e., that there exist non-QF rings R such that each countably generated right R-module embeds in a free module.

In this paper we show that it suffices to take c equal to the cardinal of R, namely, if R is a ring of cardinal $|R| = c$ such that each c-generated right R-module embeds in a free module, then R is QF. Menal also asked [9, Question 1], more generally, whether the existence of an infinite cardinal c such that every right R-module defined by a set of relations of cardinality $\leq c$ embeds in a free module already implies that R is QF. Our result implies that the answer to this question is positive when $c \geq |R|$. From a slightly different point of view we may ask which is the smallest cardinal c that guarantees that if every right R-module of cardinal $\leq c$ embeds in a free module, then R is already a QF ring. Our result then says that, if R is infinite, $c \leq |R|$. This seems the best answer that one can reasonably hope for in general.

A right CF ring need not be QF [2, 14] but, because every right FGF right artinian ring is QF [5], the FGF problem will have a positive answer if every right CF ring turns out to be right artinian. This seems unlikely in general but we obtain a rather weak sufficient condition that also extends the result of Menal mentioned above. Let $E = E(R_R)$, $S = \mathrm{End}(E_R)$, and $J = J(S)$. We show that if R is right CF and every proper submodule of E/JE is contained in a maximal submodule, then R is right artinian. These rings need not be QF and, in fact, we show that there exist right CF rings such that $E(R_R)$ is finitely generated—and hence R is right artinian with a right Morita duality—but R is not QF. With the same techniques we also obtain a sufficient condition for a right FGF ring to be QF involving only countably generated right modules: a right FGF ring such that each nonzero countably generated right R-module has a simple quotient is QF.

Throughout this paper, all rings R will be associative and with identity, and Mod-R will denote the category of right R-modules. We will write M_R whenever we want to emphasize that M is a right R-module. The notation $L \subseteq_e M$ will mean that L is an essential submodule of M. We refer to [1, 15] for all undefined notions used in the text.

2. RESULTS

Let S be a ring and $\{C_k\}_{k \in K}$ a family of pairwise nonisomorphic simple right S-modules. Recall from [7, 8] that this family is said to be idempotent-orthogonal (resp. idempotent-semiorthogonal) when there exists a family $\{e_k\}_{k \in K}$ of idempotents of S satisfying that $C_k e_k \neq 0$ for each $k \in K$ and that $C_j e_k = 0$ (resp., $C_j e_k = 0$ or $C_k e_j = 0$) for each $j, k \in K$ such that $j \neq k$. These families played a relevant role in [7, 8] as we shall now explain. Given a ring R and a right R-module M we will denote by $\Omega(R)$ (resp. $C(M)$) a set of representatives of the isomorphism classes of simple right R-modules (resp. of the simple submodules of the module M). From the proof of [8, Theorem 2.5] we can extract the following lemma which will be useful later on:

LEMMA 2.1. *Let R be a ring, E_R an injective module with $S = \mathrm{End}(E_R)$ and $J = J(S)$, and suppose that $|\Omega(R)| \leq |C(E)|$. If for every idempotent-semiorthogonal family $\{C_k\}_{k \in K}$ of simple right S/J-modules we have that $|K| \leq |\Omega(R)|$, then E_R is a cogenerator of* Mod-R *and has finitely generated essential socle.*

Our next task is to look for idempotent-semiorthogonal families of simple right S/J-modules whose cardinal is less than or equal to $|\Omega(R)|$. It was shown in [7] that if the cyclic submodules of the injective envelope $E(R_R)$ are essentially embeddable in projective modules, then the hypothesis on idempotent-semiorthogonal families of the preceding lemma holds for $E = E(R_R)$. We are going to see that a similar result can be obtained, more generally, if the essential embeddings considered are into modules with a weaker form of projectivity that we now define, and that will be only used as a merely technical device. If M is a right R-module, we say that M is *c-projective* if for each epimorphism $p: X \to Y$ and each homomorphism $f: M \to Y$ there exists for each $x \in M$ a homomorphism $h: xR \to X$ such that $(p \circ h)(x) = f(x)$.

Let R be a ring, E_R an injective module, $S = \mathrm{End}(E_R)$, and $J = J(S)$. Consider an idempotent-semiorthogonal family $\{C_k\}_{k \in K}$ of simple right S/J-modules with associated idempotents $\{e_k\}_{k \in K}$. Since the idempotents of S/J lift modulo J, there exist idempotents $\{s_k\}_{k \in K}$ of S such that $e_k = s_k + J$. Let, for each $k \in K$, $E_k = s_k E \subseteq E$ and $p_k: S \to C_k$ the epimorphism defined by $p(1) = c_k$, where $c_k \in C_k$ is an element such that $c_k s_k \neq 0$. Then we have:

LEMMA 2.2. *Let E_R be an injective module, $S = \mathrm{End}(E_R)$, $J = J(S)$, and $\{C_k\}_{k \in K}$ an idempotent-semiorthogonal family of simple right S/J-modules. Suppose that for each $k \in K$ there exists a c-projective submodule P_k of $E_k = s_k E$ such that $(p_k \otimes_S E)(P_k)$ has a nonzero simple quotient. Then $|K| \leq |\Omega(R)|$.*

Proof. We adapt the proof of [7, Lemma 2.4] to this more general situation. Let, for each $k \in K$, $i_k: E_k \to E$ and $h_k: P_k \to E_k$ be the canonical inclusions. Let $N_k = (p_k \otimes_S E)(P_k) = \mathrm{Im}((p_k \otimes_S E) \circ i_k \circ h_k)$ and denote by $q_k: P_k \to N_k$ and $w_k: N_k \to C_k \otimes_S E$ the canonical projection and injection, respectively. By hypothesis N_k has a nonzero simple quotient U_k, with canonical projection $\pi_k: N_k \to U_k$. Choosing one of these quotients for each $k \in K$, we define a map from K to $\Omega(R)$ by assigning $k \mapsto [U_k]$, where $[U_k]$ denotes the isomorphism class of U_k. To complete the proof we show that this map is injective.

Suppose that $[U_j] = [U_k]$ for $j, k \in K$ and let $\varphi : U_j \to U_k$ be an isomorphism. If $\alpha_k : U_k \to E(U_k)$ denotes the inclusion, for each $k \in K$, we obtain by injectivity an R-homomorphism $\phi : E(U_j) \to E(U_k)$ satisfying that $\phi \circ \alpha_j = \alpha_k \circ \varphi$. Also, $\alpha_k \circ \pi_k$ has an extension π'_k to $C_k \otimes_S E$, so that $\alpha_k \circ \pi_k = \pi'_k \circ w_k$. Let now $y_j \in P_j$ be an element such that $(\pi_j \circ q_j)(y_j) \neq 0$. Since P_j is c-projective, there exists a homomorphism $\psi : y_j R \to P_k$ such that if $u_j : y_j R \to P_j$ is the inclusion, then $\pi_k \circ q_k \circ \psi = \varphi \circ \pi_j \circ q_j \circ u_j$. By injectivity we also have an endomorphism $\tau : E \to E$, i.e., an element $\tau \in S$, such that $\tau \circ i_j \circ h_j \circ u_j = i_k \circ h_k \circ \psi$. Observe then that $\phi \circ \alpha_j = \alpha_k \circ \varphi$ is a monomorphism, and hence the morphism $\phi \circ \alpha_j \circ \pi_j \circ q_j \circ u_j : y_j R \to E(U_k)$ is nonzero, so that we have

$$\begin{aligned} 0 \neq \phi \circ \alpha_j \circ \pi_j \circ q_j \circ u_j &= \alpha_k \circ \varphi \circ \pi_j \circ q_j \circ u_j \\ &= \alpha_k \circ \pi_k \circ q_k \circ \psi = \pi'_k \circ w_k \circ q_k \circ \psi = \pi'_k \circ (p_k \otimes_S E) \circ i_k \circ h_k \circ \psi \\ &= \pi'_k \circ (p_k \otimes_S E) \circ \tau \circ i_j \circ h_j \circ u_j. \end{aligned}$$

Now, the argument can be completed exactly as in [7, Lemma 2.4]. If $j \neq k$ and $C_k s_j = 0$, consider the homomorphism $p_k \circ \tau_* \circ i_{j_*} : s_j S \to C_k$, where $\tau_* = \mathrm{Hom}_R(E, \tau)$ and $i_{j_*} = \mathrm{Hom}_R(E, i_j)$, and set $x := (p_k \circ \tau_*)(1) \in C_k$. Then $(p_k \circ \tau_* \circ i_{j_*})(s_j) = (p_k \circ \tau_*)(s_j) = x s_j \in C_k s_j = 0$. Tensoring with ${}_S E$ we then see that $(p_k \otimes_S E) \circ (\tau_* \otimes_S E) \circ (i_{j_*} \otimes_S E) = 0$ and, since $\tau_* \otimes_S E \cong \tau$ and $i_{j_*} \otimes_S E \cong i_j$, that $(p_k \otimes_S E) \circ \tau \circ i_j = 0$, which gives a contradiction and shows that $j = k$. ∎

Now we exploit Lemma 2.2 to obtain situations in which the cardinality of every idempotent-semiorthogonal family of simple right S/J-modules is less than or equal to that of the set of isomorphism classes of simple right R-modules.

LEMMA 2.3. *Let P_R be a finitely generated projective module such that $E_R = E(P_R)$ is c-projective and suppose that every proper submodule of E/JE is contained in a maximal submodule. Then, for every idempotent-semiorthogonal family $\{C_k\}_{k \in K}$ of simple right S/J-modules, $|K| \leq |\Omega(R)|$.*

Proof. Consider an idempotent-semiorthogonal family $\{C_k\}_{k \in K}$ of simple right S/J-modules with associated idempotents $\{e_k\}_{k \in K}$ and let, for each $k \in K$, s_k be an idempotent of S such that $e_k = s_k + J$. Then $C_k s_k \neq 0$ for each $k \in K$ and so there exists an element $c_k \in C_k$ such that $c_k s_k \neq 0$. Let $p_k : S \to C_k$ be the homomorphism of Mod-S defined by $p_k(1) = c_k$ and $E_k = s_k E$, with canonical inclusion $i_k : E_k \to E$. Setting

$i_{k_*} = \mathrm{Hom}_R(E, i_k)$, we see that $(p_k \otimes_S E) \circ i_k : E_k \to C_k \otimes_S E$ can be identified with $(p_k \circ i_{k_*}) \otimes_S E$ and hence it is an epimorphism, so that $(p_k \otimes_S E)(E_k) = C_k \otimes_S E$. By [6, Proposition 1.3] we have that $C_k \otimes_S E \neq 0$ and since this module is a quotient of E/JE, our hypothesis implies that it has a simple quotient. Thus we can apply Lemma 2.2 with $P_k = E_k$ to conclude that $|K| \leq |\Omega(R)|$. ∎

The usefulness of considering c-projective modules instead of projective ones will be apparent once we prove our next result. Indeed, in order to be able to apply Lemma 2.3 we show that if the cyclic submodules of E_R embed in c-projectives then E_R is already c-projective, even if these embeddings are not supposed to be essential. We have:

PROPOSITION 2.4. *Let E_R be an injective module. Then the following conditions are equivalent*:

(i) *E is c-projective.*

(ii) *Every cyclic submodule of E_R is essentially embeddable in a c-projective module.*

(iii) *Every cyclic submodule of E_R is embeddable in a c-projective module.*

Proof. That (i) implies (ii) is clear because a direct summand of a c-projective module is c-projective and it is also obvious that (ii) implies (iii). To prove that (iii) implies (i), let $f : E \to Y$ be an homomorphism, $p : X \to Y$ an epimorphism, and $x \in E$ with inclusion $u : xR \to E$. By hypothesis, there exists a monomorphism $v : xR \to M$, where M is a c-projective module. Since E is injective there exists a homomorphism $w : M \to E$ such that $u = w \circ v$. The c-projectivity of M then gives a morphism $g : xR \to X$ such that $f \circ w \circ v = p \circ g$ and so $f \circ u = p \circ g$, showing that E is c-projective. ∎

We are now ready to obtain a rather weak but sufficient condition for a right CF ring to be right artinian.

THEOREM 2.5. *Let R be a ring and $E = E(R_R)$. If R is right CF and every proper submodule of E/JE is contained in a maximal submodule, then R is right artinian.*

Proof. Since every cyclic right R-module embeds in a free module, it is enough to show that R_R has finitely generated essential socle. By Lemma 2.3 and Proposition 2.4 we have that the cardinality of every idempotent-orthogonal family $\{C_k\}_{k \in K}$ of simple right S/J-modules is less than or equal to $|\Omega(R)|$. Then we can use Lemma 2.1 to show that E_R and hence R_R has finite essential socle. ∎

COROLLARY 2.6. *Let R be a ring and $E = E(R_R)$. If R is right CF and E/JE is finitely generated, then R is right artinian.*

Corollary 2.6 extends Menal's result which asserts that a right CF ring R such that $E(R_R)$ is projective is already a QF ring [9, Corollary 9]. Indeed, it follows from this corollary that if R is right CF and $E(R_R)$ is finitely generated, then R is a right artinian ring with a right Morita duality. These rings need not be QF, however, as the following example shows.

EXAMPLE 2.7. Let as in [2, p. 70] p be a prime number and P the prime field of p elements. Let $K = P(X)$ be the field of rational functions over P and consider K as a (K, K)-bimodule where the right K-module structure is the natural one and the left K-module structure is given by the Frobenius endomorphism of K, that is, $a \cdot x = a^p x$ for $a, x \in K$. Let R be the trivial extension of K by the bimodule ${}_K K_K$, that is, $R = K \oplus K$ as abelian group, with multiplication given by $(a, x)(b, y) = (ab, a^p y + bx)$ for $(a, x), (b, y) \in R$. It is shown in [2] that R is a (two-sided) artinian ring which is right CF but not QF. We show that $E(R_R)$ is finitely generated. To see this observe that if $\varphi : K \to R$ denotes the canonical ring homomorphism, then the restriction of scalars functor $\varphi_* : \text{Mod-}R \to \text{Mod-}K$ has a right adjoint $\varphi^! : \text{Mod-}K \to \text{Mod-}R$ given by $\varphi^!(M) = \text{Hom}_K(R, M)$ for $M \in \text{Mod-}K$ (cf. [15, p. 105]). Since φ_* preserves monomorphisms, it is easily checked that $\varphi^!$ preserves injectivity and so $\varphi^!(M)$ is an injective right R-module for every M in Mod-K. In particular, $E_R := \varphi^!(K)$ is an injective right R-module and we show that $E_R \cong E(R_R)$. It is easily seen that $E_R \cong K \oplus K$ with the right R-module structure given by $(u, v)(a, x) = (a^p u, av + xu)$ for $(u, v) \in E_R$ and $(a, x) \in R$. We can define an R-homomorphism $u : R_R \to E_R$ by $u(a, x) = (a^p, x)$. It is easy to check that u is a monomorphism and that $\operatorname{Im} u$ is essential in E_R, so that we indeed have $E_R \cong E(R_R)$. Finally, to see that E_R is finitely generated just note that $K = K^p(X)$ is a p-dimensional vector space over K^p. Let $\{u_1, \dots, u_p\}$ be a K^p-basis of K. Then if $(u, v) \in E_R$ with $u = \sum_{i=1}^p a_i^p u_i$ and we set $v_1 = v u_1^{-1}$, $v_2 = v_3 = \cdots = v_p = 0$, we see that $(u, v) = (\sum_{i=1}^p a_i^p u_i, \sum_{i=1}^p v_i u_i) = \sum_{i=1}^p (u_i, 0)(a_i, v_i)$, so that the elements $(u_1, 0), \dots, (u_p, 0)$ generate E_R.

In [13, Theorem 3.4] it was shown that a right CF right perfect ring is right artinian. Recall that a ring R is called a right max ring (or a right B-ring [4]) when every nonzero right R-module has a maximal submodule; right perfect rings are right max rings but a right max ring need not be semiperfect. In fact, if R is a right V-ring, that is, a ring such that each simple right R-module is injective, then all the right R-modules have zero radical and so R is a right max ring. Thus, in contrast with the rings considered in [12, 13], right max rings need not even be semiregular (see

[3]). As an immediate consequence of Theorem 2.5 we obtain:

COROLLARY 2.8. *Every right CF right max ring is right artinian.*

We are now going to see that for right FGF rings it is enough to assume that countably generated nonzero modules have maximal submodules. With this goal in mind we show:

PROPOSITION 2.9. *Let R be a right FGF ring and N a finitely generated right R-module. Then N is contained in a countably generated c-projective submodule of $E(N)$.*

Proof. Let N be a finitely generated module and $E_R = E(N)$. We define a c-projective submodule of E_R containing N by induction. Starting with $M_1 = N$ with inclusion $u_1 : M_1 \to E$ and considering a monomorphism $v_1 : M_1 \to F_1$, where F_1 is a finitely generated free module, we obtain by injectivity a morphism $w_1 : F_1 \to E$ such that $w_1 \circ v_1 = u_1$. Let $M_2 = \operatorname{Im} w_1$ and, inductively, $M_{k+1} = \operatorname{Im} w_k$ for each $k \geq 1$. Then the M_k are finitely generated submodules of E and $N = M_1 \subseteq M_2 \subseteq \cdots \subseteq M_k \subseteq \cdots$, so that $M := \cup_{k=1}^{\infty} M_k$ is a countably generated submodule of E. We show that M is a c-projective module. To see this let L be a finitely generated submodule of M with inclusion $u : L \to M$, $p : X \to Y$ an epimorphism, and $f : M \to Y$ a homomorphism. Since L is a finitely generated submodule of $M := \cup_{k=1}^{\infty} M_k$, there exists an index $j \geq 1$ such that $L \subseteq M_j$. By construction, the inclusion $u_j : M_j \to E$ factors in the form $u_j = w_j \circ v_j$ with $v_j : M_j \to F_j$ a monomorphism into a free module F_j. Thus u factors through F_j and so does $f \circ u$. This gives the required morphism $g : L \to X$ satisfying that $f \circ u = p \circ g$ and shows that M is indeed c-projective. ∎

COROLLARY 2.10. *Let R be a right FGF ring such that each countably generated nonzero right R-module has a maximal submodule. Then R is QF.*

Proof. Let $\{C_k\}_{k \in K}$ be an idempotent-semiorthogonal family of simple right S/J-modules, where $S = \operatorname{End}(E(R_R))$. If, as in Lemma 2.1, $E_k = s_k E$, then $(p_k \otimes_S E)(E_k) \neq 0$ and by Proposition 2.9 there exists a countably generated c-projective submodule P_k of E_k such that $(p_k \otimes_S E)(P_k) \neq 0$. Since this module is countably generated it has a simple quotient by hypothesis and so Lemma 2.2 implies that $|K| \leq |\Omega(R)|$. Then $E(R_R)$ is a cogenerator and has finitely generated essential socle by Lemma 2.1. Thus R is right artinian and hence QF by [5]. ∎

We will now show that in the situation of Theorem 2.5 the hypothesis that E/JE is finitely generated is automatically satisfied if we assume that sufficiently large modules embed in free modules. First we find an upper bound for $|E/JE|$.

LEMMA 2.11. *Let R be a ring such that each 2-generated submodule of $E = E(R_R)$ embeds in a free module. Then $|E/JE| \leq \max\{\aleph_0, |R|\}$.*

Proof. First we show that $|E/JE| \leq |S/J|$. Since $E/JE \cong (S/J) \otimes_S E$, it is enough to prove that every element of $(S/J) \otimes_S E$ is of the form $(s + J) \otimes 1$ with $s \in S$ and $1 \in R \subseteq E$. If we consider an element of the form $(s + J) \otimes e$ with $e \in E$, then the R-homomorphism $f : R \to E$ given by $f(1) = e$ extends by injectivity to an endomorphism s' of E such that $s'(1) = f(1) = e$. Thus $(s + J) \otimes e = (s + J) \otimes s'(1) = (ss' + J) \otimes 1$. It is then clear that every element of $(S/J) \otimes_S E$ is of this form.

Next we show that $|S/J| \leq \max\{\aleph_0, |R|\}$. Let, for each $n \geq 1$, M_n be the set of all injective homomorphisms from R to R^n. For each $f \in M_n$ let $\Delta_{n,f} := \{g \in \operatorname{Hom}_R(R, R^n) \mid \operatorname{Im} g \cap \operatorname{Im} f \subseteq_e \operatorname{Im} g\}$. We define a map $\varphi_{n,f} : \Delta_{n,f} \to S/J$ as follows. Let $g \in \Delta_{n,f}$ and $u : R \to E$ the canonical inclusion. Since f is a monomorphism, there exists a homomorphism $h : R^n \to E$ such that $h \circ f = u$ and, using again the injectivity of E, we obtain an endomorphism s of E such that $s \circ u = h \circ g$. Then we define $\varphi_{n,f}(g) := s + J$. To see that $\varphi_{n,f}$ is indeed a well defined map, suppose that $h' : R^n \to E$ and $s' \in S$ satisfy that $h' \circ f = u$, $s' \circ u = h' \circ g$. Then $L = g^{-1}(\operatorname{Im} f)$ is essential in R_R and hence in E_R. Calling $v : L \to R$ to the inclusion we have a homomorphism $g' : L \to R$ such that $f \circ g' = g \circ v$ and hence $s \circ u \circ v = h \circ g \circ v = h \circ f \circ g' = u \circ g'$ and, similarly, $s' \circ u \circ v = h' \circ g \circ v = h' \circ f \circ g' = u \circ g'$. Thus we see that the restrictions of s and s' to L agree and so $s - s'$ is an endomorphism of E whose kernel is an essential submodule. Therefore we have that $s - s' \in J$ and so $s + J = s' + J$.

Now, we claim that for each $s \in S$ there exist $n \geq 1$, $f \in M_n$, and $g \in \Delta_{n,f}$ such that $\varphi_{n,f}(g) = s + J$. Let $x = s(1) \in E$ and $\alpha : R \to xR + R$ the homomorphism given by left multiplication with x. If $v : xR + R \to E$ is the canonical inclusion, then $s \circ u = v \circ \alpha$. Because every 2-generated module embeds in a free module, there exist $n \geq 1$ and a monomorphism $w : xR + R \to R^n$. Let $f = w|_R$ and $g = w \circ \alpha$. Since R is essential in E, we have that $xR \cap R$ is essential in xR and hence $\operatorname{Im} g \cap \operatorname{Im} f = w(xR) \cap w(R) \subseteq_e \operatorname{Im} g = w(xR)$. This shows that $g \in \Delta_{n,f}$. By injectivity we obtain a morphism $h : R^n \to E$ such that $h \circ w = v$ so that, in particular, $h \circ f = v|_R = u$. Thus $s \circ u = v \circ \alpha = h \circ w \circ \alpha = h \circ g$ and so we have that $\varphi_{n,f}(g) = s + J$, proving our claim.

Hence it follows that $|S/J| \leq |\cup_{n \geq 1}(\cup_{f \in M_n} \Delta_{n,f})|$ Let $c = \max\{\aleph_0, |R|\}$. Since $|M_n| \leq |\operatorname{Hom}_R(R, R^n)| = |R^n| \leq c$ and, similarly, $|\Delta_{n,f}| \leq c$ for each $n \geq 1$ and each $f \in M_n$, we have that $|S/J| \leq c$. ∎

Now we are ready to give the promised condition for the finite generation of E/JE.

LEMMA 2.12. *Let R be an infinite ring, $E = E(R_R)$, $S = \operatorname{End}(E_R)$, and $J = J(S)$. If every quotient of $R_R^{(R)}$ embeds in a free module, then E/JE is finitely generated.*

Proof. First note that by Lemma 2.11, $|E/JE| \leq c = |R|$. Let $q : E \to E/JE$ be the canonical projection. Then there exists a c-generated submodule M of E such that $q(M) = E/JE$ and, replacing M by $M + R$ if necessary, we can assume that $R \subseteq M$. Denote by $v : R \to M$ the inclusion. By hypothesis we have a monomorphism $w : M \to R^{(I)}$ for some set I and so there exists a finite subset F of I such that $\operatorname{Im}(w \circ v) \subseteq R^F$. Then, if $\pi : R^{(I)} \to R^F$ denotes the projection, $\pi \circ w \circ v$ is a monomorphism. Because v is essential we have that $\pi \circ w$ is also a monomorphism. Thus the inclusion $\alpha : M \to E$ factors, by injectivity, in the form $\alpha = h \circ \pi \circ w$ for some $h : R^F \to E$. Therefore $E/JE = (q \circ \alpha)(M) = (q \circ h \circ \pi \circ w)(M) \subseteq (q \circ h)(R^F) \subseteq E/JE$. From this it follows that $E/JE = (q \circ h)(R^F)$ is finitely generated. ∎

We now give our main result.

THEOREM 2.13. *Let R be a ring such that every quotient of $R_R^{(R)}$ embeds in a free module. Then R is QF.*

Proof. If R is finite, then every 2-generated right R-module embeds in a free module and so the result follows from [16, Proposition 1.24]. If R is infinite the result follows from Corollary 2.6 and Lemma 2.12. ∎

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