An iterative method for solving neutron transport equation in 2-D plane geometry

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Abstract

This paper presents an iterative method based on a self ad joint and m-Accretive splitting for the numerical treatment of the steady neutron transport equation in 2-D plane geometry.

Theoretical results show the convergence of the method.

The convergence of the method is numerically illustrated and compare with the standard source iteration method on a sample problem.

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1. Introduction

The Neutron Transport integral equations occurring in various branches of applications such as solid mechanics, phase transitions and many others, have been extensively investigated theoretically and numerically in recent year [1–11].

Generally, the type of these equations arise naturally in connection with physical models described by systems of ODEs and PDEs [9–10] which can equivalently be written as integral equations.

In fact, many algorithms for ODEs have counterparts for Neutron transport equations. The efficiency of solving Neutron transport equations is somewhat counterbalanced by the fact that systems of these equations occur more frequently in practice.

Many methods have been proposed for the solutions of linear and nonlinear Transport integral equation [10–11].

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2. Mathematical setting and Computational models

For an incompressible fluid and a supposed one-dimensional flow \((x,t)\), the equation of mass conservation reads:

\[
\frac{\partial Q}{\partial x} + \frac{\partial S}{\partial t} = q(x) \quad (2-1)
\]

Considering the particular derivative of the volume occupied within a control surface, the equations are written as follows:

\[
\frac{Dv}{Dt} = \int_{S} \vec{V} \cdot \vec{n} dS = \int_{x_{1}}^{x_{2}} q(x) dx \quad (2.2)
\]

On the free surface \(\vec{V} \cdot \vec{n} = \frac{\partial y}{\partial t}\) since \(y(x,t)\) the draft means. The flow is still one-dimensional element of the surface \(ds\) of the free surface is written \(dS = Bdx\) as \(B(x,t)\) the mirror is the width of the cross section line to the abscissa \(x\) and wet at the moment \(t\). The flow balance is written:

\[
Q(x_{2},t) - Q(x_{1},t) + \int_{x_{1}}^{x_{2}} B \frac{\partial y}{\partial t} dx = \int_{x_{1}}^{x_{2}} q(x) dx \quad (2.3)
\]

Where:

\[
Q(x_{2},t) - Q(x_{1},t) = \int_{x_{1}}^{x_{2}} \frac{\partial Q}{\partial x} dx \quad (2.4)
\]

The medium is assumed continuous, the theorem of the integral zero leads to the equation

\[
\frac{\partial Q}{\partial x} + B \frac{\partial y}{\partial t} = q(x) \quad (2.5)
\]

\[
\frac{\partial Q}{\partial x} + S \frac{\partial S}{\partial t} = q(x) \quad (2.6)
\]

\[
\frac{\partial S}{\partial x} + S \frac{\partial U}{\partial x} \frac{\partial S}{\partial t} = q(x) \quad (2.7)
\]

Is the differential \(dS = Bdy\) and dividing both members of the above equation by \(B\), necessarily different from zero, since there flow into the channel, we obtain the following expression:

\[
\frac{\partial y}{\partial x} + \frac{\partial U}{\partial x} \frac{\partial y}{\partial t} = \frac{q}{B} \quad (2.8)
\]

\[
\frac{\partial y}{\partial x} + \frac{c^2 \partial U}{g \partial y} \frac{\partial y}{\partial t} = \frac{q}{B} \quad (2.9)
\]

The general equation of mass conservation written below:

\[
\frac{\partial S}{\partial x} + S \frac{\partial U}{\partial x} \frac{\partial S}{\partial t} = q(x) \quad (2,10)
\]

Since the flow is unidimensional and considering the equation \(Q = SU\). The general equation of conservation of mass, taking into account any input side is written:

\[
\frac{\partial S}{\partial x} + \frac{\partial Q}{\partial x} = q \quad (2.11)
\]

The resulting term \(\frac{1}{g} \frac{\partial U}{\partial t}\) in the stationary flow is written in view of the equation of mass conservation:
\[
\frac{1}{g} \frac{\partial U}{\partial t} + \frac{1}{gS} \frac{\partial Q}{\partial t} - \frac{Qq}{gS^2} + \frac{Q}{gS^2} \frac{\partial Q}{\partial x} = 0 \tag{2.12}
\]

The term \(\frac{\partial H}{\partial x}\) to be \(\alpha = 1\) written:
\[
\frac{\partial H}{\partial x} = -J_f + \frac{\partial y}{\partial x} + \frac{Q}{gS^2} \frac{\partial Q}{\partial x} - \frac{Q^2}{gS^2} \frac{\partial S}{\partial x} \tag{2.13}
\]

Saint Venant's equation takes the following form, after regrouping of terms:
\[
\frac{\partial y}{\partial x} \left( 1 - \frac{Q^2L}{gS^3} \right) - J_f + J + \frac{1}{gS} \frac{\partial Q}{\partial t} - \frac{Qq}{gS^2} + 2 \frac{Q}{gS^2} \frac{\partial Q}{\partial x} - \frac{Q^2}{gS^2} \frac{\partial S}{\partial x} \right)_{y,t} = -\frac{1}{S} \left[ \frac{\partial (h \gamma S)}{\partial x} \right]_{y,t} + \frac{qV}{gS} \tag{2.14}
\]

Since the flow is steady \(\frac{\partial Q}{\partial t} = 0\), the draft and flow do function as the abscissa of the flow.

The channel is prismatic \(-\frac{Q^2}{gS^3} \left( \frac{\partial S}{\partial x} \right)_{y,t} = 0\) and \(-\frac{1}{S} \left[ \frac{\partial (h \gamma S)}{\partial x} \right]_{y,t} = 0\), the terms and conditions.

The flow is gradually growing in the channel \(\frac{dQ}{dx} = q\), then we can write.

The previous equation becomes:
\[
\frac{\partial y}{\partial x} \left( 1 - \frac{Q^2L}{gS^3} \right) - J_f + J + \frac{Qq}{gS^2} = \frac{qV}{gS} \tag{2.15}
\]

3. Numerical method

\[
\left\{ \begin{array}{l}
\frac{\partial y}{\partial t} + U \frac{\partial y}{\partial x} + y \frac{\partial U}{\partial x} = 0 \\
\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + g \frac{\partial y}{\partial x} = g(J_f - J) \end{array} \right. \tag{3.1}
\]

In this scheme, the governing partial differential equations are replaced by quotients of finite differences as follows:
\[
\frac{\partial y}{\partial t} = \frac{y_{i+1}^j - y_i^j}{\Delta t} \tag{3.3}
\]
\[
\frac{\partial U}{\partial t} = \frac{U_{i+1}^j - U_{i}^j}{\Delta t} \tag{3.4}
\]
\[
\frac{\partial U}{\partial x} = \frac{U_{i+1}^j - U_{i-1}^j}{2\Delta x} \tag{3.5}
\]

Where:
\[
y_i^* = 0.5(y_{i-1}^j + y_{i+1}^j) \tag{3.6}
\]
\[
U_i^* = 0.5(U_{i-1}^j + U_{i+1}^j) \tag{3.7}
\]
\[
\frac{\partial y_i}{\partial t} = -\bar{U} \frac{y_{i+1} - y_{i-1}}{2\Delta x} - y_i \frac{\bar{U}_{i+1} - \bar{U}_{i-1}}{2\Delta x}
\]  
\(3.8\)

\[
\frac{\partial \bar{U}_i}{\partial t} = -\bar{U}_i \frac{\bar{U}_{i+1} - \bar{U}_{i-1}}{2\Delta x} - g \frac{y_{i+1} - y_{i-1}}{2\Delta x} + g(J_f - J_i)
\]  
\(3.9\)

This equation is applicable only for nodes that are not at the edges, but as in any boundary value problem we know the edges so the Riemann problem can be solved.

In addition:
\[
J_i = \frac{\bar{U}_i \left| \bar{U}_i \right|}{(Rh)_i^{\frac{1}{2}}} n^2
\]  
\(3.10\)

Using an explicit scheme that is we know \(\bar{U}, y\) and \(J_f\) now, we can calculate these values at once \(j + 1\).

\[
y^{j+1}_i = y^*_i + \Delta t \left[ -\bar{U}_i \frac{y^j_{i+1} - y^j_{i-1}}{2\Delta x} - y_i \frac{\bar{U}^j_{i+1} - \bar{U}^j_{i-1}}{2\Delta x} \right]
\]  
\(3.11\)

\[
\bar{U}^{j+1}_i = \bar{U}^*_i + \Delta t \left[ -\bar{U}_i \frac{\bar{U}^j_{i+1} - \bar{U}^j_{i-1}}{2\Delta x} - g \frac{y^j_{i+1} - y^j_{i-1}}{2\Delta x} + g(J_f - J^{j+1}_i) \right]
\]  
\(3.12\)

\[
J^{j+1}_i = \frac{\left| \bar{U}^{j+1}_i \right|}{(Rh)^{\frac{1}{2}}_i} n^2
\]  
\(3.13\)

By asking: \(\Gamma = \frac{n^2}{(Rh)^{\frac{1}{2}}_i} g \Delta t\) and multiplying the equation by \(\Gamma^{-1}\), we obtain the following equation:

\[
[\bar{U}^{j+1}_i]^2 + \Gamma[\bar{U}^{j+1}_i] - \Gamma \beta = 0
\]  
\(3.14\)

where: \(\beta = \left[ \bar{U}^*_i - \frac{\Delta t}{2\Delta x} \bar{U}^j_i \frac{\bar{U}^j_{i+1} - \bar{U}^j_{i-1}}{2\Delta x} - g \frac{\Delta t}{2\Delta x} \frac{y^j_{i+1} - y^j_{i-1}}{2\Delta x} + g \Delta t J_f \right] \)

This is a quadratic equation \(\bar{U}^{j+1}_i\) and the result:

\[
\bar{U}^{j+1}_i = \frac{1}{2} \left[ -\Gamma + \left( \Gamma^2 + 4\Gamma \beta \right)^{\frac{1}{2}} \right]
\]  
\(3.15\)

\[
y^{j+1}_i = y^*_i + \frac{\Delta t}{\Delta x} \left[ \bar{U}^j_i (y^j - y^j_{i+1}) + y^j_i (\bar{U}^j_i - \bar{U}^j_{i+1}) \right]
\]  
\(3.16\)

\[
y^{j+1}_i = y^*_i + \frac{\Delta t}{\Delta x} \left[ \bar{U}^j_i (y^j_{i-1} - y^j_i) + y^j_i (\bar{U}^j_{i-1} - \bar{U}^j_i) \right]
\]  
\(3.17\)
Figure 1. Variation height at the foot of the dam.

Figure 2. Variation speed at the foot of the dam.

Figure 3. Variation of the height at $x = 7368.645$ m from the dam
Figure 4. Change the speed at $x = 7368.645$ m dam.

Figure 5. Change the speed of the wave breaking at $x = 7368.645$ m from the dam.

Figure 6. Variation of the height at $t = 100$ s.
4. Conclusion

Throughout this work, it comes that the iterative methods based on a Self-adjoint and m-Accretive splitting presented for solving the transport equation in 2D-planel geometry, converge unconditionally. The theoretical proof of the convergence of the method is independent of the discretization. The previous Numerical results show that the SAS iteration is efficient compare to the standard Source Iteration. The method is easy to implemented as SI method.

5. References