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Inadmissible Forcing

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A structure is E-closed if it is closed under all partial E-recursive functions from V into V, a set theoretic extension of Kleene's partial recursive functions of finite type in the normal case. Let $L(\kappa)$ be E-closed and $\sum_1$ inadmissible. Then $L(\kappa)$ has reflection properties useful in the study of generic extensions of $L(\kappa)$. Every set generic extension of $L(\kappa)$ via countably closed forcing conditions is E-closed. A class generic construction shows: if $L(\kappa)$ is countable, and inside $L(\kappa)$ the greatest cardinal $\mathrm{gc}(\kappa)$, has uncountable cofinality, then there exists a $T \subseteq \mathrm{gc}(\kappa)$ such that $L(\kappa, T) = E(T)$, the least E-closed set with T as a member. A partial converse is obtained via a selection theorem that implies $E(X)$ is $\sum_1$ admissible when X is a set of ordinals and the greatest cardinal in the sense of $E(X)$ has countable cofinality in $E(X)$. © 1987 Academic Press, Inc.

1. INTRODUCTION

The notion of partial recursive function with numerical arguments was extended by Kleene [1, 2] so as to allow arguments of finite type. His extension was defined by schemes which give rise to infinitely long computations. In the so-called normal case the equality predicate is treated as if it were recursive. Normann [3] and Moschovakis independently extended the normal Kleene theory so as to allow all sets to occur as arguments. Thus $\{e\}(x)$, the eth partial recursive function with argument x, has a meaning for all $x \in V$.

Normann's definition is inductive. $\{e\}$ is defined by interpreting e as the Gödel number of a scheme. The schemes include rudimentary functions such as pairing and union, composition, effective bounding, and also enumeration $(f(e, x) \simeq \{e\}(x))$. The attempt to compute $\{e\}(x)$ via schemes produces a tree that is well founded iff $\{e\}(x)$ is defined. Computation trees are discussed in Section 2, but it is worth noting now that the unique source of infinitely long computations is the *effective bounding*

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scheme: *if* $\{m\}(x)$ is defined, and if $\{n\}(y)$ is defined for all $y \in \{m\}(x)$, then $\{2^m \cdot 3^n\}(x)$ is defined and is equal to

$$\{\{n\}(y) \mid y \in \{m\}(x)\}.^1$$

If $\{e\}(x)$ is defined, then $\{e\}(x)$ said to convergence (in symbols $\{e\}(x)\downarrow$). Otherwise $\{e\}(x)$ diverges ($\{e\}(x)\uparrow$).

Let A be a transitive set. A is *E-closed* if

$$x \in A \ \&\ \{e\}(x)\downarrow \rightarrow \{e\}(x) \in A.$$

Every $\sum_1$ admissible set is E-closed but not conversely. The *E-closure* of a set x, denoted $E(x)$, is the least E-closed set that contains $\{x\} \cup \mathrm{TC}(x)$.[2] The classic example of an inadmissible, E-closed set is $E(2^\omega)$. The least example is

$$E(\omega) \cup E(E(\omega)) \cup E(E(E(\omega))) \cup \cdots.$$

Some of the results of [4] were obtained by showing certain generic extension of $E(2^\omega)$ are E-closed. The language of [4] was that of Kleene [1], so for clarity it must be pointed out that $E(2^\omega)$ corresponds to recursion in ${}^3E$ as follows: $z \in E(2^\omega)$ iff z is codable by a relation on 2^ω recursive in ${}^3E$, b for some $b \in 2^\omega$.

Set forcing over a Σ_1 admissible structure poses few problems. It is well known that if A is $\sum_1$ admissible and G is generic over A with respect to some set P (in A) of forcing conditions, then $A(G)$ is $\sum_1$ admissible. Class forcing ($P \subseteq A$) over A can be more intricate, as in Abramson [5] or Steel [6].

Set forcing over an E-closed, inadmissible structure poses a new problem. Let t be a term that refers to a generic object. What does it mean to force $\{e\}(t)$ to converge? The answer varies greatly from one notion of forcing to the next. It is a consequence of Gandy selection that if a Levy-style, generic collapse of ω_1 to ω is adjoined to $E(\omega_1)$, then the resulting extension is not E-closed. In [4] it was shown that every Cohen-generic extension of $E(\omega_1)$ is E-closed. Here Cohen-generic refers to forcing conditions that are countable initial segments of characteristic functions of subsets of ω_1. The causal difference between the above Levy and Cohen extensions was pointed out by Slaman: the set of countable Cohen conditions on ω_1 is *countably closed*, that is,

$$(n)[p_n \geqslant p_{n+1}] \rightarrow (Eq)(n)[p_n \geqslant q].$$

[1] For the sake of exposition the bounding scheme given here differs somewhat from that of Normann [3].

[2] $\mathrm{TC}(x)$ is the transitive closure of x.

The countable closure condition is not necessary for E-closed, generic extensions. The countable chain condition also suffices by a mode of argument [8] not employed in the present paper.

The set forcing construction of [4] was extended to an arbitrary E-closed, inadmissible $L(\kappa)$ in [9]. The machinery of [9] is reviewed and amplified in Sections 2 and 3, and then applied to a class forcing construction in Section 4 to prove

THEOREM 4.8. *Let $L(\kappa)$ be countable, E-closed and $\sum_1$ inadmissible. Suppose the greatest cardinal of $L(\kappa)$ has uncountable cofinality in $L(\kappa)$. Then there exists an $S \subseteq \omega_1^{L(\kappa)}$ such that $L(\kappa, S)$ is the E-closure of S.*

(According to a well-known fine structure fact, Proposition 2.5,

$$L(\kappa) \models [\text{There is a greatest cardinal.}],$$

where $L(\kappa)$ is $\sum_1$ inadmissible.)

Theorem 4.8 resembles the following result of admissibility theory [7]: If $L(\kappa)$ is countable and $\sum_1$ admissible, then there exists a $T \subseteq \omega$ such that $L(\kappa, T)$ is the least $\sum_1$ admissible set with T as a member. A partial converse to 4.8 is obtained in Section 5 via a selection theorem.

COROLLARY. 5.2. *Let X be a set of ordinals. If in $E(X)$ the greatest cardinal has cofinality ω, then $E(X)$ is $\sum_1$ admissible.*

If $\{e\}(x)$ does not converge, then its computation tree has at least one infinite descending path. We call any such path a Moschovakis witness to the divergence of $\{e\}$ at x. They were introduced in Moschovakis [10] to show $E(2^\omega)$ is not $\sum_1$ admissible, and have since proved essential to the study of recursion in higher type objects. A structure A is said to admit M. witnesses if for all $e < \omega$ and $x \in A$: if $\{e\}(x)$ diverges, then some member of A is an M. witness for $\{e\}(x)$.

LEMMA 2.4 [9]. *If $L(\kappa)$ is E-closed but not $\sum_1$ admissible, then $L(\kappa)$ admits Moschovakis witnesses.*

The above reflection property is precisely what is needed to make countably closed set forcing succeed over an E-closed structure.

THEOREM 3.5. *Suppose A is E-closed and admits Moschovakis witnesses. Let $P \in A$ be countably closed in A. If G is P-generic, then $A(G)$ is E-closed.*

The paper ends with some open questions and conjectures.

2. Existence of Moschovakis Witnesses

In this section it is shown that many inadmissible, E-closed structures admit Moschovakis witnesses. The proof extracts a reflecting ordinal from the failure of $\sum_1$ admissibility, and then draws on a characterization of reflection devised by Harrington [11] for recursion in objects of finite type.

y is said to be *E-recursive in* x if there exists an e such that $y = \{e\}(x)$. Whether or not an arbitrary z belongs to y can be decided by first computing $\{e\}(x)$ and then checking whether or not z belongs to $\{e\}(x)$. The second step is legitimate according to the schemes of E-recursion .

B is said to be *E-recursively enumerable in* x if there exists an e such that

$$B = \{z \mid \{e\}(z, x)\downarrow\}.$$

From now on the prefix E will often be dropped.

The enumeration scheme yields the usual Kleene recursion theorems and thereby the method of defining partial recursive functions by effective transfinite recursion.

Let $<_R$ be a well-founded relation and I a recursive function from ω into ω. Suppose for all $e < \omega$ and all x in the field of $<_R$,

$$\{I(e)\}(x)\downarrow$$

whenever $\{e\}(y)\downarrow$ for all $y <_R x$. Then there exists a partial E-recursive function $\{c\}$ such that $\{c\}(x)\downarrow$ for all x in the field of $<_R$ and $\{I(c)\} \simeq \{c\}$.

The *universal computation tree* $>_U$ is defined by reference to the schemes of E-recursion. Each element of the field of $>_U$ is of the form $\langle e, x\rangle$ where $e < \omega$ and $x \in V$. $a >_U b$ is read: b is a *subcomputation* of a, and is defined by: there exists a finite sequence $z_0, \ldots, z_n$ such that $a = z_0$, $z_n = b$, and z_{i+1} is an *immediate subcomputation* of z_i $(i < n)$. The definition of immediate subcomputation has only two interesting clauses. They correspond to the effective bounding scheme of Section 1.

(1) $\langle m, x\rangle$ is an immediate subcomputation of $\langle 2^m \cdot 3^n, x\rangle$.

(2) If $\{m\}(x)$ is defined, and $y \in \{m\}(x)$, then $\langle n, y\rangle$ is an immediate subcomputation of $\langle 2^m \cdot 3^n, x\rangle$.

The remaining clauses, save one, correspond to the finitary schemes and so give rise to finitary branching in $>_U$. The final clause states: if c is not the Godel number of a scheme, then $\langle c, x\rangle$ is an immediate subcomputation of $\langle c, x\rangle$.

It is readily verified that the relation "b is an immediate subcomputation of a" is recursively enumerable, and that $\{e\}(x)\downarrow$ iff $>_U$ below $\langle e, x\rangle$ is well founded. The method of definition by effective transfinite recursion yields a partial recursive function f such that for all e and x, if $\{e\}(x)\downarrow$, then $f(e, x)$ is the set theoretic object consisting of $>_U$ below $\langle e, x\rangle$. Thus an E-closed set includes not only the end values of computations but also the computations themselves. (If $\{e\}(x)\downarrow$, then it seems appropriate to call $>_U$ below $\langle e, x\rangle$ the computation for $\{e\}(x)$.)

Similarly there exists a partial recursive function $|\{e\}(x)|$ such that if $\{e\}(x)\downarrow$, then $|\{e\}(x)|$ is the ordinal height of $>_U$ below $\langle e, x\rangle$. For example, $|\{2^m \cdot 3^n\}(a)| = \sigma$ if

(1) $(E\gamma)_{\gamma<\sigma}[|\{m\}(a)| = \gamma]$,

(2) $(x)(E\tau)_{\tau<\sigma}[x \in \{m\}(a) \rightarrow |\{n\}(x)| = \tau]$, and

(3) $(\tau)_{\tau<\sigma}(Ex)[(x \in \{m\}(a) \,\&\, |\{n\}(x)| \geqslant \tau) \vee |\{m\}(a)| = \tau]$.

An ordinal δ is said to be recursive in x if there exists a c such that $\delta = \{c\}(x)$. Thus $|\{e\}(x)|$ is recursive in x if $\{e\}(x)\downarrow$.

An essential result in E-recursion theory is the Gandy Selection Theorem [19]: There exists a partial recursive function $\phi(e, x)$ such that for all $e < \omega$,

$$(En)_{n<\omega}[\{e\}(n, x)\downarrow] \leftrightarrow \phi(e, x)\downarrow \,\&\, \{e\}(\phi(e, x), x)\downarrow.$$

Suppose $P(x, y)$ is a recursively enumerable predicate and

$$(x)(Ey)[y \text{ is recursive in } x \,\&\, P(x, y)].$$

Then by Gandy selection there exists a recursive function f such that $(x)\, P(x, f(x))$.

An E-closed set A admits *Moschovakis witnesses* if for all $e < \omega$ and $x \in A$: if $\{e\}(x)\uparrow$, then there is an infinite descending path through $>_U$ below $\langle e, x\rangle$ in A.

Three more definitions are needed for the main lemma (2.3) on the existence of Moschovakis witnesses in E-closed sets.

κ_0^x is the supremum of all ordinals recursive in x.

κ^x is the supremum of all ordinals γ such that γ is recursive in a, x, where $a \in \mathrm{TC}(x)$.

If $\{e\}(x)\downarrow$ and $|\{e\}(x)| = \delta$, then the computation tree for $\{e\}(x)$ is first-order definable over

$$L(\delta, \{x\} \cup \mathrm{TC}(x)).$$

Consequently each set recursive in x belongs to $L(\kappa_0^x, \{x\} \cup \mathrm{TC}(x))$, and

$$E(x) = L(\kappa^x, \{x\} \cup \mathrm{TC}(x)).$$

An ordinal γ is said to be *x-reflecting* if for every $\sum_1$ formula ϕ,

$$L(\gamma, \{x\} \cup \mathrm{TC}(x)) \models \phi(x)$$

iff

$$L(\kappa_0^x, \{x\} \cup \mathrm{TC}(x)) \models \phi(x).$$

κ_r^x is the greatest x-reflecting ordinal. To appreciate the definition of κ_r^x, let $D(u, v)$ be a Δ_0 formula (with free variables u and v), and suppose

$$L(\kappa_r^x, \{x\} \cup \mathrm{TC}(x)) \models (Ev)\, D(x, v).$$

Then

$$L(\delta, \{x\} \cup \mathrm{TC}(x)) \models (Ev)\, D(x, v),$$

for some δ recursive in x. Suppose further that there is a well ordering $>$ of $\mathrm{TC}(x)$ recursive in x. Then the presence of an existential witness for D below κ_r^x implies there is a witness recursive in x, namely the least one relative to the well ordering of $L(\delta, \{x\} \cup \mathrm{TC}(x))$ extending $>$.

LEMMA 2.1. *Suppose A is recursive in x and $\mathscr{B}$ is co-recursively enumerable in x. If $A \cap \mathscr{B}$ is nonempty, then there exists a $c \in A \cap \mathscr{B}$ such that $\kappa_r^x \geqslant \kappa_r^{x,c}$ (Harrington–Kechris basis theorem [11]).*

Proof. First, observe that $\kappa_r^{x,c} > \kappa_r^x$ implies $\kappa_0^{x,c} > \kappa_r^x$. Otherwise $\sum_1$ statements about x would reflect from $\kappa_r^{x,c}$ down to κ_r^x. Thus it suffices to find a $c \in A \cap B$ such that $\kappa_r^x \geqslant \kappa_0^{x,c}$. Now suppose no such c exists. Consider the following recursive (in x) enumeration of A:

(I) If $c \in A - \mathscr{B}$, then enumerate c.

(II) If $c \in A$ and $\kappa_r^x < \kappa_0^{x,c}$, then enumerate c.

(Note that κ_r^x can be effectively defined from x and any ordinal greater than κ_r^x. Thus (II) applies when there is an ordinal recursive in x, c and greater than κ_r^x.)

By effective bounding there is a γ recursive in x that bounds the height of all computations occurring in the above enumeration of A. Since $\gamma < \kappa_0^x \leqslant \kappa_r^x$, it follows that (II) never applies. Hence, every $c \in A$ is enumerated according (I) and so $A \cap \mathscr{B}$ is empty. ∎

LEMMA 2.1a. *Suppose there is a well ordering of* $\mathrm{TC}(x)$ *which is recursive in $\langle x, z\rangle$. Then there is a well ordering of $\{y \mid y$ is an immediate subcomputation of $\langle e, x\rangle\}$ which is (uniformly) recursive in $\langle x, z\rangle$. Moreover, if y is an immediate subcomputation of x then there is a well ordering of* $\mathrm{TC}(y)$ *which is (uniformly) recursive in $\langle x, y, z\rangle$.*

Proof. Assume $e=2^m\cdot 3^n$. If the only immediate subcomputation of $\langle e, x\rangle$ is $\langle m, x\rangle$ then the lemma is immediate. Suppose then that $\{m\}(x)\downarrow$. The value of $\{m\}$ at x and the entire computation of $\{m\}(x)$ from x is contained in $L(|\{m\}(x)|, \{x\}\cup \mathrm{TC}(x))$, which is recursive in x. The $\langle x, z\rangle$-recursive well-ordering of $\mathrm{TC}(x)$ induces a well ordering W of $L(|\{m\}(x)|, \{x\}\cup \mathrm{TC}(x))$. W restricted to $\{m\}(x)$ induces a well ordering of the immediate subcomputations of $\langle e, x\rangle$.

If $\langle e', y\rangle$ is an immediate subcomputation of x and $\langle e', y\rangle$ is not equal to $\langle m, x\rangle$ then $e'=n$ and $y\in\{m\}(x)$. So $\mathrm{TC}(y)$ is a subset of $L(|\{m\}(x)|, \{x\}\cup \mathrm{TC}(x))$ since this set is transitive and contains $\{m\}(x)$. W restricted to the elements of $\mathrm{TC}(y)$ is a well ordering of $\mathrm{TC}(y)$ and is recursive in $\langle x, y, z\rangle$. ∎

LEMMA 2.2. *Suppose some well ordering of* $\mathrm{TC}(x)$ *is recursive in* x. *If* $\{e\}(x)\uparrow$, *then a Moschovakis witness for* $\{e\}(x)$ *belongs to* $L(\kappa_r^x+1, \{x\}\cup \mathrm{TC}(x))$.

Proof. Related to arguments of Moschovakis [10] and Harrington [11]. The witness $\lambda t\,|\,\langle e_t, x_t\rangle$ is defined by recursion on t. $\langle e_0, x_0\rangle$ is $\langle e, x\rangle$. Assume that $\langle e_t, x_t\rangle$ has been defined so that

(a) $\{e_t\}(x_t)\uparrow$,

(b) $x_t\in L(\kappa_r^x, \{x\}\cup \mathrm{TC}(x))$,

(c) $\kappa_r^x \geqslant \kappa_r^{x_0,\dots,x_t}$,

(d) there is a well ordering of $\mathrm{TC}(x_t)$ uniformly recursive in $\langle x_0,\dots, x_t\rangle$.

For simplicity assume $e_t=2^m\cdot 3^n$. By the inductive assumption (d) and Lemma 2.1a let $<_w$ be recursive in $\langle x_0,\dots, x_t\rangle$ and well order $\{z\,|\,z$ is an immediate subcomputation of $\langle e_t, x_t\rangle\}$.

If $\{m\}(x_t)\uparrow$ let $\langle e_{t+1}, x_{t+1}\rangle = \langle m, x_t\rangle$. Assumption (c) on x_t implies that $\kappa_r^x\geqslant\kappa_0^{x_t}$. Consequently, $\{m\}(x_t)\downarrow$ if and only if $|\{m\}(x_t)|<\kappa_r^x$. Thus one can look down from κ_r^x and see whether or not $\{m\}(x_t)\uparrow$. The inductive hypothesis are immediate in this case.

Suppose $\{m\}(x_t)\downarrow$. Then define $\langle e_{t+1}, x_{t+1}\rangle$ to be $\langle n, y\rangle$ where $\langle n, y\rangle$ is the $<_w$ least pair $\langle n, u\rangle$ so that

$$u\in\{m\}(x_t)\ \&\ |\{n\}(n)|\geqslant\kappa_r^x.$$

Since $|\{m\}(x_t)|<\kappa_r^{x_0,\dots,x_t}\leqslant\kappa_r^x$, $y\in L(\kappa_r^x+1, \{x\}\cup \mathrm{TC}(x))$. It has to be shown that

$$\{n\}(y)\uparrow\ \&\ \kappa_r^{x_0,\dots,x_t}\geqslant\kappa_r^{x_0,\dots,x_t,y}.$$

By Lemma 2.1 there is a $z \in \{m\}(x_t)$ such that

$$\{n\}(z)\uparrow \ \& \ \kappa_r^{x_0,\dots,x_t} \geqslant \kappa_r^{x_0,\dots,x_t,z}.$$

Let $\langle n, z_0 \rangle$ be the $<_w$ least pair so that z_0 is such a z. It suffices to show that $y = z_0$. Clearly, $\langle n, y \rangle \leqslant_w \langle n, z_0 \rangle$ since $|\{n\}(z_0)| = \infty > \kappa_r^x$. Suppose $\langle n, y \rangle <_w \langle n, z_0 \rangle$. Consider (as in the proof of Lemma 2.1) the following recursive (in $x_0,\dots, x_t, z_0$) enumeration of $W \upharpoonright \langle n, z_0 \rangle = \langle \{n, z\rangle \mid \langle n, z \rangle <_w \langle n, z_0 \rangle\}$. By induction, $W \upharpoonright \langle n, z_0 \rangle$ is recursive in $x_0,\dots, x_t, z_0$.

(I) If $\{n\}(z)\downarrow$, then enumerate z.

(II) If $\kappa_r^{x_0,\dots,x_t} < \kappa_r^{x_0,\dots,x_t,z}$, then enumerate z.

(Note that $\kappa_r^{x_0,\dots,x_t} < \kappa_r^{x_0,\dots,x_t,z}$ implies that $\kappa_r^{x_0,\dots,x_t} < \kappa_0^{x_0,\dots,x_t,z}$.)

By effective bounding there is a γ recursive in $x_0,\dots, x_t, z_0$ that bounds the height of all computations occuring in the above enumeration. It follows that (II) never applies, since otherwise

$$\kappa_r^{x_0,\dots,x_t} \leqslant \gamma < \kappa_0^{x_0,\dots,x_t,z_0} \leqslant \kappa_r^{x_0,\dots,x_t}.$$

Hence (I) applies to every z so that $\langle n, z \rangle <_w \langle n, z_0 \rangle$. In particular, if $y \neq z_0$ then $|\{n\}(y)| \leqslant \gamma$. But

$$|\{n\}(y)| \geqslant \kappa_r^x \geqslant \kappa_r^{x_0,\dots,x_t,z_0} > \gamma$$

according to assumption (c). Thus $z_0 = y$.

Finally to complete the induction, Lemma 2.2 uniformly provides a well ordering of TC(y) which is recursive in $x_0,\dots, x_t, y$. ▮

LEMMA 2.3. *Suppose $L(\kappa)$ is E-closed but not $\sum_1$ admissible. Then $\kappa_r^x < \kappa$ for all $x \in L(\kappa)$.*

Proof. Let $\kappa_r^x \geqslant \kappa$ for some fixed x in $L(\kappa)$ with the object of showing that $L(\kappa)$ is $\sum_1$ admissible.

Since any element of $L(\kappa)$ can be well ordered in $L(\kappa)$,

$$L(\kappa) \models (EW)[W \text{ is a well ordering of } \mathrm{TC}(x)].$$

By assumption $\kappa_r^x \geqslant \kappa$ so there is a γ which is recursive in x such that

$$L(\gamma) \models (EW)[W \text{ is a well ordering of } \mathrm{TC}(x)].$$

Thus the L-least well-ordering of W_x of TC(x) is recursive in x.

To prove that $L(\kappa)$ satisfies $\sum_1$-bounding, suppose

$$L(\kappa) \models (u)_{u \in \mathbf{d}}(Ev)\, \mathscr{F}(u, v),$$

where $\mathscr{F}$ is Δ_0 with parameter p. Let γ be less than κ so that there is a well ordering $W_{p,d}$ of $\mathrm{TC}(\langle p, d\rangle)$ which is recursive in p, d, γ. Then there is a well ordering of $\mathrm{TC}(\langle p, d, \gamma\rangle)$ which is recursive in p, d, γ. To simplify the discussion, the parameters p, d, γ together are referred to as p^*.

For each b in d

$$\kappa_r^{x, p^*, b} \geqslant \kappa_r^x.$$

This last follows from (1)–(3).

(1) Let θ^z be the least ordinal, if there is one, needed to construct (from z) Moschovakis witnesses for $\{e\}(z)$ for all e such that $\{e\}(z)\uparrow$. Then $\theta^z \geqslant \kappa_r^z$. If not, then, by reflection, θ^z would be recursive in z; this would make the complete recursively enumerable in z subset of ω recursive, and absurdity.

(2) $\theta^{\langle x, p^*\rangle, b} \leqslant \kappa_r^{\langle x, p^*\rangle, b}$ for each b in d by Lemma 2.2. (The parameter γ was introduced to provide the well ordering needed to apply Lemma 2.2.)

(3) For any u and v, $\theta^u \leqslant \theta^{u,v}$, since a Moschovakis witness for a divergent u computation $\{e\}(u)$ can be construed as a Moschovakis witness for a divergent $\langle u, v\rangle$-computation $\{e^*\}(u, v)$.

Thus $\kappa_r^{\langle x, p^*\rangle, b} \geqslant \theta^{\langle x, p^*\rangle, b} \geqslant \theta^x \geqslant \kappa_r^x \geqslant \kappa$. Hence for each $b \in d$, there is a c such that

$$L(\kappa) \models \mathscr{F}(\mathbf{b}, \mathbf{c})$$

and c is recursive (by reflection) in x, p^*, b. The set of all e such that

$$\{e\}(x, p^*, b)\downarrow \quad \text{and} \quad L(\kappa) \models \mathscr{F}(\mathbf{b}, \{e\}(x, p^*, b))$$

is recursively enumerable in x, p^*, b. By Gandy selection (Sect. 2) one such e can be chosen by an effective method which is uniform in x, p^*, b. It follows there is a partial recursive function f such that for each $b \in d$,

$$f(x, p^*, b)\downarrow \quad \text{and} \quad L(\kappa) \models \mathscr{F}(b, f(x, p^*, b)).$$

By effective bounding $\{f(x, p^*, b) \mid b \in d\}$ is recursive in x, p^*, d and so must be an element of $L(\kappa)$ since $L(\kappa)$ is E-closed. ▮

LEMMA 2.5 [9]. *If $L(\kappa)$ is E-closed but not $\sum_1$ admissible, then $L(\kappa)$ admits Moschovakis witnesses.*

Proof. For $z \in L(\kappa)$ let γ be the least ordinal so that there is a well ordering of $\mathrm{TC}(z)$ which is recursive in z, γ. By Lemma 2.4, $\kappa_r^{z,\gamma} < \kappa$. By

Lemma 2.3, if $\{e\}(z, \gamma)$ is divergent, then there is a Moschovakis witness to this effect in $L(\kappa_r^{z,\gamma}+1, \mathrm{TC}(z \cup \gamma) \cup \{z, y\})$. In particular, the relevant Moschovakis witnesses for z are elements of $L(\kappa_r^{z,\gamma}+1, \mathrm{TC}(z \cup \gamma) \cup \{z, \gamma\})$ which is a subset of $L(\kappa)$.

PROPOSITION 2.5 (well known). *Suppose $L(\kappa)$ is E-closed but not $\sum_1$ admissible. Then*

$$L(\kappa) \models [\textit{There exists a greatest cardinal.}]$$

Proof. Let γ be a cardinal in the sense of $L(\kappa)$. A standard downward Lowenheim–Skolem argument shows that $L(\gamma)$ is a $\sum_1$ substructure of $L(\kappa)$. If f is $\sum_1$ over $L(\kappa)$ and its domain and defining parameters belong to $L(\gamma)$, then its range belongs to $L(\gamma)$. ∎

The greatest cardinal of $L(\kappa)$, when it exists, is denoted by $\mathrm{gc}(\kappa)$.

3. Generic Extensions of *E*-Closed Sets

Let A be an E-closed set, and G a subset of some element of A. Define $A(G)$ to be the set of all z such that for some e and a:

$$z = \{e\}(a, G), \qquad a \in A \quad \text{and} \quad |\{e\}(a, G)| \in A.$$

Suppose $\mathscr{P}$ is a partial ordering of forcing conditions such that $\mathscr{P} \in A$ and is countable closed in A. In this section, it will be shown that $A(G)$ is E-closed, if G is $\mathscr{P}$-generic over A, and A admits Moschovakis witnesses.

The forcing language $\mathscr{L}_A$ has the power to describe the computation of $\{e\}(a, G)$ when a and $|\{e\}(a, G)|$ belong to A. The primitive terms of $\mathscr{L}_A$ are $\mathbf{a}$ $(a \in A)$ and $\mathscr{G}$. A general term is $\{e\}(t_1, ..., t_n)$, where $t_1, ..., t_n$ are terms. Each term t is equivalent to one of so-called normal form $\{e\}(\mathbf{a}, \mathscr{G})$. A unique normal form for t can be found effectively from t by elementary syntatical manipulations. $\mathscr{L}_A$ also includes among its primitive symbols: σ $(\sigma \in A)$, $\in$, $|\ |$, variables, quantifiers and propositional connectives.

Let $\mathscr{P} = \langle P, \geqslant \rangle \in A$ be a partial ordering of forcing conditions $p, q, r, ...$; $p \geqslant q$ means p is extended by q (i.e., p says less than q about G).

The forcing relation $p \Vdash |\{e\}(\mathbf{a}, \mathscr{G})| = \sigma$ and the set $\mathscr{T}(p, e, a, \mathscr{G}, \sigma)$ of terms are defined simultaneously by recursion on $\sigma \in A$. Among the elements of $\mathscr{T}(p, e, a, \mathscr{G}, \sigma)$ are terms that name the elements of $\{e\}(a, \mathscr{G})$ when $\mathscr{G}$ is generic, $p \in G$ and p forces $|\{e\}(\mathbf{a}, \mathscr{G})|$ to be σ. Each major clause of the recursion corresponds to one of the Normann schemes, as in the definition of $<_U$, the universal computation tree of Section 2. $\mathscr{T}(p, e, a, \mathscr{G}, \sigma)$ shall be used to represent a set of terms and also as a term itself. The effective bounding scheme is handled as follows.

Define $p \Vdash |\{2^m \cdot 3^n\}(a, \mathscr{G})| = \sigma$ by:

$$(E\gamma)_{\gamma<\sigma}[p \Vdash |\{m\}(a, \mathscr{G})| = \gamma];$$

$$p \Vdash (u)(E\tau)_{\tau<\sigma}[u \in \mathscr{T}(p, m, a, \mathscr{G}, \gamma)$$
$$\& \; u \in \{m\}(a, \mathscr{G}) \to |\{n\}(u)| = \tau];$$

$$p \Vdash (\tau)_{\tau<\sigma}(Eu)[(u \in \mathscr{T}(p, m, a, \mathscr{G}, \gamma)$$
$$\& \; u \in \{m\}(a, \mathscr{G}) \; \& \; |\{n\}(u)| \geqslant \tau) \vee |\{m\}(a, \mathscr{G})| = \tau].$$

Define $\mathscr{T}(p, 2^m \cdot 3^n, a, \mathscr{G}, \sigma)$ to be

$$\{\{n\}(s) \mid s \in \mathscr{T}(p, m, a, \mathscr{G}, \gamma)\}.$$

The quantifiers and propositional connectives are forced in standard fashion. Thus p forces $(Ex)\,\mathscr{F}(x)$ if p forces $\mathscr{F}(i)$ for some appropriate i, p forces $(x)\,\mathscr{F}(x)$ if p forces $\sim(Ex)\sim\mathscr{F}(x)$, and p forces $\sim\mathscr{F}$ if $\mathscr{F}$ is not forced by any $q \leqslant p$. One detail above is the definition of

$$q \Vdash s \in \{m\}(a, \mathscr{G})$$

when $p \leqslant q$ and $s \in \mathscr{T}(p, m, a, \mathscr{G}, \gamma)$. It is filled in by adding one more step to the recursion. Define

$$q \Vdash \{n\}(s) \in \{2^m \cdot 3^n\}(a, \mathscr{G})$$

to be $q \Vdash \{s \in \{m\}(a, \mathscr{G}) \; \& \; (E\tau)_{\tau<\sigma}(|\{n\}(s)| = \tau)]$.

The *weak forcing relation*, $p \Vdash {}^*F$, is given by $p \Vdash \sim\sim\mathscr{F}$.

$p \Vdash |\{e\}(t)| = \sigma$ means $p \Vdash |\{e_t\}(a_t, \mathscr{G})| = \sigma$, where $\{e_t\}(a_t, \mathscr{G})$ is the normal form of $\{e\}(t)$. Similarly

$$\mathscr{T}(p, e, t, \sigma) = \mathscr{T}(p, e_t, a_t, \mathscr{G}, \sigma).$$

LEMMA 3.1. *$p \Vdash |\{e\}(t)| = \sigma$, $\mathscr{T}(p, e, t, \sigma)$ and $q \Vdash s \in \{e\}(t)$ (where $p \geqslant q$ and $s \in \mathscr{T}(p, e, t, \sigma)$) are recursive in σ, $\mathscr{P}$ (uniformly in σ).*

Proof. Be effective transfinite recursion on σ. ∎

The relation $>_V$ is needed for the development of an effective bound on $|\{e\}(t)|$ when p weakly forces $|\{e\}(t)|$ into A. An approximate version of

$$\langle p, e, t_1 \rangle >_V \langle q, n, t_2 \rangle$$

is simply $p \leqslant q$ and $q \Vdash {}^*\langle e, t_1\rangle >_U \langle n, t_2\rangle$. U is the universal computation tree of Section 2. A typical detail of the precise definition of $>_V$ is

$$q \Vdash \langle 2^m \cdot 3^n, t_1\rangle >_U \langle n, t_2\rangle$$

if there is a $\gamma \in A$ such that $q \Vdash |\{m\}(t_1)| = \gamma$,

$$t_2 \in \mathscr{T}(q, m, t_1, \gamma) \qquad \text{and} \qquad q \Vdash t_2 \in \{m\}(t_1).$$

$>_V$ is the forcing counterpart of $>_U$. It includes all possible values of $>_U$ for all generic G's. It will be seen that $p \Vdash$ "b is an immediate subcomputation of a" is a recursively enumerable relation on appropriate triples $\langle p, a, b\rangle$;

$$p \Vdash {}^*(E\sigma)[|\{e\}(t)| = \sigma],$$

as a relation on p, e, and t, is recursively enumerable in $\mathscr{P}$; and that $>_V$ is well founded below $\langle p, e, t\rangle$ when p weakly forces $|\{e\}(t)|$ into A. These three facts are interwoven in an essential way with the existence of an effective bound on $|\{e\}(t)|$.

LEMMA 3.2. *Suppose $>_V$ is well founded below $\langle p, e, t\rangle$. If*

$$p \Vdash {}^*(E\sigma)[|\{e\}(t)| = \sigma],$$

then there exists a γ recursive in p, t, $\mathscr{P}$ such that

$$p \Vdash {}^*|\{e\}(t)| \leqslant \gamma.$$

Proof. γ is computed by an effective transfinite recursion on the height of $\langle p, e, t\rangle$ in $>_V$. The case of maximum interest occurs when $e = 2^m \cdot 3^n$. By recursion there is a δ recursive in p, t, $\mathscr{P}$ such that

$$p \Vdash {}^*|\{m\}(t)| \leqslant \delta.$$

By Lemma 3.1 the set K, defined by

$$\langle p_0, \sigma\rangle \in K \leftrightarrow p \geqslant p_0 \ \&\ p_0 \Vdash |\{m\}(t)| = \sigma,$$

is recursive in p, t, $\mathscr{P}$. Assume $\langle p_0, \sigma\rangle \in K$ and $s \in \mathscr{T}(p_0, m, t, \sigma)$. Then

$$p_0 \Vdash {}^*(E\beta)[s \in \{m\}(t) \rightarrow |\{n\}(s)| = \beta].$$

Suppose $p_0 \geqslant q$. Then there is an r such that $q \geqslant r$ and either

$$r \Vdash s \in \{m\}(t) \qquad \text{or} \qquad r \Vdash s \notin \{m\}(t).$$

By Lemma 3.1 the set of all $r \leqslant q$ that force $s \in \{m\}(t)$ is recursive in $q, s, t, \sigma, \mathscr{P}$. For each such r there is by recursion a ρ recursive in $r, s, \mathscr{P}$ such that

$$r \Vdash^* |\{n\}(s)| \leqslant \rho.$$

The effective bounding priciple and Gandy selection yield a strict upper bound γ on ρ (for *all* relevant r and s) such that γ is recursive in $p, t, \mathscr{P}$. It is safe to assume $\delta < \gamma$. ∎

LEMMA 3.3. *Assume A admits Moschovakis witnesses and $\mathscr{P}$ is countably closed in A. If*

$$p \Vdash^* (E\sigma)[|\{e\}(t)| = \sigma],$$

then $>_V$ is well founded below $\langle p, e, t \rangle$.

Proof. Let $\{g\}(p, e, t)$ $(=\gamma)$ be the partial recursive function defined in the proof of Lemma 3.2. If $\{g\}$ converges on $\langle p, e, t\rangle$ then $>_V$ is well founded below $\langle p, e, t\rangle$. This is so because the value of $\{g\}$ on any $w \leqslant_V \langle p, e, t\rangle$ is a strict upper bound for the values of $\{g\}$ on the predecessors of w in $>_V$. Thus g assigns rank.

Suppose $\{g\}(p, e, t)\uparrow$. Let $z \in A$ be a Moschovakis witness for $\langle g, \langle p, e, t\rangle\rangle$. Thus $z_0 = \langle g, \langle p, e, t\rangle\rangle$, and $z_n >_U z_{n+1}$ for all n. There must be a z_m such that

$$z_m = \langle g, \langle q, n, s\rangle\rangle \qquad \text{and} \qquad \langle p, e, t\rangle >_V \langle q, n, s\rangle.$$

The existence of such *a* z_m is a consequence of the details of the definition of $\{g\}$ by effective transfinite recursion on $>_V$. The immediate predecessors of $\langle g, \langle p, e, t\rangle\rangle$ in $>_U$ need not have the form of z_m, but every Moschovakis witness, as it winds its way down below $\langle g, \langle p, e, t\rangle\rangle$, must encounter elements of the form z_m infinitely often. Thus each Moschovakis witness to the divergence of $\{g\}(p, e, t)$ has a built-in, infinite, descending sequence in $>_V$ below $\langle p, e, t\rangle$. It follows there is a sequence $\langle p_n, e_n, t_n\rangle$ $(n \geqslant 0)$ in A such that

$$\langle p_0, e_0, t_0\rangle = \langle p, e, t\rangle$$

and

$$\langle p_n, e_n, t_n\rangle >_V \langle p_{n+1}, e_{n+1}, t_{n+1}\rangle.$$

Since $\mathscr{P}$ is countably closed in A, there is a q such that for all n, $p_n \geqslant q$. But then

$$q \Vdash^* (n)[\langle e_n, t_n\rangle >_U \langle e_{n+1}, t_{n+1}\rangle].$$

Thus q weakly forces the existence of an Moschovakis witness to the divergence of $\{e\}(t)$, and so p $(\geqslant q)$ cannot weakly force $\{e\}(t)$ to convergence. ▮

LEMMA 3.4. *Assume A admits Moschovakis witnesses, and $\mathscr{P}$ is countably closed in A. If*

$$p \Vdash {}^*(E\sigma)[|\{e\}(t)| = \sigma],$$

then there exists a γ recursive in p, t, $\mathscr{P}$ such that

$$p \Vdash {}^*|\{e\}(t)| \leqslant \gamma.$$

Proof. Combine Lemmas 3.2 and 3.3. ▮

Let G be a path through $\mathscr{P}$. G is $\mathscr{P}$-generic with respect to a sentence $\mathscr{F}$ if there is a $p \in G$ such that either $p \Vdash \mathscr{F}$ or $p \Vdash \sim \mathscr{F}$. G is *$\mathscr{P}$-generic over A* if:

(i) G is generic with respect to every sentence of the form

$$|\{e\}(t)| = \sigma \qquad \text{or} \qquad (E\sigma)[|\{e\}(t)| = \sigma].$$

(ii) Suppose there is a $p \in G$ such that $p \Vdash |\{m\}(t)| = \sigma$. Then G is generic with respect to

$$(u)(E\tau)[u \in \mathscr{T}(p, m, t, \sigma) \\ \& \ u \in \{m\}(t) \rightarrow |\{n\}(u)| = \tau],$$

and to every sentence of the form $s \in \{m\}(t)$ for all $s \in \mathscr{T}(p, m, t, \sigma)$.

THEOREM 3.5. *Suppose A is E-closed and admits Moschovakis witnesses. Let $\mathscr{P} \in A$ be countably closed in A. If G is $\mathscr{P}$-generic over A, then $A(G)$ is E-closed.*

Proof. Let $a \in A$. Suppose $|\{e\}(t)| \leqslant \kappa$, where κ is the least ordinal not in A, with the intent of showing $|\{e\}(t)| < \kappa$. As usual assume $e = 2^m \cdot 3^n$. Then there are $p \in G$ and $\sigma < \kappa$ such that

$$p \Vdash |\{m\}(t)| = \sigma$$

and

$$p \Vdash (u)(E\tau)[u \in \mathscr{T}(p, m, t, \sigma) \\ \& \ u \in \{m\}(t) \rightarrow |\{n\}(u)| = \tau].$$

For each $s \in \mathscr{T}(p, m, t, \sigma)$ and $q \leqslant p$, there is an $r \leqslant q$ such that either

(i) $r \Vdash s \in \{m\}(t)$ or

(ii) $r \Vdash s \notin \{m\}(t)$.

The set of all $r \leqslant q$ such that (1) holds is recursive in $q, s, t\ \sigma, \mathscr{P}$ by Lemma 3.1. If r satisfies (i) then

$$r \Vdash {}^*|\{n\}(s)| \leqslant \rho$$

for some ρ recursive in $r, s, \mathscr{P}$ by Lemma 3.4. The effective bounding principle, as in proof of Lemma 3.2, yields a bound γ on ρ (for *all* relevant r and s) such that γ is recursive in $p, t, \mathscr{P}$. Since A is E-closed, $\gamma \in A$. Thus p forces $|\{e\}(t)|$ to be at worst $\max(\sigma, \gamma)$. ∎

The proof of Theorem 3.5 shows: if p forces $|\{e\}(a, \mathscr{G})| \leqslant \kappa$, then p forces $|\{e\}(a, \mathscr{G})|$ to be less than $\kappa_0^{p, a, \mathscr{P}}$. Further information can be obtained if A is of the form $L(\kappa)$. For example, for each $a \in L(\kappa)$ and p, there must be a $q \geqslant p$ such that q forces $K_r^{a, \mathscr{G}} \leqslant \kappa_r^{p, a, \mathscr{P}}$. Such matters are discussed in [9] and [13].

The present section concludes with an application of Theorem 3.5 needed for the main result of the next section. Recall Proposition 2.5. The next result states that under suitable hypotheses, the greatest cardinal of an E-closed structure can be collapsed to ω_1.

CORROLARY 3.6. *Assume $L(\kappa)$ is countable, E-closed and $\sum_1$ inadmissible. Suppose* $\mathrm{gc}(\kappa)$, *the greatest cardinal in the sense of $L(\kappa)$, has uncountable cofinality in $L(\kappa)$. Then there exists an onto*

$$f\colon\ \omega_1^{L(\kappa)} \to \mathrm{gc}(\kappa)$$

such that $L(\kappa, f)$ is E-closed, its greatest cardinal is $\omega_1^{L(\kappa)}$, and $\omega_1^{L(\kappa)} = \omega_1^{L(\kappa, f)}$.

Proof. Let $\mathscr{P}$ be the set of all $p \in L(\kappa)$ of the form

$$p\colon\ \delta \to \mathrm{gc}(\kappa) \qquad (\delta < \omega_1^{L(\kappa)}).$$

$\mathscr{P}$ is an element of $L(\kappa)$ by a standard condensation argument, and $\mathscr{P}$ is countably closed in $L(\kappa)$, since $\mathrm{cf}(\mathrm{gc}(\kappa)) > \omega$ in $L(\kappa)$. By Lemma 2.4; $L(\kappa)$ admits Moschovakis witnesses. Let

$$f\colon \omega_1^{L(\kappa)} \to \mathrm{gc}(\kappa)$$

be $\mathscr{P}$-generic. Then f is an onto map, and by Theorem 3.5, $L(\kappa, f)$ is E-closed. $\omega_1^{L(\kappa)} = \omega_1^{L(\kappa, f)}$ because $\mathscr{P}$ is countably closed. ∎

4. E-POINTED PERFECT FORCING

Let $L(\kappa)$ be countable, E-closed and $\sum_1$ inadmissible. By Proposition 2.9, $L(\kappa)$ has a greatest cardinal, namely $\mathrm{gc}(\kappa)$. Assume $\mathrm{gc}(\kappa)$ has uncountable cofinality in $L(\kappa)$. In this section it will be shown that there exists a $T \subseteq \mathrm{gc}(\kappa)$ such that $L(\kappa, T)$ is the E-closure of T. T will be generic with respect to uncountable, E-pointed, perfect forcing conditions. The ideas underlying the proof of Theorem 3.5 will guide the proof that $L(\kappa, T)$ is E-closed. Some modifications are necessary because the collection of forcing conditions is now a "class" rather than a "set." Pointed, perfect forcing was applied in [7]. Iterated perfect forcing was developed by Baumgartner and Laver [14] for countable conditions, and by Kanamori [15] for uncountable conditions.

By Corollary 3.6 there is an f such that

$$L(\kappa, f) \models [\text{There is only one uncountable cardinal.}]$$

and $L(\kappa, f)$ is E-closed. The forcing conditions will belong to $L(\kappa, f)$, and the desired T will be generic over $L(\kappa, f)$, and the desired T will be generic over $L(\kappa, f)$. Let ω_1 denote the greatest cardinal of $L(\kappa, f)$. Since f can be construed as a subset of ω_1, it follows that each bounded subset of ω_1 (in $L(\kappa, f)$) is constructible from a countable (in $L(\kappa, f)$) initial segment of f via an ordinal less than ω_1. Thus Seq, defined by

$$s \in \mathrm{Seq} \leftrightarrow s\colon \alpha \to \{0, 1\} \;\&\; \alpha < \omega_1 \;\&\; s \in L(\kappa, f),$$

is a member of $L(\kappa, f)$.

Suppose $p \subseteq \mathrm{Seq}$. If $s \in p$, then s is said to *split* in p if $s * \langle 0 \rangle$ and $s * \langle 1 \rangle$ belong to p. (If $\mathrm{domain}(s) = \alpha$, then $\mathrm{domain}(s * \langle 0 \rangle) = \alpha + 1$ and $(s * \langle 0 \rangle)(\alpha) = 0$.) p is *perfect* if:

(a) $(s)_{s \in p}(\alpha)_{\alpha \in \mathrm{dom}(s)}[(s \restriction \alpha) \in p]$.

(b) $(s)_{s \in p}(Et)_{t \in p}[s \subseteq t \;\&\; t \text{ splits in } p]$.

(c) $(\alpha)_{\alpha < \omega_1}(s)[(\mathrm{domain}(s) = \alpha \;\&\; \alpha \text{ is a limit } \&\; (\beta)_{\beta < \alpha}((s \restriction \beta) \in p)) \to s \in p]$.

(d) $(f)_{f \in p^\omega}[(m)(n)(n < m \to f(n) \subseteq f(m)) \;\&\; (n)(f(n) \text{ splits in } p) \to \bigcup_n f(n) \text{ splits in } p]$.

Clause (d) says that p splits on a *closed* unbounded subset of each branch through p. A branch is a function $g\colon \omega_1 \to 2$ such that every countable initial segment of g belongs to p. Let $[p]$ be the collection of all branches through p.

p is *E-pointed* if

$$(E\delta)_{\delta < \omega_1}(T)[T \in [p] \to p \leqslant_E T, \delta, f].$$

In short P can be computed from any branch through p.

Let P be the class of all E-pointed, perfect p's in $L(\kappa, f)$. Note that $P \notin L(\kappa, f)$. Let $p, q, r, \ldots,$ denote elements of P. $p \geqslant q$ (read p is extended by q) if $p \supseteq q$. Define $\mathscr{P}$ to be $\langle P, \geqslant \rangle$.

PROPOSITION 4.1. *$\mathscr{P}$ is countably closed in $L(\kappa, f)$.*

Proof. Let p_n $(n < \omega)$ be an infinite descending sequence in $\mathscr{P}$, and let q be $\bigcap \{p_n \mid n < \omega\}$. q satisfies clause (d) of the definition of perfect, because the filter of closed unbounded subsets of ω_1 is countably closed. To see that q is E-pointed, observe that each countable sequence of countable ordinals is E-recursive in some countable ordinal. Thus the δ_n's needed to compute the p_n's from a common branch T can be combined into a single δ that serves to compute q from T.

Suppose $X, Y \subseteq \omega_1$. X and Y are said to have the same *degree* if there exist $\alpha, \beta < \omega_1$ such that

$$X \leqslant_E Y, \quad \alpha, f \qquad \text{and} \qquad Y \leqslant_E X, \beta, f.$$

PROPOSITION 4.2. *The degrees of the branches through p are just the degrees greater than or equal to that of p.*

Proof. Suppose $p \leqslant_E X, \alpha, f$ for some $\sigma < \omega_1$. A $T \in [p]$ is defined by recursion on ω_1. If $T \restriction \gamma$ does not split in p, then $T(\gamma) = 0$ if $T \restriction \gamma * \langle 0 \rangle \in p$, and $= 1$ otherwise. Assume $T \restriction \gamma$ splits in p. Let β be the order type of

$$\{\delta \mid \delta < \gamma \;\&\; T \restriction \delta \text{ splits in } p\}.$$

Then

$$T(\gamma) = \begin{cases} 0 & \text{if } \beta \in X \\ 1 & \text{otherwise.} \end{cases}$$

T winds its way through P in the same manner that X does through 2^{ω_1}. Consequently

$$T \leqslant_E p, X \qquad \text{and} \qquad X \leqslant_E p, T.$$

The initial assumption on X implies the degree of T is at most that of X. The E-pointedness of p implies the degree of X is at most that of T. ▌

In the light of Proposition 4.2, an E-pointed p represents a cone of degrees whose vertex is the degree of p. If $T \in [p]$, then p "forces" the degree of T to be at least that of p without "forcing" any bound on the degree of T.

p is said to be *E-pointed via δ* $(\delta < \omega_1)$ if

$$(T)[T \in [p] \rightarrow p \leqslant_E T, \delta, f].$$

Proposition 4.3. *Suppose $Y \subseteq \omega_1$ and p is E-pointed via δ. Then there exists a q such that $p \geqslant q$, $q \leqslant_E p, Y$, and $Y \leqslant_E q, f, \delta$.*

Proof. $\jmath \in q$ is defined by recursion on the domain of $\jmath$ with the intent of coding Y into every branch of $[q]$.

Suppose $\jmath \in q$ and $\jmath$ does not split in p. Then $\jmath * \langle 0 \rangle \in q$ if $\jmath * \langle 0 \rangle \in p$. Otherwise $\jmath * \langle 1 \rangle \in q$.

Suppose $\jmath$ splits in p. Let τ be the order type of

$$\{\gamma \mid \gamma < \text{domain}(\jmath) \;\&\; \jmath \restriction \gamma \text{ splits in } p\}.$$

If $\tau = \beta + 1$, then

$$\jmath * \langle 0 \rangle \in q \quad \text{if} \quad \beta \in Y,$$

$$\jmath * \langle 1 \rangle \in q \quad \text{if} \quad \beta \in Y.$$

If τ is a limit, then $\jmath$ splits in q.

By construction $p \leqslant q$. To see q is E-pointed, fix $T \in [q]$. Then $T \in [p]$, and so $p \leqslant_E T, f, \delta$. By construction $Y \leqslant_E T, p$. Thus $Y \leqslant_E T, f, \delta$. $q \leqslant_E p, Y$, so $q \leqslant_E T, f, \delta$.

The left-most branch of $[q]$ is recursive in q, hence $Y \leqslant_E q, f, \delta$. ▮

The language $\mathscr{L}(\kappa, f, \mathscr{T})$ has the power to analyze the computation of $\{e\}(x)$ when $x \in \mathscr{L}(\kappa, f, T)$ and $|\{e\}(x)| < \kappa$. The primitive terms are: $\mathscr{T}, f, a$ for each $a \in L(\kappa)$, and σ for each ordinal $\sigma < \kappa$. If $t_1, \ldots, t_n$ are terms, then $\{e\}(t_1, \ldots, t_n)$ is a term. The language also includes: $\in$, $|\;|$, variables, quantifiers and propositional connectives. The relations

$$L(\kappa, f, T) \models |\{e\}(t)| = \sigma,$$

$s \in \mathscr{T}(e, t, \sigma)$ and

$$L(\kappa, f, T) \models s \in \{e\}(t)$$

are defined simultaneously by recursion on σ. $\mathscr{T}(e, t, \sigma)$ is a set of terms that suffice to name the elements of $\{e\}(t)$ when $|\{e\}(t)| = \sigma$. For example, suppose $e = 2^m \cdot 3^n$.

Then $L(\kappa, f, T) \models |\{2^m \cdot 3^n\}(t)| = \sigma$ if there is a $\gamma < \sigma$ such that the following are true in $L(\kappa, f, T)$:

$$|\{m\}(t)| = \gamma,$$

$$(u)(E\tau)_{\tau < \sigma}[u \in \mathscr{T}(m, t, \gamma) \;\&\; u \in \{m\}(t) \rightarrow |\{n\}(u)| = \tau],$$

$$(\tau)_{\tau < \sigma}(Eu)[(u \in \mathscr{T}(m, t, \gamma) \;\&\; u \in \{m\}(t) \;\&\; |\{n\}(u)| \geqslant \tau) \vee |\{m\}(t)| = \tau].$$

$$\mathscr{T}(2^m \cdot 3^n, t, \sigma) = \{\{n\}(s) \mid (E\gamma)_{\gamma < \sigma}(s \in \mathscr{T}(m, t, \gamma))\};$$

and

$$L(\kappa, \not f, T) \models \{n\}(s) \in \{2^m \cdot 3^n\}(t)$$

if $s \in \mathcal{T}(m, t, \sigma)$ and $L(\kappa, f, T) \models s \in \{m\}(t)$.

A sentence $\mathcal{F}$ is said to be *ranked* (and of rank at most σ) if it is of the form $|\{e\}(t)| = \sigma$, or $s \in \{e\}(t)$, where $s \in \mathcal{T}(e, t, \sigma)$. The *forcing relation* $p \Vdash \mathcal{F}$ *is defined by*

$$(T)[T \in [p] \rightarrow L(\kappa, f, T) \models \mathcal{F}]$$

when $\mathcal{F}$ is ranked, and in a standard fashion when $\mathcal{F}$ is unranked (e.g., has an unbounded ordinal quantifier).

The weak forcing relation $p \Vdash {}^*\mathcal{F}$ is given by $(q)_{p \geqslant q}(Er)_{q \geqslant r}[r \Vdash \mathcal{F}]$.

The reason for defining forcing for ranked sentences in terms of truth is to avoid quantification over a class of forcing conditions. The next lemma shows that the definition of $\Vdash$ is workable.

LEMMA 4.4. *Suppose $\mathcal{F}$ has rank at most σ. Let W be a well ordering of ω_1 of height σ. Then for each p, there is a q such that $p \leqslant q$, either $q \Vdash \mathcal{F}$ or $q \Vdash \sim\mathcal{F}$, and $q \leqslant_E p$, W, $\ulcorner \mathcal{F} \urcorner$ (uniformly).*

Proof. By Proposition 4.3 there is an r such that $p \geqslant r$, $r \leqslant_E W, \ulcorner \mathcal{F} \urcorner$ and $\langle W, \ulcorner \mathcal{F} \urcorner \rangle \leqslant_E r, f, \delta_0$ for some $\delta_0 < \omega_1$. q will be constructed by local forcing. Each branch through q will be generic in the sense of $\Vdash^r_\sigma$. The forcing conditions associated with $\Vdash^r_\sigma$ are the elements of r. The only sentences eligible for forcing are those of rank at most σ from the language $\mathcal{L}(\kappa, \not f, \mathcal{T})$. The definition of $\Vdash^r_\sigma$ is in essence the same as that of $\Vdash$, given at the beginning of Section 3. The only changes are the restrictions to r and σ.

Let $W^0 = \{\mathcal{F}_\sigma \mid \delta < \omega_1\}$ be a well ordering of all sentences of $\mathcal{L}(\kappa, \not f, \mathcal{T})$ that arise in the unraveling of the definition of $\Vdash \mathcal{F}$. Note that $W^0 \leqslant_E W, \ulcorner \mathcal{F} \urcorner$. A map h: Seq $\rightarrow r$ is defined by recursion on the length α of $t \in$ Seq. The range of h is the desired q.

$\alpha = 0$: $h(t)$ is the shortest $s \in r$ such that either $s \Vdash^r_\sigma \mathcal{F}$ or $s \Vdash^r_\sigma \sim\mathcal{F}$. (Thus $h(0)$ decides whether q forces $\mathcal{F}$ or $\sim\mathcal{F}$.)

$\alpha = \beta + 1$, where β is not a limit: let $\langle s_0, s_1 \rangle$ be the least pair of incomparable extensions of $h(t \restriction (\beta + 1))$ in r such that for each $i < 2$, either

$$s_i \Vdash^r_\sigma \mathcal{F}_\beta \qquad \text{or} \qquad s_i \Vdash^r_\sigma \sim \mathcal{F}_\beta.$$

Define $h((t \restriction (\beta + 1))^* \langle i \rangle) = s_i$.

$$\alpha = \lambda: \qquad h(t) = \bigcup \{h(t \restriction \gamma) \mid \gamma < \lambda\},$$

$$\alpha = \lambda + 1: \qquad h(t \restriction \lambda)^* \langle i \rangle = h(t \restriction \lambda)^* \langle i \rangle.$$

(Note that $h(t \upharpoonright \lambda)$ must split in r, because $h(t \upharpoonright (\beta+1])$ splits when β is not a limit.)

With the aid of Lemma 3.1, $q \leqslant_E r$, W, $\ulcorner \mathscr{F} \urcorner$, hence $q \leqslant_E p$, W, $\ulcorner \mathscr{F} \urcorner$ (uniformly). q is perfect because it is a homeomorphic image of Seq. To see q is E-pointed, fix $T \in [q]$. Then $T \in [p]$, and so $p \leqslant_E$, T, f, δ_1 for some $\delta_1 < \omega_1$. Thus $q \leqslant_E T, f, \delta_1, r, \delta_0$. But $T \in [r]$, so $r \leqslant_E T, f, \delta_2$ for some $\delta_2 < \omega_1$. Consequently $q \leqslant_E T, f, \delta_0, \delta_1, \delta_2$. ∎

As in Section 3 the relation $>_V$ is invoked to obtain an effective bound on $|\{e\}(t)|$ when p weakly forces $|\{e\}(t)| < \kappa$. The proof of the next lemma is similar to that of Lemma 3.4 save for a fusion construction needed to overcome the failure of $\mathscr{P}$ to be an element of $L(\kappa)$.

LEMMA 4.5. *Assume* $\ulcorner t \urcorner \subseteq \omega_1$. *Suppose* $>_V$ *is well founded below* $\langle p, e, t \rangle$. *If*

$$p \Vdash^* (E\sigma)[|\{e\}(t)| = \sigma],$$

then there exists $\langle q, \gamma, W(\gamma) \rangle$ *recursive in* $p, \ulcorner t \urcorner, f$ *such that* $p \geqslant q$,

$$q \Vdash |\{e\}(t)| = \gamma,$$

and $W(\gamma)$ *is a well ordering of* ω_1 *of height* γ.

Proof. $\langle q, \gamma, W(\gamma) \rangle$ is computed by an effective transfinite recursion on the height of $\langle p, e, t \rangle$ in $>_V$. As always assume $e = 2^m \cdot 3^n$. By recursion there exists $\langle p_0, \gamma_0, W(\gamma_0) \rangle$ recursive in p, $\ulcorner t \urcorner$, f such that $p \geqslant p_0$,

$$p_0 \Vdash |\{m\}(t)| = \gamma_0,$$

and $W(\gamma_0)$ is a well ordering of ω_1 of height γ_0. By Proposition 4.3 there is a p_1 such that $p_0 \geqslant p_1$, $p_1 \leqslant_E p_0$, $W(\gamma_0)$, $\ulcorner t \urcorner$, and $\langle W(\gamma_0), \ulcorner t \urcorner \rangle \leqslant_E p_1, f, \delta_0$ for some $\delta_0 < \omega_1$.

Let $W^0(\gamma_0) = \{t_\alpha \mid \alpha < \omega_1\}$ be a well ordering of all terms that belong to $\mathscr{T}(m, t, \gamma_0)$. A straightforward definition of $W^0(\gamma_0)$ yields: $W^0(\gamma_0) \leqslant_E W(\gamma_0)$, $\ulcorner t \urcorner$; and $\ulcorner t_\alpha \urcorner \subseteq \omega$, for all $\alpha < \omega_1$.

A contracting sequence $\{q_\alpha \mid \alpha < \omega_1\}$ of forcing conditions is defined by effective transfinite recursion on α.

$$q_0 = p_1, \qquad q_\lambda = \bigcap \{q_\alpha \mid \alpha < \lambda\},$$

$$q_{\alpha+1} = \bigcup \{q^s_{\alpha+1} \mid s \in q_\alpha \cap 2^{\alpha+1}\}.$$

The definition of $q^s_{\alpha+1}$ has two parts. First, by Lemma 4.4, there is an r_α such that:

$$q_\alpha \geqslant r_\alpha, \qquad r_\alpha \cap 2^{\alpha+1} = \{s\};$$

$$r_\alpha \leqslant_E q_\alpha, s, W(\gamma_0), \ulcorner t \urcorner, \ulcorner t_\alpha \urcorner;$$

and

(i) $r_\alpha \Vdash t_\alpha \in \{m\}(t)$ or

(ii) $r_\alpha \Vdash t_\alpha \notin \{m\}(t)$.

If (ii) holds, then $q^s_{\alpha+1} = r_\alpha$. If (i) holds, then

$$\langle p, e, t\rangle >_V \langle r_\alpha, n, t_\alpha\rangle;$$

and so by recursion on $>_V$ there is $\langle q^s_{\alpha+1}, \gamma^s_{\alpha+1}, W(\gamma_{\alpha+1})\rangle$ recursive in r_α, $\ulcorner t_\alpha \urcorner$, f (uniformly by Gandy selection) such that $r_\alpha \geqslant q^s_{\alpha+1}$,

$$q^s_{\alpha+1} \Vdash |\{n\}(t_\alpha)| = \gamma^s_{\alpha+1},$$

and $W(\gamma_{\alpha+1})$ is a well ordering of height $\gamma_{\alpha+1}$. Define

$$q_\infty = \bigcap\{q_\alpha \mid \alpha < \omega_1\}$$

and

$$\begin{aligned}\gamma_\infty = {} & \text{strict upper bound of} \\ & \{\gamma_0, \gamma^s_{\alpha+1} \mid \alpha < \omega_1 \;\&\; s \in q_\alpha \cap 2^{\alpha+1}\}.\end{aligned}$$

Then $q_\infty \leqslant p_1$, $W(\gamma_0)$, $\ulcorner t \urcorner$, $W^0(\gamma_0) \leqslant_E p$, $\ulcorner t \urcorner$, f. To see q_∞ is E-pointed, fix $T \in [q_\infty]$. Then $T \in p_1$, hence $p_1 \leqslant_E$, T, f, δ_1 for some countable δ_1 not depending on T. Thus $q_\infty \leqslant T, f, \delta_1, \delta_0$. To check that $[q_\infty]$ is perfect, let

$$C_\alpha = \{\beta \mid T \restriction \beta \text{ splits in } q_\alpha\}$$

for all $\alpha \leqslant \omega_1$. C_α $(\alpha \leqslant \omega_1)$ is closed and unbounded, hence $C_{\omega_1} = \bigcap\{C_\alpha \mid \alpha < \omega_1\}$ is closed. To verify that C_{ω_1} is unbounded, fix $\delta_0 < \omega_1$. Choose $\delta_{n+1} \in C_{\delta_n}$ so that $\delta_{n+1} > \delta_n$. Let $\delta_\omega = \bigcup\{\delta_n \mid n < \omega\}$. Then

$$\delta_\omega \in \bigcap\{C_\beta \mid \beta \in \delta_\omega\} = C_{\delta_\omega},$$

and so $T \restriction \delta_\omega$ splits in C_{δ_ω}. For all $\alpha \geqslant \delta_\omega$,

$$q_\alpha \cap 2^{\delta_\omega+1} = q_{\delta_\omega} \cap 2^{\delta_\omega+1}.$$

Hence $\delta_\omega \in C$.

Since $q_\infty \Vdash |\{e\}(t)| \leqslant \gamma_\infty$, it follows from a local forcing construction as in Lemma 4.4 that there exists a q such that $q_\infty \geqslant q$,

$$q \Vdash (E\beta)_{\beta \leqslant \gamma_0}[|\{e\}(t)| = \beta],$$

and $q \leqslant_E q_\infty$, $W(\gamma_\infty)$, $\ulcorner t \urcorner$. The desired γ is the definite value of $|\{e\}(t)|$ forced by q. $W(\gamma_\infty)$ can be obtained effectively from the $W(\gamma_{\alpha+1})$'s. ∎

A closer examination of Lemmas 3.2 and 4.5 will clarify the proof of Lemma 4.6. In both 3.2 and 4.5 a partial recursive function $\{g\}$ was defined by effective transfinite recursion. It was shown by induction that if $<_V$ is well founded below $\langle p, e, t\rangle$ then $\{g\}(p, e, t)\downarrow$ and its value includes an ordinal γ and a condition $q \leqslant p$ such that q forces $|\{e\}(t)|$ to be γ.

There is a second viewpoint. The partial recursive function $\{g\}$ is defined formally via a fixed point construction without reference to $<_V$. If $\{g\}(p, e, t)\downarrow$, then $<_U$ below $\langle g, \langle p, e, t\rangle\rangle$ is a well-founded relation closely associated with $<_V$ below $\langle p, e, t\rangle$. The arguments of 3.2 and 4.5 applied to $<_U$ below $\langle g, \langle p, e, t\rangle\rangle$ show that the value of $\{g\}(p, e, t)$ includes a γ and a q as above.

The second approach yields no new information concerning set forcing, because in that case $<_U$ below $\langle g, e, t\rangle\rangle$ is essentially the same as $<_V$ below $\langle p, e, t\rangle$. It is in the context of class forcing that further insight is gained. If $\mathscr{P}$ is a class, then the immediate predecessors of $\langle p, e, t\rangle$ in $<_V$ form a class rather than a set, and this is so even if $<_V$ below $\langle p, e, t\rangle$ is well founded with height less than κ. The fusion construction of Lemma 4.5 supplies a q, and a set version of $<_V$ below $\langle q, e, t\rangle$ extracted from $<_U$ below $\langle g, \langle p, e, t\rangle\rangle$. The set version of $<_V$ below $\langle q, e, t\rangle$ is dense in the full class version of $<_V$ below $\langle q, e, t\rangle$.

According to the second viewpoint the two cases that occur in the analysis of $\langle p, e, t\rangle$ are $\{g\}(p, e, t)$ converges or diverges, rather than $<_V$ is well founded or ill founded below $\langle p, e, t\rangle$.

LEMMA 4.6. *If* $\ulcorner t \urcorner \subseteq \omega_1$ *and*

$$p \Vdash^* (E\sigma)[|\{e\}(t)| = \sigma],$$

then $\{g\}(p, e, t)\downarrow$.

Proof. Virtually identical with that of Lemma 3.3. If $\{g\}(p, e, t)\uparrow$, then there is a Moschovakis witness to the divergence of $\{g\}(p, e, t)$ in $L(\kappa, f)$ by Lemma 2.4. The witness yields an infinite descending sequence in $<_V$ below $\langle p, e, t\rangle$. Since $\mathscr{P}$ is countably closed by Lemma 4.1, the infinite descending sequence contracts to a q below p that forces $\{e\}(t)$ to diverge. ∎

LEMMA 4.7. *If* $\ulcorner t \urcorner \subseteq \omega_1$ *and*

$$p \Vdash^* (E\sigma)[|\{e\}(t)| = \sigma],$$

then there exist $\langle q, \gamma, W(\gamma)\rangle$ *recursive in* $p, \ulcorner t \urcorner f$ *such that* $p \geqslant q$,

$$q \Vdash |\{e\}(t)| = \gamma,$$

and $W(\gamma)$ *is a wellordering of* ω_1 *of height* γ.

Proof. By Lemmas 4.5 and 4.6, and the intervening remarks. ∎

Let $T \subseteq \omega_1$. T is $\mathscr{P}$-generic on a sentence $\mathscr{F}$ of $\mathscr{L}(\kappa, f, \mathscr{T})$ if there is a p such that $T \in [p]$, and either $p \Vdash \mathscr{F}$ or $p \Vdash \sim \mathscr{F}$. T is $\mathscr{P}$-generic if:

(i) T is $\mathscr{P}$-generic on every $\mathscr{F}$ that is ranked or of the form $(E\sigma)[|\{e\}(t)| = \sigma]$ and

(ii) if $T \in [p]$ and $p \Vdash |\{m\}(t)| = \sigma$, then there is a $q \leqslant p$ such that $T \in [p]$ and q forces the following sentence or its negation,

$$(u)(E\tau)[u \in \mathscr{T}(m, t, \sigma) \;\&\; u \in \{m\}(t) \to |\{n\}(u)| = \tau].$$

THEOREM 4.8. *Let* $L(\kappa)$ *be countable, E-closed and* $\sum_1$ *inadmissible. Suppose the greatest cardinal of* $L(\kappa)$ *has uncountable cofinality in* $L(\kappa)$. *Then there exists an* $S \subseteq \omega_1^{L(\kappa)}$ *such that* $L(\kappa, S)$ *is the E-closure of* S.

Proof. Let T be $\mathscr{P}$-generic as above. Then S is $\langle f, T\rangle$ encoded as a subset of $\omega_1^{L(\kappa)}$. By Proposition 4.3 and the genericity of T, each $Y \in 2^{\omega_1} \cap L(\kappa, f)$ is recursive in T, f, δ for some countable δ. Hence $E(S) \supseteq L(\kappa, f, T)$. To check $E(S) \subseteq L(\kappa, f, T)$, suppose $\ulcorner t \urcorner \leqslant \omega_1$ and

$$L(\kappa, f, T) \models |\{2^m \cdot 3^n\}(t)| \leqslant \kappa.$$

Then there is a p such that $T \in [p]$, $p \Vdash (E\sigma)[|\{m\}(t)| = \sigma]$, and

$$p \Vdash (u)(E\tau)[u \in \mathscr{T}(m, t, \sigma) \;\&\; u \in \{m\}(t)$$
$$\to |\{n\}(u)| = \tau].$$

By Lemma 4.7 there is $\langle p_0, \gamma_0, W(\gamma_0)\rangle$ recursive in $p, \ulcorner t \urcorner, f$ such that $p \geqslant p_0$,

$$p_0 \Vdash |\{m\}(t)| = \gamma_0,$$

and $W(\gamma_0)$ is a wellordering of ω_1 of height γ_0. Now the fusion argument of Lemma 4.5 can be repeated to obtained q_∞ and γ_∞ such that $p_0 \geqslant q_\infty$,

$$q_\infty \Vdash^* |\{2^m \cdot 3^n\}(t)| \leqslant \gamma_\infty,$$

and $\gamma_\infty \leqslant_E p, \ulcorner t \urcorner, f$. Thus q_∞ weakly forces $|\{2^m \cdot 3^n\}(t)|$ to be less than κ. The set of such q_∞'s is dense in p, and so p weakly forces $(E\sigma)[|\{2^m \cdot 3^n\}(t)| = \sigma]$. (Note that p does not in general force a universal bound on $|\{2^m \cdot 3^n\}(t)|$ for all generic elements of $[p]$. ∎

5. Selection

In this section a selection theorem is proved to obtain a partial converse to Theorem 4.8. The theorem is inspired by a result of Y. Moschovakis [18] (also cf. Kirousis [20]). Let $E(R(\alpha))$ be the E-closure of the set of all sets of rank less than α. Moschovakis showed: if α has countable cofinality in $E(R(\alpha))$, then $E(R(\alpha))$ is $\sum_1$ admissible. Corollary 5.2 states: let X be a set of ordinals; if in $E(X)$ the greatest cardinal has cofinality ω, then $E(X)$ is $\sum_1$ admissible. Corollary 5.3 is the intended partial converse to Theorem 4.8. Let X be a set of ordinals. Some fundamental facts concerning the structure of $E(X)$ will prove helpful below.

(R0) Let $a, b, c, \ldots, \in \sup X$. For all z, $z \in E(X)$ iff $z \leqslant_E X, a$ for some a.

(R1) Each element of $E(X)$ can be coded by a subset of $\sup X$. More precisely, if $z \in E(X)$, then there exists a $z^* \subseteq \sup X$ such that z is recursive in z^*, and $z^* \in E(X)$. z^* is defined by a recursion on the length of computation of z from x.

(R2) In $E(X)$ there is a greatest cardinal.

Call it $\mathrm{gc}(E(X))$. It is safe to assume $\mathrm{gc}(E(X))$ is $\sup X$. More precisely, there exists an $X_0 \subseteq \sup X$ such that $E(X_0) = E(X)$ and $\mathrm{gc}(E(X_0)) = \sup X_0$.

Let $A \subseteq \sup X$. Recall from Section 2 that A is said to be recursively enumerable in X if there exists an e such that for all $a \in \sup X$,

$$a \in A \leftrightarrow \{e\}(a, X)\downarrow.$$

Each a corresponds to a node $\langle e, \langle a, X \rangle \rangle$ on the universal computation tree $>_U$ defined in Section 2. a belongs to A iff $>_U$ is well founded beneath $\langle e, \langle a, X \rangle \rangle$. For simplicity the node $\langle e, \langle a, X \rangle \rangle$ will be written a. Thus $a \in A$ iff $a\downarrow$. Fact (R1) makes it possible to think of each node below $\langle e, \langle a, X \rangle \rangle$ as being of the form $\langle e_0, \langle b, X \rangle \rangle$ for some $b \in \sup X$, or more simply b. Define

$$\min A = \min\{|a| \mid a \in A\}.$$

$|a|$ is the height of $>_U$ beneath a. $|a|$ is defined by recursion to be the strict least upper bound ($\sup^+$) of all $|b|$ such that b is an immediate subcomputation of a (as defined in Sect. 2).

Theorem 5.1. *Let X be a set of ordinals. Suppose in $E(X)$ there is an ascending sequence $\{\kappa_j \mid j < \omega\}$ of cardinals (in the sense of $E(X)$) such that*

$$\sup X = \mathrm{gc}(E(X)) = \sup\{\kappa_j \mid j < \omega\}.$$

If $A \subseteq \sup X$ is nonempty and recursively enumerable in X, then min A *is recursive in X, $\{\kappa_j \mid j < \omega\}$ (uniformly).*

Proof. The predicate, b is an immediate subcomputation of a, is recursively enumerable. The following modification, b is an immediate computation of a via β, is recursive. The idea is to let β bound the height of the computation needed to show b is an immediate subcomputation of a. For example:

(1_β) $\langle m, u\rangle$ is an immediate subcomputation of $\langle 2^m \cdot 3^n, u\rangle$ via β.

(2_β) If $\{m\}(u)\downarrow$, $|\{m\}(u)| \leqslant \beta$, and $v \in \{m\}(u)$, then $\langle n, v\rangle$ is an immediate subcomputation of $\langle 2^m \cdot 3^n, u\rangle$ via β.

The theorem is proved by an effective transfinite recursion on min A, henceforth called the main recursion. There are countably many cases woven together at the finish by the Gandy selection principle.

Case $2^i \cdot 3^j$. Suppose $i < j$ and there is an $a \in A \cap \kappa_i$ such that

$$\begin{aligned}\min A &= \min(A \cap \kappa_i) = |a| \\ &= \sup{}^+\{|b| \mid b < \kappa_j \;\&\; b \text{ is immediate subcomputation of } a\}.\end{aligned}$$

In this case $\min(A \cap \kappa_i)$ is computed by a recursion of length κ_{j+1}. Fix $\alpha < \kappa_{j+1}$ and assume $\beta(\gamma)$ has been computed for all $\gamma < \alpha$. Define

$$\beta^-(\alpha) = \sup\{\beta(\gamma) \mid \gamma < \alpha\}.$$

The definition of $\beta(\alpha)$ has two subcases. In the first, $\beta^-(\alpha) + 1 \geqslant \min(A \cap \kappa_i)$ and $\beta(\alpha)$ is defined to be $\min(A \cap \kappa_i)$. In the second,

$$\beta^-(\alpha) + 1 < \min(A \cap \kappa_i),$$

and $\beta(\alpha)$ is computed as follows. Let Z_α be the set of all b such that

$b < \kappa_j$, $b\downarrow$, $\beta^-(\alpha) < |b|$ and b is an immediate subcomputation of some $a \in \kappa_i$ via $\beta^-(\alpha)$.

Z_α is nonempty thanks to the a mentioned in the supposition of Case $2^i \cdot 3^j$. Z_α is recursively enumerable in $\beta^-(\alpha)$, X, κ_i, hence the main recursion can be applied to compute min Z_α. Define

$$\beta(\alpha) = \min Z_\alpha.$$

It follows that $\{\beta(\alpha) \mid \alpha < \kappa_{j+1}\}$ is recursive in X and the κ_n sequence. Note that for all sufficiently large α, $\beta(\alpha)$ is $\min(A \cap \kappa_i)$. If not, then for all sufficiently large α, the second subcase holds. For each such α there is a $b \in \kappa_j$ such that

$$\beta(\alpha) = |b| \qquad \text{and} \qquad \beta(\alpha) > \beta(\gamma) \quad \text{for all } \gamma < \alpha.$$

Each such b is associated with at most one α, but there are κ_{j+1} α's.

Case 5^i. Suppose min $A = \min(A \cap \kappa_i)$. Let $\gamma(j, a)$ be the supremum$^+$ of all $|b|$ such that $b < \kappa_j$, and b is an immediate subcomputation of a. $\gamma(j, a)$ is a partial function, and is defined iff every $b < \kappa_j$ (and an immediate subcomputation of a) converges. Suppose further that

$$a \in A \cap \kappa_i \mathbin{\&} |a| = \min A \rightarrow \gamma(j, a) < \min A$$

for all $j < \omega$. In this case $\min(A \cap \kappa_i)$ is computed by a recursion of length κ_{i+1}. Fix $\alpha < \kappa_{i+1}$ and assume $\beta(\gamma)$ has been computed for all $\gamma < \alpha$. Define

$$\beta^-(\alpha) = \sup\{\beta(\gamma) \mid \gamma < \alpha\}.$$

The definition of $\beta(\alpha)$ has two subcases. In the first, $\beta^-(\alpha) + 1 \geqslant \min(A \cap \kappa_i)$ and $\beta(\alpha)$ is $\min(A \cap \kappa_i)$. In the second,

$$\beta^-(\alpha) + 1 < \min(A \cap \kappa_i),$$

and $\beta(\alpha)$ is computed as follows. Let $\gamma(j, a, \beta)$ be the supremum$^+$ of all $|b|$ such that $b < \kappa_j$ and b is an immediate subcomputation of a via b. $\gamma(j, a, \beta)$ is partial recursive in j, a, β, and is defined iff every $b < \kappa_j$ (and an immediate subcomputation of a via β) converges. Let Y_α be the set of $\langle j, a \rangle$ such that

$$j < \omega \mathbin{\&} a \in \kappa_i \mathbin{\&} \beta^-(\alpha) < \gamma(j, a, \beta^-(\alpha)).$$

Y_α is nonempty by virtue of the suppositions that define Case 5^i. Y_α is recursively enumerable in $\beta^-(\alpha)$, X, κ_i, hence the main recursion can be applied to compute

$$\beta(\alpha) = \min\{\gamma(j, a, \beta^-(\alpha)) \mid \langle j, a \rangle \in Y_\alpha\}.$$

It follows that $\{\beta(\alpha) \mid \alpha < \kappa_{i+1}\}$ is recursive in X and the κ_n sequence. For all sufficiently large α, $\beta(\alpha)$ is $\min(A \cap \kappa_i)$. If not, then the second subcase obtains for all sufficiently large α. For each such α there is a $\langle j, a \rangle \in \omega \times \kappa_i$ such that

$$\beta(\alpha) = \gamma(j, a, \beta^-(\alpha)) \qquad \text{and} \qquad \beta(\alpha) > \beta(\gamma) \quad (\gamma < \alpha).$$

Each such $\langle j, a \rangle$ is associated with at most two α's, but there are κ_{i+1} α's. For a given $\langle j, a \rangle$ the first α might occur at β_0 when the only immediate subcomputation of a via β_0 is $\langle m, a, X\rangle$, and the second at β_1 when $\beta_1 > |\langle m, a, X\rangle|$.

An effective procedure has been defined for each case q above. It converges iff the suppositions of the case are true . If it converges, then it converges to min A. Let B be the set of all q such that the procedure for case q converges. Recall the Gandy selection principle from the proof of Lemma 2.3. B is a nonempty subset of ω recursively enumerable in X, $\{\kappa_i \mid i<\omega\}$. By Gandy selection, an element of B can be computed from X, $\{\kappa_i \mid i<\omega\}$. ∎

The proof of Theorem 5.1 establishes more than is stated. $E(X)$ can be replaced by an arbitrary E-closed structure $\mathscr{E}$ with an additional predicate. The only structural fact needed for the proof is somewhat weaker than the existence of a greatest cardinal of cofinality ω in $\mathscr{E}$. As above no assumption need be made about the power set operation inside $\mathscr{E}$ (cf. [13] for details).

COROLLARY 5.2. *Let X be a set of ordinals. If in $E(X)$ the greatest cardinal has cofinality ω, then $E(X)$ is $\sum_1$ admissible.*

Proof. Let $D(a, z)$ be Δ_0. Suppose

$$(b)_{b\in X}(Ez)_{z\in E(X)}D(b, z)$$

in the hope of bounding z. Then

$$(b)_{b\in X}(Ea)_{a\in X}(Ee)[\{e\}(X, a)\downarrow \;\&\; D(b, \{e\}(X, a))].$$

The set

$$\{|\{e\}(X, a)| \mid \{e\}(X, a)\downarrow \;\&\; D(b, \{e\}(X, a))\}$$

is recursively enumerable in X, b. Thus its min is recursive in X, b, $\{\kappa_i \mid i<\omega\}$ by Theorem 5.1. Consequently z is a computable function of b, hence bounded in $E(X)$. ∎

COROLLARY 5.3. *Suppose $L(\kappa)$ is E-closed and not $\sum_1$ admissible. If $\delta<\kappa$, $S\subseteq\delta$ and $E(S)=L(\kappa, S)$, then in $L(\kappa, S)$ the greatest cardinal has uncountable cofinality.*

Proof. Apply Corollary 5.2.

COROLLARY 5.4. *Suppose $L(\kappa)$ is E-closed and not $\sum_1$ admissible. Let $\{\omega_n^{L(\kappa)} \mid n\leqslant\omega\}$ be the set of all infinite cardinals in the sense of $L(\kappa)$. Then there do not exist $\delta<\kappa$ and $S\subseteq\delta$ such that $E(S)=L(\kappa, S)$.*

Proof. Suppose δ and S exist. Let $\mathrm{gc}(S)$ be the greatest cardinal in the sense of $E(S)$. By remark (R2) at the beginning of Section 5, it is safe to assume $S \subseteq \mathrm{gc}(S)$. It follows from Corollary 5.3 that $\mathrm{gc}(S) = \omega_n^{L(\kappa)}$ for some positive $n < \omega$, and that $\omega_n^{L(\kappa)}$ is regular in $E(S)$. There is a $t \in E(S)$ such that

$$t\colon\ \omega_\omega^{L(\kappa)} \xrightarrow[1\ 1]{\text{onto}} \omega_n^{L(\kappa)}.$$

t will be used to define a violation of the E-closedness of $E(S)$.

Suppose $Z \subseteq \omega_\omega^{L(\kappa)}$ and $Z \in L(\kappa)$. Z can be effectively coded by a countable subset of $\omega_\omega^{L(\kappa)}$ as follows. Let $z(n)$ be the least triple $\langle \gamma, p, n\rangle$ such that $Z \cap \omega_n$ is first order definable over $L(\gamma)$ via formula n with parameter p. A standard downward Skolem–Lowenheim argument shows $\langle \gamma, p, n\rangle \in \omega_{n+1}^{L(\kappa)}$. Thus $\{z(n) \mid n < \omega\}$ encodes Z, and the processes of encoding and decoding are effective.

Let Y be a countable subset of $\omega_\omega^{L(\kappa)}$ in $L(\kappa)$. Then $t(Y)$ is a bounded subset of $\omega_n^{L(\kappa)}$ in $E(S)$, since $\omega_n^{L(\kappa)}$ is uncountable and regular in $E(S)$. By a well-known theorem of Hajnal, $t(Y) \in L(\beta, S)$ for some $\beta < \omega_n^{L(\kappa)}$.

Thus every subset of $\omega_\omega^{L(\kappa)}$ in $L(\kappa)$ can be coded by a countable subset of $\omega_\omega^{L(\kappa)}$ in $E(S)$, which is constructible from S below $\omega_n^{L(\kappa)}$. This process, reversed and slightly modified, violates the E-closedness of $E(S)$. Let K be the set of all bounded subsets of $\omega_n^{L(\kappa)}$ in $E(S)$, and let $w \in K$. Define $h(w)$ as follows. If $t^{-1}[w]$ encodes a subset of $w_\omega^{L(\kappa)}$ that is the graph of a well-founded relation, then $h(w)$ is the ordinal height of that relation. Otherwise $h(w) = 0$.

h is recursive in S, t, but maps $K \in E(S)$ onto κ. ∎

6. Further Questions and Results

(Q1) Is the converse of Theorem 4.8 true? To make the question definite, let $L(\kappa)$ be countable and $\sum_1$ inadmissible, and assume there exist $\delta < \kappa$ and $S \subseteq \delta$ such that $L(\kappa, S) = E(S)$. Does it follow that $\mathrm{gc}(\kappa)$, the greatest cardinal of $L(\kappa)$, has uncountable cofinality in $L(\kappa)$? Corollary 5.3 requires the greatest cardinal of $L(\kappa, S)$ to have uncountable cofinality in $L(\kappa, S)$. It seems possible for $\mathrm{gc}(\kappa)$ to have countable cofinality in $L(\kappa)$, and to collapse in $L(\kappa, S)$ to a cardinal of uncountable cofinality.

(Q2) Suppose κ, δ, and S are as in (Q1). Can the degree of S be made minimal? In the present setting "minimal" means:: for all $R \in E(S)$, if $S \notin E(R)$, then $L(\kappa) \nsubseteq E(R)$. An affirmative answer is morally certain via the methods of [7]. There it was shown that if α is countable, greater than ω and $\sum_1$ admissible, then there exists an $S \subseteq \omega$ such that

(i) $E(S) = L(\alpha, S)$ and

(ii) for all $R \in E(S)$, if $S \notin E(R)$, then $L(\alpha) \nsubseteq E(R)$.

(Q3) Suppose $L(\kappa)$ is uncountable, E-closed and $\sum_1$ inadmissible. Assume $\mathrm{gc}(\kappa)$ has uncountable cofinality in $L(\kappa)$. Do there exist $\delta < \kappa$ and $S \subseteq \delta$ such that $L(\kappa, S) = E(S)$? S. Friedman [16, 17] has found a virtually complete answer to this question when $V = L$ and cardinality of $L(\kappa)$ is δ.

The analysis of $E(X)$ made in the various sections of the present paper depends strongly on the assumption that X is a set of ordinals. Slaman [13] has shown: Let $L(\kappa)$ be countable and E-closed. Then there exists a countable $X \subseteq 2^{\omega}$ such that

(a) $L(\kappa, X) = E(X)$;

(b) $L(\kappa, X)$ does not admit Moschovakis witnesses; and

(c) if $L(\kappa)$ is $\sum_1$ admissible but not the E-closure of any $Z \in L(\kappa)$, then $L(\kappa, X)$ is $\sum_1$ admissible but the notions of E-recursive enumerability and boldface $\sum_1$ do not agree on $L(\kappa, X)$.

REFERENCES

1. S. C. KLEENE, Recursive functionals and quantifiers of finite type I, *Trans. Amer. Math. Soc.* **91** (1959), 1–52.
2. S. C. KLEENE, Recursive functionals and quantifiers of finite type II, *Trans. Amer. Math. Soc.* **108** (1963), 106–142.
3. D. NORMANN, "Set Recursion, in Generalized Recursion Theory, II," pp. 303–320, North-Holland, Amsterdam, 1978.
4. G. E. SACKS, Post's problem, absoluteness and recursion in finite types, *in* "The Kleene Symposium," pp. 201–222, North-Holland, Amsterdam, 1980.
5. F. G. ABRAMSON, Sacks forcing does not always produce a minimal upper bound, *Advan. in Math.* **31** (1979), 110–130.
6. J. STEEL, Forcing and tagged trees, *Ann. Math. Logic* **15** (1978).
7. G. E. SACKS, Countable admissible ordinals and hyperdegrees, *Advan. in Math.* **20** (1976), 213–262.
8. G. E. SACKS, Selection and forcing, in preparation.
9. G. E. SACKS, On the limits of recursive enumerability, in preparation.
10. Y. N. MOSCHOVAKIS, Hyperanalytic predicates, *Trans. Amer. Math. Soc.* **138** (1967), 249–282.
11. L. HARRINGTON, "Contributions to Recursion Theory in Higher Types," PhD thesis, M.I.T., Cambridge, Mass., 1973.
12. J. MOLDSTAD, "Computations in Higher Types," Lecture Notes in Math. Vol. 574, Springer-Verlag, Berlin, 1977.
13. T. A. SLAMAN, "Aspects of E-Recursion," PhD thesis, Harvard, Univ. Cambridge, Mass., 1981.

14. J. BAUMGARTNER AND R. LAVER, Iterated perfect-set forcing, *Ann. Math. Logic* **17** (1979), 271–288.
15. A. KANAMORI, Perfect-set forcing for uncountable cardinals, *Ann. Math. Logic* **19** (1980), 97–114.
16. S. FRIEDMAN, Uncountable admissibles. I. Forcing, *Trans. Amer. Math. Soc.*, in press.
17. S. FRIEDMAN, Uncountable admissibles. II. Compactness, *Israel J. Math.*, in press.
18. Y. N. MOSCHOVAKIS, On the Grilliot–Harrington–MacQueen theorem, *in* "Logic Year 1979–80," Lecture Notes in Math., Vol. 859, pp. 246–267, Springer-Verlag, Berlin, 1981.
19. J. E. FENSTAD, "General Recursion Theory," Springer-Verlag, Berlin, 1980.
20. L. KIROUSIS, A selection theorem (abstract), *J. Symbolic Logic* **46** (1981), 38.