Subforests of bipartite digraphs — the minimum degree condition

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Abstract

Let $D$ be a bipartite digraph and let $F$ be an oriented forest of size $k - 1$. We consider two conditions on the minimum indegree and the minimum outdegree of the digraph $D$ guaranteeing that $D$ contains $F$. These conditions extend older results concerning oriented trees of size $k - 1$.

We consider only finite graphs and finite digraphs, without loops and multiple edges and arcs. Our terminology and notation are standard unless otherwise stated.

Let $G$ be a graph with vertex set $V(G)$. The minimum degree of $G$ is denoted by $\delta(G)$. A forest is an acyclic graph and a tree is a connected forest.

Let $D = (X; Y; A(D))$ be a bipartite digraph with vertex set $V(D)$ and arc set $A(D)$ such that $V(D) = X \cup Y$, $X \cap Y = \emptyset$ and $A(D) \subset (X \times Y \cup Y \times X)$. For any vertex $v \in V(D)$, by $d_D^+(v), d_D^-(v)$ we denote outdegree or indegree (respectively) of the vertex $v$ in the digraph $D$. Set $N_D^+(v) = \{w \in V(D): (v, w) \in A(D)\}$ and $N_D^-(v) = \{w \in V(D): (w, v) \in A(D)\}$ for $v \in V(D)$. For any subset $W$ of $V(D)$ we define $\delta_W^+(D)$ ($\delta_W^-(D)$) to be $\min_{v \in W} \{d_D^+(v)\}$ ($\min_{v \in W} \{d_D^-(v)\}$) and $\Delta_W^+(D)$ ($\Delta_W^-(D)$) to be $\max_{v \in W} \{d_D^+(v)\}$ ($\max_{v \in W} \{d_D^-(v)\}$). An oriented graph is a digraph without symmetric arcs. An oriented forest is an oriented graph such that the graph obtained from it by replacing arcs by edges is a forest. An oriented tree is an oriented graph such that the graph obtained from it by replacing arcs by edges is a tree. By a leaf of an oriented forest we mean a vertex incident with only one arc of the forest. An oriented tree is an oriented graph in which the graph obtained from it by replacing arcs by edges is a tree. By a leaf of an oriented tree we mean a vertex incident with only one arc of the tree. An oriented tree $T = (U, W; A(T))$ is balanced if $|U| = |W|$.

Observation. Let $T = (U, W; A(T))$ be a balanced oriented tree. Then there is at least one leaf $l_u \in U$ and there is at least one leaf $l_w \in W$.

Proof. Let $T = (U, W; A(T))$ be an oriented tree such that $|U| = |W|$. If there are no leaves in $U$, then every vertex of $U$ is incident with at least two arcs. Hence the oriented tree $T$ has at least $2|U| = |V(T)|$ arcs, a contradiction. □
Definition 1. Let $T$ be an oriented tree. We say that $T$ is leaves-uniform if either for every leaf $l$ of $T$ $d^+_T(l) = 1$ or for every leaf $l$ of $T$ $d^-_T(l) = 1$.

For short we will use the following terminology. Let $D_1$ and $D_2$ be two digraphs and let $V_1 \subset V(D_1)$ and $V_2 \subset V(D_2)$. We will call an injection $f : V(D_1) \to V(D_2)$ an embedding of $D_1$ in $D_2$ if $(v, w) \in A(D_1)$ implies $(f(v), f(w)) \in A(D_2)$ for $v, w \in V(D_1)$. The existence of such an embedding means that $D_2$ contains $D_1$ as a subdigraph. If for an embedding $f$ of $D_1$ in $D_2$, $(V_1) \subset V_2$ we will write that $D_2$ contains $D_1$ such that $V_1 \subset V_2$. We will say that $D_1$ is a subdigraph of $D_2$ indicated by $W \subset V(D_2)$ if $V(D_1) = W$ and $A(D_1) = (W \times W) \cap A(D_2)$.

Several sufficient conditions on the minimum indegree and the minimum outdegree of a bipartite digraph, involving the existence of paths of various lengths are given in [1]. Sufficient conditions on the minimum indegree and the minimum outdegree of an oriented graph for the existence of long paths are considered in [6].

It is a well-known fact that [3]:

Lemma 1. Any graph $G$ with minimum degree at least $k-1$ contains every tree with $k-1$ edges.

This result was extended by Brandt [2] for forests:

Theorem 1. Suppose $F$ is a forest with $k-1$ edges, $G$ is a graph with $\delta(G) \geq k-1$ and $|V(F)| \leq |V(G)|$. Then $G$ contains $F$.

Similar problem, but for bipartite graphs and for forests understood as bipartite graphs with fixed numbers of vertices in two sets, was considered by Orchel [7].

In [8] we studied the analogue of Lemma 1 for bipartite digraphs and we obtained:

Theorem 2. Given $k > 0$, let $D = (X, Y; A(D))$ be a bipartite digraph with $\delta^-_X(D) \geq k-1$, $\delta^-_Y(D) \geq k-1$ and $\delta^+_Y(D) \geq \lfloor k/2 \rfloor$, $\delta^+_Y(D) \geq \lceil k/2 \rceil$. Then $D$ contains every oriented tree of order $k$.

Theorem 3. Given $k > 0$, let $D = (X, Y; A(D))$ be a bipartite digraph with $\delta^+_X(D) \geq k-1$, $\delta^+_X(D) \geq \lfloor (k-1)/2 \rfloor$ and $\delta^+_Y(D) \geq \lfloor (k-1)/2 \rfloor$, $\delta^+_Y(D) \geq k-1$. Then $D$ contains every oriented tree of order $k$.

We define four oriented trees on $k$ vertices. Let $S^+$ and $S^-$ be oriented stars (oriented trees such that the graphs obtained from them by replacing arcs by edges are stars) with one vertex $v$ such that $d_{S^+}^-(v) = k-1$ or $d_{S^-}^-(v) = k-1$, respectively. Let $DS^+$ be an oriented tree with two vertices $u, v$ joined by an arc $(u, v)$ such that $d_{DS^+}^-(u) = \lfloor (k-2)/2 \rfloor + 1$ and $d_{DS^+}^+(v) = \lfloor (k-2)/2 \rfloor$. Let $DS^-$ be an oriented tree such that $(x, y)$ is an arc of $DS^-$ if and only if $(y, x)$ is an arc of $DS^+$ (see Fig. 1).

Degree bounds in above theorems are sharp because of $S^+$, $S^-$ and because of $DS^+$, $DS^-$. For Theorem 2, if either $\delta^+_X(D) = \Delta^+_X(D) = k-2$ ($\Delta^+_Y(D) < k-1$) or
\( \delta^+_{Y}(D) = \Delta^+_{Y}(D) = \lfloor k/2 \rfloor - 1 \) then either \( D \) does not contain \( S^+ \) or \( D \) does not contain \( DS^+ \) when \( k \) is odd, respectively. When \( k \) is even, the digraph \( D \) with \( \delta^+_{Y}(D) = \Delta^+_{Y}(D) = k/2 - 1 \) and with only symmetric arcs does not contain \( DS^+ \). For Theorem 3, if either \( \delta^+_{Y}(D) = \Delta^+_{Y}(D) = k - 2 \) \((\Delta^+_{Y}(D) < k - 1)\) or \( \delta^+_{Y}(D) = \Delta^+_{Y}(D) = \lfloor (k - 1)/2 \rfloor - 1 \) then either \( D \) does not contain \( S^+ \) or \( D \) does not contain \( DS^+ \), respectively. The same holds for the minimum indegree for both results.

We now consider oriented forests of size \( k - 1 \), \( k > 0 \) and look for analogous results of those of Theorems 2 and 3, replacing oriented trees by oriented forests. We have to add some assumptions associated with the fact that an oriented forest can be the union of independent arcs or that an oriented forest can have an isolated vertex as a tree component.

As an extension of Theorem 2, we prove the following.

**Theorem 4.** Given \( k > 0 \), let \( F \) be an oriented forest with \( k - 1 \) arcs. Let \( D = (X, Y; A(D)) \) be a bipartite digraph with \( \delta^+_{X}D \geq k - 1 \), \( \delta^+_YD \geq k - 1 \) and \( \delta^+_YD \geq \lfloor k/2 \rfloor \), \( \delta^-_YD \geq \lceil k/2 \rceil \), such that \( |X| \geq k - 1 \), \( |V(F)| \leq |V(D)| \). Then \( D \) contains \( F \).

**Proof.** We first observe that

**Lemma 2.** Let \( F \) be an oriented forest with \( k - 1 \) arcs, consisting of an even number of balanced oriented trees. Let \( D = (X, Y; A(D)) \) be a bipartite digraph with \( \delta^+_X D \geq k - 1 \), \( \delta^+_Y D \geq k - 1 \) and \( \delta^+_Y D \geq \lfloor k/2 \rfloor \), \( \delta^-_Y D \geq \lceil k/2 \rceil \), such that \( |X| \geq k - 1 \). Then \( D \) contains \( F \).

**Proof of Lemma 2.** We will use theorem of a system of distinct representatives [4,5]. Given sets \( S_1, \ldots, S_j \), we say any element \( x_i \in S_i \) is a representative for the set \( S_i \) which contains it, \( i = 1, \ldots, j \). A collection of distinct representatives for sets \( S_1, \ldots, S_j \) is called a system of distinct representatives of the sets.
Theorem 5 (of a system of distinct representatives). A collection $S_1, S_2, \ldots, S_j$, $j \geq 1$ of finite nonempty sets has a system of distinct representatives if, and only if, the union of any $l$ of these sets contains at least $l$ elements for each $l$ ($1 \leq l \leq j$).

Let $F$ be an oriented forest with $k - 1$ arcs, consisting of $2n$ ($n \geq 1$) oriented trees $T_i = (U_i, W_i; A(T_i))$ such that $|U_i| = |W_i| = u_i \geq 1$ ($i = 1, \ldots, 2n$). Set $u = \sum_{i=1}^{2n} u_i$. Since $k - 1 = \sum_{i=1}^{2n} (2u_i - 1)$, we have $k = 2r + 1$ ($r \geq 1$). Observe that $r = u - n$, $u \geq 2n$ and then $r \geq n$. For $i \in \{1, \ldots, 2n\}$, let $l_i$ be a leaf of the oriented tree $T_i$ and let $s_i$ be the only neighbour of $l_i$ in $T_i$. Set $l_i \in U_i$ and then $s_i \in W_i$ for $i = 1, \ldots, 2n$. Without loss of generality, we may assume that at least $n$ vertices $l_i$ ($i = 1, \ldots, 2n$) has indegree equal to 1 in $T_i$ and that it holds for $l_1, \ldots, l_n$.

Let $D = (X, Y; A(D))$ be a bipartite digraph such that $\delta^+_X D \geq 2r$ and $\delta^+_Y D \geq 2r$, $\delta^-_Y D \geq 2r$ and $\delta^-_Y D \geq r$ and $|X| \geq 2r$.

The proof will be divided into two parts.

Part 1. We will show that there are $n$ vertices $x_1, x_2, \ldots, x_n \in X$ and $n$ vertices $y_1, y_2, \ldots, y_n \in Y$ such that for any $m$, $1 \leq m \leq n$, for any $\{i_1, i_2, \ldots, i_m\} \subset \{1, 2, \ldots, n\}$,

$$\left| \bigcup_{j=1}^{m} N^+_D(y_{i_j}) \cup \{x_i : i = 1, \ldots, n\} \right| \geq r + m.$$ 

For every vertex $y \in Y$ we choose a set $N_y$ such that $N_y \subset N^+_D(y)$ and $|N_y| = r$. Observe first that there are at least $k - 1 > n$, not necessarily different, sets $N_y$ ($y \in Y$). From sets $N_y$ ($y \in Y$) we will choose $n$ sets $N_i$ and $n$ vertices $x_i \in X$ ($i = 1, \ldots, n$) such that for any $m$ ($1 \leq m \leq n$), for any $\{i_1, i_2, \ldots, i_m\} \subset \{1, 2, \ldots, n\}$ holds:

$$(*) \quad \left| \bigcup_{j=1}^{m} N_{i_j} \cup \{x_i : i = 1, \ldots, n\} \right| \geq r + m.$$ 

As $N_1$ we take one of the above defined sets $N_y$ ($y \in Y$). Since $|X \setminus N_1| \geq r$, it is clear that $(*)$ is satisfied if $n = 1$. Let $n > 1$. We take another set $N_y$ ($y \in Y$) and denote it by $N_2$. If $|N_1 \cup N_2 \cup \{x_1\}| \geq r + 2$ then we set the vertex $x_2 \in (X \setminus (N_2 \cup \{x_1\})) \neq \emptyset$. If $|N_1 \cup N_2 \cup \{x_1\}| = r + 1$ then since $|X \setminus (N_2 \cup N_1 \cup \{x_1\})| \geq r - 1 \geq n - 1 > 0$, we set the vertex $x_2 \in (X \setminus (N_1 \cup N_2 \cup \{x_1\}))$. Observe that $(*)$ is satisfied if $n = 2$ and $m \leq 2$.

Suppose that we have chosen $c$ sets $N_1, \ldots, N_c$ and $c$ vertices $x_1, \ldots, x_c$ such that $(*)$ is satisfied if $n = c$ and $m \leq c$.

Let $n > c$. We take another, not still chosen set $N_y$ ($y \in Y$) and denote it by $N_{c+1}$.

Observe that it is possible in this construction that $N_{i_1} = N_{i_2}$ and $i_1 \neq i_2$, $i_1, i_2 \in \{1, \ldots, c\}$. In the proof we mean that two sets $N_{i_1}$, $N_{i_2}$ are different if $i_1 \neq i_2$, $i_1, i_2 \in \{1, \ldots, c\}$.

Now, we will set the vertex $x_{c+1}$. Consider $m$ ($1 \leq m \leq c$) sets $N_{i_1}, \ldots, N_{i_m}$. If $|\bigcup_{j=1}^{m} N_{i_j} \cup \{x_i : i = 1, \ldots, c\} \cup N_{c+1}| > r + m$ then, independently of the choice of the vertex $x_{c+1}$, $(*)$ is satisfied for sets $N_{i_1}, \ldots, N_{i_m}, N_{c+1}$ and $n = c + 1$. 

Let \(|\bigcup_{i=1}^{m} N_i \cup \{x_i: i = 1, \ldots, c\}| = r + m\) and \(N_{c+1} \subset \bigcup_{i=1}^{m} N_i \cup \{x_i: i = 1, \ldots, c\}\). Such a sequence of sets \(N_i (i \in \{1, \ldots, c\})\) we will call combination. For every \(m (1 \leq m \leq c)\) we consider all combinations of \(m\) sets \(N_i (i \in \{1, \ldots, c\})\). We will prove that it is possible to set the vertex \(x_{c+1}\) such that for any \(m (1 \leq m \leq c)\), for any combination \(N_i, N_j, \ldots, N_a, x_{c+1} \notin \bigcup_{i=1}^{m} N_i \cup \{x_i: i = 1, \ldots, c\}\).

By maximal combination we mean such a combination \(Z_1\) that there is no another combination \(Z_2\) such that every set of combination \(Z_1\) is a set of combination \(Z_2\). It is clear that it is sufficient to consider maximal combinations.

For the maximal combination \(Z\) of \(m\) sets \(N_i, i \in \{1, \ldots, c\}\), by \(M_m\) we will denote the union of sets of \(Z\). Let \(X_c = \{x_i: i = 1, \ldots, c\}\).

Let \(Z_1\) and \(Z_2\) be two different maximal combinations of \(m_1\) and \(m_2\), respectively, sets \(N_i, i \in \{1, \ldots, c\}\). Then \(|M_{m_1} \cup X_c| = r + m_1, N_{c+1} \subset (M_{m_1} \cup X_c)\) and \(|M_{m_2} \cup X_c| = r + m_2, N_{c+1} \subset (M_{m_2} \cup X_c)\). By the definition of maximal combination, we obtain that the sequence of such sets \(N_i, i \in \{1, \ldots, c\}\) that \(N_i\) is a set of \(Z_1\) or \(N_i\) is a set of \(Z_2\) is not a combination. Let us suppose that there are \(s\) sets \(N_i, i \in \{1, \ldots, c\}\) common to both maximal combinations \(Z_1\) and \(Z_2\). By \(S\) we denote the union of these sets. Then \(N_{c+1} \subset (M_{m_1} \cup M_{m_2} \cup X_c)\) and \(|M_{m_1} \cup M_{m_2} \cup X_c| > r + m_1 + m_2 - s\). Observe that \(|S \cup X_c| \geq r + s\). It is clear that \((S \cup X_c) \subset (M_{m_1} \cup X_c), i = 1, 2,\) and consequently \(|(M_{m_1} \cap M_{m_2}) \cup X_c| \geq r + s\). Hence \(|M_{m_1} \cup M_{m_2} \cup X_c| \leq r + m_1 + m_2 - s\), a contradiction. Thus, any two different maximal combinations do not have common sets \(N_i (i \in \{1, \ldots, c\}\).

Let \(z\) be the number of maximal combinations of sets \(N_i (i \in \{1, \ldots, c\}, 1 \leq z \leq c)\). Let \(Z_j\) be the maximal combination of \(m_j\) sets \(N_i, i \in \{1, \ldots, c\}, j = 1, \ldots, z\). Then \(N_{c+1} \subset (M_{m_j} \cup X_c)\) and \(|M_{m_j} \cup X_c| = r + m_j, j = 1, \ldots, z\). Hence \(|\bigcup_{j=1}^{z} M_{m_j} \cup X_c| \leq |N_{c+1}| + \sum_{j=1}^{z} (M_{m_j} \cup X_c) \backslash N_{c+1} = r + \sum_{j=1}^{z} m_j\). Since every two different maximal combinations do not have common sets, \(\sum_{j=1}^{z} m_j\) is the number of sets \(N_i (i \in \{1, \ldots, c\}\) belonging to maximal combinations, and then \(\sum_{j=1}^{z} m_j \leq c\). We obtain that \(|X_c \backslash \bigcup_{j=1}^{z} M_{m_j} \cup X_c| \geq r - c > 0\) and we set the vertex \(x_{c+1}\) such that \(x_{c+1} \in X_c \backslash \bigcup_{j=1}^{z} M_{m_j} \cup X_c\).

Thus we can choose \(n\) sets \(N_i\) and \(n\) vertices \(x_i, i = 1, \ldots, n\), such that \((*)\) is satisfied for \(m = n\).

Let \(y_i, i = 1, \ldots, n\) be vertices of \(Y\) such that \(N_i = N_{y_i}\). Observe that if we replace each set \(N_{y_i}\) by set \(N^+_D(y_i)\), for any \(m (1 \leq m \leq n)\), for any \(\{i_1, i_2, \ldots, i_m\} \subset \{1, 2, \ldots, n\}\) holds:

\[
\left| \bigcup_{j=1}^{m} N^+_D(y_{i_j}) \cup \{x_i: i = 1, \ldots, n\} \right| \geq r + m.
\]

Part 2. We will prove that \(D\) contains \(F\).

Let \(F_1\) be the oriented forest, consisting of \(2n\) oriented trees \(T_i\) such that \(V(T_i) = \{s_i\}\) for \(i = 1, \ldots, n\) and \(V(T_{n+i}) = \{l_i\}\) for \(i = n + 1, \ldots, 2n\). Let \(F_{II}\) be the oriented forest obtained from \(F\) by removing leaves \(l_1, \ldots, l_n\). It is obvious that the oriented forest \(F_1\) is a subforest of \(F_{II}\) and the oriented forest \(F_{II}\) is a subforest of \(F\).

The proof will be divided into three steps. First we will fix the embedding \(f_1\) of \(F_1\) in \(D\), then we will extend it to the embedding \(f_{II}\) of \(F_{II}\) in \(D\), and next we will
extend $f_{II}$ to the embedding $f$ of $F$ in $D$. To extend these embeddings we will use the following fact.

**Lemma 3.** Let $F = (U_F, W_F; A(F))$ be an oriented forest consisting of $c$ oriented trees $T_i = (U_i, W_i; A(T_i))$, $U_i \subset U_F$, $W_i \subset W_F$, $i = 1, \ldots, c$. Let $D = (X, Y; A(D))$ be a bipartite digraph with $\delta^+_F(D) \geq |W_F|$, $\delta^+_F(D) \geq |U_F|$, $\delta^-_F(D) \geq |U_F|$, $\delta^-_F(D) \geq |U_F|$. Let $F' = (U_F', W_F'; A(F'))$ be an oriented forest consisting of $c$ oriented trees $T'_i = (U'_i, W'_i; A(T'_i))$, $U'_i \subset U_F'$, $W'_i \subset W_F'$ such that $T'_i$ is a subdigraph of $T_i$ indicated by $V(T'_i)$, $U'_i \subset U_i$, $W'_i \subset W_i$, $i = 1, \ldots, c$. If there exists some embedding $f_1$ of $F'$ in $D$ such that $f_1(U_F') \subset X$, $f_1(W_F') \subset Y$, then $f_1$ can be extended to some embedding $f$ of $F$ in $D$ such that $f(U_F) \subset X$ and $f(W_F) \subset Y$.

**Proof of Lemma 3.** If $|V(F)| = |V(F')|$ then $F = F'$ and $f = f_1$. Let $|V(F)| > |V(F')|$. Then for some oriented tree $T_{ii}$, $i_0 \in \{1, \ldots, c\}$ there is the vertex $u \in V(T_{ii}) \setminus V(T'_{ii})$ such that the subdigraph of $T_{ii}$ indicated by $V(T'_{ii}) \cup \{u\}$ is an oriented tree. Let us denote it by $T''_{ii}$. Let $F''$ denote the oriented forest consisting of $c$ oriented trees $T''_{ii}$, $T''_i$, $i \in \{1, \ldots, c\}$, $i \neq i_0$. Without loss of generality we can assume that $u \in U_{ii}$. Hence $|U'_{ii}| < |U_{ii}|$. Let $w$ denote the only neighbour of $u$ in $T''_{ii}$. It is obvious that $w \in W_{ii}$ and hence $f_1(w) \in Y$. Since $d^+_D(f_1(w)) = |U'_{ii}| + \delta^+_F(D) - |U'_{ii}| > 0$ and $d^-_D(f_1(w)) = |U'_{ii}| + \delta^-_F(D) - |U'_{ii}| > 0$, it follows that independently of the orientation of the arc joining vertices $u$ and $w$ in $T_{ii}$, there is the vertex $x \in X$ such that the function $f_2$ defined in the following way: $f_2(v) = f_1(v)$ for $v \in V(F')$ and $f_2(u) = x$ is an extension of the embedding $f_1$ to an embedding of $F''$ in $D$. Observe that $|V(F'')| = |V(F')| + 1$ and $f_2(U_F \cup \{u\}) \subset X$, $f_2(W_F') \subset Y$. Repeating the above reasoning we extend given embedding to an embedding of a forest with one more vertex. In this way, we extend $f_1$ to the embedding $f$ of $F$ in $D$ such that $f(U_F) \subset X$ and $f(W_F) \subset Y$. □

1. We define the embedding $f_1$ in the following way: $f_1(s_i) = y_i$ for $i = 1, \ldots, n$ and $f_1(l_i) = x_i-n$ for $i = n+1, \ldots, 2n$ when $y_j$, $x_j$, $j = 1, \ldots, n$, are the vertices determined in Part 1.

2. Let $T_{ii} = (U_{ii}, W_{ii}; A(T_{ii}))$ be the oriented tree of the oriented forest $F_{ii}$, $i = 1, \ldots, 2n$. Let $U_{ii} = U_i \setminus \{l_i\}$ for $i = 1, \ldots, n$, $U_{ii} = U_i$ for $i = n+1, \ldots, 2n$ and $W_{ii} = W_i$ for $i = 1, \ldots, 2n$. Observe that $s_i \in W_{ii}$ ($i = 1, \ldots, 2n$), $l_i \in U_{ii}$ ($i = n+1, \ldots, 2n$). Hence $\sum_{i=1}^{2n} |U_{ii}| = u-n = r \leq \delta^+_F(D)$ (or $\delta^-_F(D)$) and $\sum_{i=1}^{2n} |W_{ii}| = u+r+n \leq 2r \leq \delta^+_F(D)$ (or $\delta^-_F(D)$). Therefore by Lemma 3 $f_1$ can be extended to the embedding $f_{ii}$ of $F_{ii}$ in $D$ such that $f_{ii}(U_{ii}) \subset X$ and $f_{ii}(W_{ii}) \subset Y$ ($i = 1, \ldots, 2n$).

3. Since $\sum_{i=1}^{2n} |U_{ii}| = r$ and $\{x_j: j = 1, \ldots, n\} \subset \bigcup_{i=1}^{2n} U_{ii}$, we obtain that for any $m \leq n$ and for any $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$ holds:

$$\left| \bigcup_{j=1}^{m} N_D^+(y_{i_j}) \setminus \bigcup_{i=1}^{2n} U_{ii} \right| \geq m.$$

By Theorem 5 (of a system of distinct representatives) we conclude that there are $n$ different vertices $x'_j \in (X \setminus f_{II}(V(F_{II})))$ such that $x'_j \in N_D^+(y_i)$, $i = 1, \ldots, n$. Since
Let \(f_u(s_i) = y_i\), \(d_T^-(l_i) = 1\), \((s_i, l_i) \in A(T_i)\), \(i = 1, \ldots, n\), we can extend \(f_u\) to \(f\) in such a way that \(f(l_i) = x'_i\), \(i = 1, \ldots, n\), which completes the proof of Lemma 2. \(\square\)

In the proof we will use Theorem 2. In fact, in [8], we proved more:

**Lemma 4.** Let \(T = (U, W; A(T))\), \(|U| \leq |W|\) be an oriented tree of order \(n\). Let \(D = (X, Y; A(D))\) be a bipartite digraph with \(\delta_X^+(D) \geq n - 1\), \(\delta^+_Y(D) \geq n - 1\) and \(\delta^+_Y(D) \geq [n/2]\), \(\delta^+_Y(D) \geq [n/2]\). Then \(D\) contains \(T\) such that \(U \subseteq X\) and \(W \subseteq Y\).

Let us go back to the proof of Theorem 4. Fix \(k > 0\). Let \(F\) be an oriented forest of size \(k - 1\), consisting of \(c \geq 1\) oriented trees \(T_1, \ldots, T_c\).

Let \(D = (X, Y; A(D))\) be a bipartite digraph with \(\delta_X^+(D) \geq k - 1\), \(\delta^+_Y(D) \geq k - 1\), \(\delta^+_Y(D) \geq [k/2]\), \(\delta^+_Y(D) \geq [k/2]\), such that \(|V(F)| \leq |V(D)|\) and \(|X| \geq k - 1\).

The proof is by induction on \(c\). By Theorem 2 we obtain a case when \(c = 1\).

Let \(c > 1\). Assume the property holds until \(c - 1\). Let \(T\) be one of oriented trees \(T_i\), \(i = 1, \ldots, c\). By \(F - T\) we will denote the oriented forest obtained from \(F\) by removing the oriented tree \(T\).

**Case 1:** Let \(T\) be one of oriented trees \(T_i\), \(i = 1, \ldots, c\), such that \(T\) is an isolated vertex.

By induction hypothesis \(D\) contains \(F - T\). Since \(|V(F)| \leq |V(D)|\), \(D\) contains \(F\).

**Case 2:** Let none of \(T_i\), \(i = 1, \ldots, c\), be an isolated vertex. Let \(k\) be even or let one of oriented trees \(T_i\), \(i = 1, \ldots, c\), be not balanced.

Let \(T = (U, W; A(T))\), \(u = |U|, w = |W|, 1 \leq u \leq w\) be one of oriented trees \(T_i\), \(i = 1, \ldots, c\). By Lemma 4, there is the embedding \(f\) of \(T\) in \(D\) such that \(f(U) \subseteq X\) and \(f(W) \subseteq Y\). Let \(D_i = (X_i, Y_i; A(D_i))\) be the bipartite digraph obtained from \(D\) by removing vertices of \(f(V(T))\), \(X_i \subseteq X, Y_i \subseteq Y\). Hence \(|X_i| = |X| - u, \delta^+_X(D_i) \geq k - 1 - w\), \(\delta^+_X(D_i) \geq k - 1 - w\), \(\delta^+_Y(D_i) \geq [k/2] - u\) and \(\delta^+_Y(D_i) \geq [k/2] - u\). The oriented forest \(F - T\) has \(k - (u + w)\) arcs and consists of \(c - 1\) oriented trees. Since \(k - 1 - w \geq k - (u + w)\) and if \(k\) is even or if \(u < w\) then \([k/2] - u \geq [(k + 1 - (u + w))/2]\), by induction hypothesis, \(D_i\) contains \(F - T\). Hence \(D\) contains \(F\).

**Case 3:** Let \(k\) be odd and let every oriented tree \(T_i\) \((i = 1, \ldots, c)\) be balanced.

Let \(T_i = (U_i, W_i; A(T_i))\), \(|U_i| = |W_i| = u_i\) for \(i = 1, \ldots, c\). Then \(k - 1 = \sum_{i=1}^{c} (2u_i - 1)\). Hence \(c\) is even and by Lemma 2, \(D\) contains \(F\). \(\square\)

Observe that Theorem 4 implies condition for bipartite graphs to contain every forest of size \(k - 1\).

Let \(G = (X, Y; E(G))\) be a bipartite graph with vertex set \(V(G) = X \cup Y, X \cap Y = \emptyset\) and edge set \(E(G) \subseteq \{xy: x \in X, y \in Y\}\). For any subset \(W\) of \(V(G)\) let \(\delta^+_W(G) = A^+_W(G)\) denote the minimum degree (the maximum degree) in \(G\) of a vertex from \(W\).

**Corollary 1.** Given \(k > 0\), let \(F\) be a forest of size \(k - 1\). Let \(G = (X, Y; E(G))\) be a bipartite graph with \(\delta_X(G) \geq k - 1, \delta_Y(G) \geq [k/2]\) such that \(|X| \geq k - 1, |V(F)| \leq |V(G)|\). Then \(G\) contains \(F\).
By $S$ and $DS$ we will denote the following two trees on $k$ vertices. Let $S$ be the star and let $DS$ be the tree obtained from $DS^+$ by replacing arcs by edges.

Degree bounds are sharp because of $S$ and $DS$. If either $\delta_w(G) = \Delta_w(G) = k - 2$ ($\delta_y(G) < k - 1$) or $\delta_y(G) = \Delta_y(G) = \lceil k/2 \rceil - 1$ then either $G$ does not contain $S$ or $G$ does not contain $DS$.

The next result extends Theorem 3.

**Theorem 6.** Given $k > 0$, let $F$ be an oriented forest with $k - 1$ arcs. Let $D = (X, Y; A(D))$ be a bipartite digraph with $\delta^+_X(D) \geq k - 1$, $\delta^-_X(D) \geq \lfloor (k - 1)/2 \rfloor$ and $\delta^+_Y(D) \geq \lceil (k - 1)/2 \rceil$, $\delta^-_Y(D) \geq k - 1$, such that $|V(F)| \leq |V(D)|$. Then $D$ contains $F$.

**Proof.** We need the following.

**Lemma 5.** Let $F$ be an oriented forest with $k - 1$ arcs, consisting of an odd number of balanced leaves-uniform trees, each of them of order at least $4$. Let $D = (X, Y; A(D))$ be a bipartite digraph with $\delta^+_X(D) \geq k - 1$, $\delta^-_X(D) \geq \lfloor (k - 1)/2 \rfloor$ and $\delta^+_Y(D) \geq \lceil (k - 1)/2 \rceil$, $\delta^-_Y(D) \geq k - 1$. Then $D$ contains $F$.

**Proof of Lemma 5.** The method of the proof is similar to the method of the proof of Lemma 2. We will also use Theorem 5 (of a system of distinct representatives).

Let $F$ be an oriented forest with $k - 1$ arcs, consisting of $2n + 1$ ($n \geq 0$) oriented trees $T_i = (U_i, W_i; A(T_i))$ such that $|U_i| = |W_i| = u_i \geq 2$ and every $T_i$ is leaves-uniform ($i = 1, \ldots, 2n + 1$). Set $u = \sum_{i=1}^{2n+1} u_i$. Since $k - 1 = \sum_{i=1}^{2n+1} (2u_i - 1)$, we have $k = 2r$ ($r \geq 2$). Observe that $r = u - n$, $u \geq 2(2n + 1)$ and then $r \geq 3n$. Without loss of generality, we may assume that every leaf of $T_i$, $i = 1, \ldots, n + 1$ has indegree equal to 1. By Observation 1 there is a leaf in $U_i$ and there is a leaf in $W_i$, $i = 1, \ldots, 2n + 1$. Let $l_i \in U_i$, $t_j \in W_i$ be leaves of $T_i$ and $s_i \in W_i$ be the only neighbour of $l_i$ in $T_i$ ($i \in \{1, \ldots, 2n + 1\}$, $i \neq n + 1$).

Let us consider more carefully the oriented tree $T_{n+1}$. Every leaf of $T_{n+1}$ has indegree equal to 1. If among neighbours of leaves of $T_{n+1}$ there is at least one with indegree greater than 0, then we will denote it by $s_{n+1}$. Then by $l_{n+1}$ we will denote a leaf which is a neighbour of $s_{n+1}$. Else, let $l_{n+1}$ be a leaf of $T_{n+1}$ and $s_{n+1}$ the only neighbour of $l_{n+1}$. Without loss of generality, in both cases, we may assume that $l_{n+1} \in U_{n+1}$ and $s_{n+1} \in W_{n+1}$. Let $t_{n+1}$ be a leaf in $W_{n+1}$. Hence $d^+_T(t_i) = 1 = d^-_T(t_i)$ for $i = 1, \ldots, n + 1$.

Let $D = (X, Y; A(D))$ be a bipartite digraph such that $\delta^+_X(D) \geq 2r - 1$, $\delta^-_X(D) \geq r - 1$, $\delta^+_Y(D) \geq r - 1$ and $\delta^-_Y(D) \geq 2r - 1$.

By Theorem 3 we obtain a case when $n = 0$. Let us assume that $n \geq 1$. The proof will be divided into two parts.

**Part 1.** We will show that there are $n$ vertices $x_1, x_2, \ldots, x_n \in X$ and $n$ vertices $y_1, y_2, \ldots, y_n \in Y$ such that for any $m$, $1 \leq m \leq n$, for any $\{i_1, i_2, \ldots, i_m\} \subset \{1, 2, \ldots, n\}$,

$$\left| \bigcup_{j=1}^{m} N^+_D(y_{i_j}) \cup \{x_i : i = 1, \ldots, n\} \right| \geq r - 1 + m.$$
The proof is exactly the same as for Lemma 2, Part 1, and we also prove that there are \( n \) vertices \( x_1, \ldots, x_n \in X \) and \( n \) vertices \( y_1, \ldots, y_n \in Y \) such that for any \( m, 1 \leq m \leq n \), for any \( \{i_1, i_2, \ldots, i_m\} \subset \{1, 2, \ldots, n\} \),

\[
\bigcup_{j=1}^{m} N_i \cup \{x_i : i = 1, \ldots, n\} \geq r - 1 + m,
\]

when \( N_i \) (\( i = 1, \ldots, n \)) is a subset of \( X \) such that \( N_i \subset N_i^D(y_i) \) and \( |N_i| = r - 1 \).

**Part 2.** We will prove that \( D \) contains \( F \).

Let \( F \) be the oriented forest consisting of \( 2n + 1 \) oriented trees \( T_{li} \) such that \( V(T_{li}) = \{s_i\} \) for \( i = 1, \ldots, n, n + 1 \), \( V(T_{li}) = \{l_i\} \) for \( i = n + 2, \ldots, 2n + 1 \).

Let \( F_{II} \) be the oriented forest consisting of \( 2n + 1 \) oriented trees \( T_{ll} = (U_{ll}, W_{ll}; A(T_{ll})) \), \( U_{ll} \subset U_{l} \), \( W_{ll} \subset W_{l} \), such that \( T_{ll} = T_{li} \) for \( i = 1, \ldots, n \), \( T_{ll} = T_{li} \) is obtained from \( T_{n+1} \) by removing vertices \( l_{n+1}, t_{n+1}, T_{II} = T_{ll} \) for \( i = n + 2, \ldots, 2n + 1 \). Observe that \( U_{II(n+1)} = U_{ll+1} \) for \( i = 1, \ldots, n + 1 \) and \( l_i = U_{II} \) for \( i = n + 2, \ldots, 2n + 1 \).

Let \( F_{IV} \) be the oriented forest consisting of \( 2n + 1 \) oriented trees \( T_{IV} = (U_{IV}, W_{IV}; A(T_{IV})) \), \( U_{IV} \subset U_{I} \), \( W_{IV} \subset W_{I} \), such that \( T_{IV} = T_{IV} \) is obtained from \( T_{i} \) by removing the vertex \( t_i \) for \( i = 1, \ldots, n + 1 \), \( T_{IV} = T_{i} \) for \( i = n + 2, \ldots, 2n + 1 \).

The oriented forest \( F_{J} \) is a subforest of \( F_{J+1} \), \( J = I, II, III \) and \( F_{IV} \) is a subforest of \( F \).

The proof will be divided into five steps. First we will fix the embedding \( f_1 \) of \( F_1 \) in \( D \). For \( J = I, II, III \) we will extend the embedding \( f_J \) of \( F_J \) in \( D \) to the embedding \( f_{J+1} \) of \( F_{J+1} \) in \( D \). Finally, we will extend \( f_{IV} \) to the embedding \( f \) of \( F \) in \( D \).

1. It is clear that we can fix \( f_1 \) in the following way: \( f_1(s_i) = y_i \) for \( i = 1, \ldots, n \), \( f_1(l_i) = x_{j-(s_i+1)} \) for \( i = n + 2, \ldots, 2n + 1 \) when \( x_j, y_j, j = 1, \ldots, n \) are the vertices determined in Part 1 and \( f_1(s_{n+1}) = y_{n+1} \in Y \setminus \{y_1, \ldots, y_n\} \).

2. There are two cases with respect to the indegree of the vertex \( s_{n+1} \).

   **Case A:** \( d_{T_{li}}(s_{n+1}) > 0 \). Since \( T_{II(n+1)} \) is obtained from \( T_{n+1} \) by removing leaves \( l_{n+1} \) and \( t_{n+1} \), we have \( d_{T_{II(n+1)}}(s_{n+1}) = d_{T_{li}}(s_{n+1}) = d_{T_{li}}(s_{n+1}) > 0 \). In this case by \( d \) we will denote a vertex of \( U_{II(n+1)} \) such that \( (d, s_{n+1}) \in A(T_{II(n+1)}) \).

   **Case B:** \( d_{T_{li}}(s_{n+1}) = 0 \). In this case every neighbour of any leaf of \( T_{n+1} \) has indegree equal to 0. Let \( t_{n+1} \in U_{n+1} \) be the only neighbour of \( l_{n+1} \) in \( T_{n+1} \). It is clear that \( s_{n+1} \notin l_{n+1}, t'_{n+1} \notin l_{n+1} \). Observe that \( t'_{n+1} \in U_{II(n+1)} \) and \( d_{T_{II(n+1)}}(s_{n+1}) = d_{T_{II(n+1)}}(t'_{n+1}) = 0 \). Let \( T_{II(n+1)} \) be a subtree of \( T_{II(n+1)} \) such that \( T_{II(n+1)} \) has the most possible number of vertices and \( t'_{n+1} \notin V(T_{II(n+1)}) \), \( s_{n+1} \in V(T_{II(n+1)}) \). Set \( T'_{II(n+1)} = (U'_{II(n+1)}, W'_{II(n+1)}; A(T'_{II(n+1)})) \), \( U'_{II(n+1)} \subset U \), \( W'_{II(n+1)} \subset W \). Then there is the vertex \( s'_{n+1} \) of \( T_{II(n+1)} \) such that \( s'_{n+1} \in W'_{II(n+1)} \) and \( (t'_{n+1}, s'_{n+1}) \in A(T'_{II(n+1)}) \). In this case as \( d \) we will take the vertex \( t_{n+1} \).
Let $F^*_{II}$ be the oriented forest consisting of $2n + 1$ oriented trees $T_{U_i}$, $i = 1, \ldots, n$, $n+2, \ldots, 2n+1$ and $T^*_{II(n+1)}$. Since $\sum_{i=1}^{2n+1}|U_{II_i}| < u - (n+1) = r - 1 \leq \delta^+_V(D) (\delta^+_X(D))$ and $\sum_{i=1}^{2n+1}|W_{II_i}| < u - (n+1) = r - 1 \leq \delta^-_V(D) (\delta^-_X(D))$, we obtain that by Lemma 3 in Case B $f_1$ can be extended to the embedding $g_{II}$ of $F^*_{II}$ in $D$ such that $g_{II}(U_{II_i}) \subset X$, $g_{II}(W_{II_i}) \subset Y$ for $i = 1, \ldots, n+2, \ldots, 2n+1$ and $g_{II}(U_{II(n+1)}) \subset X$, $g_{II}(W_{II(n+1)}) \subset Y$. Set $g_{II}(s'_{n+1}) = y'_{n+1}$. Let $N_{n+1}$ be a subset of $X$ such that $N_{n+1} \subset N^+_D(y'_{n+1})$ and $|N_{n+1}| = r - 1$. The choice of $N_{n+1}$ is the same in Cases A and B.

In both cases we will map the vertex $d$ on the vertex $x_{n+1} \in X$ such that $n+1$ sets $N_1, \ldots, N_n$, $N_{n+1}$ and $n+1$ vertices $x_1, \ldots, x_n, x_{n+1}$ for any $m \ (1 \leq m \leq n+1)$, for any $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n,n+1\}$ will satisfy

$$\left| \bigcup_{j=1}^{m} N_{i_j} \cup \{x_i : \ i = 1, \ldots, n,n+1\} \right| \geq r - 1 + m.$$ 

Now, we will fix $x_{n+1}$.

By Part 1 it is enough to consider $m$ sets such that $N_{i_m} = N_{n+1}$ and it is sufficient to consider maximal combinations of sets $N_1, \ldots, N_n$. Observe that it is also sufficient to prove that it is possible to set the vertex $x_{n+1}$ such that for every maximal combination $N_1, \ldots, N_n$, $x_{n+1} \notin \left( \bigcup_{i=1}^{m} N_{i} \cup \{x_i : \ i = 1, \ldots, n\} \right)$.

Let $z$ be a number of maximal combinations of $m_j$ sets $N_i, \ i \in \{1, \ldots, n\}$ and by $M_{m}$ the union of sets of $z$, $j = 1, \ldots, z$. Let $X_z = \{x_i : \ i = 1, \ldots, n\}$. Then $N_{n+1} \subset (M_{m_1} \cup X_z)$ and $|M_{m_1} \cup X_z| = r - 1 + m_j, \ j = 1, \ldots, z$. Hence $| \bigcup_{j=1}^{z} (M_{m_j} \cup X_z) | \leq |N_{n+1}| + \sum_{j=1}^{z} |(M_{m_j} \cup X_z) \setminus N_{n+1}| = r - 1 + \sum_{j=1}^{z} m_j$. Since two different maximal combinations do not have common sets $N_i (i \in \{1, \ldots, n\})$, we obtain that $\sum_{j=1}^{z}$ is the number of sets $N_i (i \in \{1, \ldots, n\})$ belonging to maximal combinations, and then $\sum_{j=1}^{z} m_j \leq n$.

Let $N^-$ denote the set $N^+_D(y_{n+1})$ (Case A) or the set $N^-_D(y'_{n+1})$ (Case B). It is clear that $x_{n+1}$ has to be a vertex of $N^-$.

Since $y_{n+1}, y'_{n+1} \in Y$, we have $|N^-| \geq 2r - 1$. Hence we obtain that $|N^- \setminus (\bigcup_{j=1}^{z} M_{m_j} \cup X_z)| \geq r - n > 0$ and we can set the vertex $x_{n+1} \in N^- \setminus (\bigcup_{j=1}^{z} M_{m_j} \cup X_z)$.

Observe that if we replace each set $N_i$ by the set $N^+_D(y_i)$, then for any $m \ (1 \leq m \leq n+1)$, for any $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n,n+1\}$ holds:

$$\left| \bigcup_{j=1}^{m} N^+_D(y_{i_j}) \cup \{x_i : \ i = 1, \ldots, n,n+1\} \right| \geq r - 1 + m.$$ 

In both cases we map the vertex $d$ on the vertex $x_{n+1}$. Since $\sum_{i=1}^{2n+1}|U_{II_i}| \leq u -(n+1) = r + 1 \leq \delta^+_V(D) (\delta^+_X(D))$ and $\sum_{i=1}^{2n+1}|W_{II_i}| \leq u -(n+1) = r - 1 \leq \delta^-_V(D) (\delta^-_X(D))$, we obtain that, by Lemma 3, $f_1$ (in Case A) or $g_{II}$ (in Case B) can be extended to $f_{II}$ in such a way that $f_{II}(d) = x_{n+1}$, $f_{II}(U_{II_i}) \subset X$, $f_{II}(W_{II_i}) \subset Y, \ i = 1, \ldots, 2n+1$.

3. Since $\sum_{i=1}^{2n+1}|U_{II_i}| = u -(n+1) = r - 1 \leq \delta^+_V(D) (\delta^+_X(D))$ and $\sum_{i=1}^{2n+1}|W_{II_i}| = u -(n+1) = r - 1 \leq \delta^-_V(D) (\delta^-_X(D))$, by Lemma 3, $f_{II}$ can be extended to the embedding $f_{III}$ of $F_{III}$ in $D$ such that $f_{III}(U_{III_i}) \subset X$, $f_{III}(W_{III_i}) \subset Y, \ i = 1, \ldots, 2n+1$. 

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4. Since \(\sum_{i=1}^{2n+1}|U_{III}| = r - 1\) and \(\{x_j: j = 1, \ldots, n + 1\} \subset \bigcup_{i=1}^{2n+1} U_{III}\), we obtain that for any \(m \leq n + 1\), for any \(\{i_1, \ldots, i_m\} \subset \{1, \ldots, n + 1\}\) holds:

\[
\left| \bigcup_{j=1}^{m} N_+^D(y_{i_j}) \right| \geq m.
\]

By Theorem 5 (of a system of distinct representatives), we can conclude that there are \(n + 1\) different vertices \(x'_i \in (X \setminus f_{III}(V(F_{III})))\) such that \(x'_i \in N_+^D(y_{i_j}), i = 1, \ldots, n + 1\).

Since \(f_{III}(s_i) = y_i, d^+_T(t_i) = 1, (s_i, t_i) \in A(T_i), i = 1, \ldots, n + 1\), we can extend \(f_{III}\) to \(f_{IV}\) such that \(f_{IV}(s_i) = y_i, i = 1, \ldots, n + 1\).

5. Let \(t''_i\) be the only neighbour of \(t_i\) in \(T_i, i = 1, \ldots, n + 1\). Observe that \(f_{IV}(t''_i) \in X\) for \(i = 1, \ldots, n + 1\). Since \(d^+_T(t_i) = 1, i = 1, \ldots, n + 1\) and \(\delta^+_X(D) \geq 2r - 1 \geq u\), \(f_{IV}\) can be extended to the embedding \(f\) of \(F\) in \(D\) such that \(f(U_i) \subset X, f(W_i) \subset Y, i = 1, \ldots, 2n + 1\).

In case that at least \(n + 1\) oriented trees \(T_i, i \in \{1, \ldots, 2n + 1\}\) have leaves with outdegree equal to 1, without loss of generality we may assume that every leaf of \(T_i, i = 1, \ldots, n + 1\) has outdegree equal to 1. In Part 1 we will prove that there are \(n\) vertices \(x_1, \ldots, x_n \in X\) and \(n\) vertices \(y_1, \ldots, y_n \in Y\) such that for any \(m, 1 \leq m \leq n\), for any \(\{i_1, i_2, \ldots, i_m\} \subset \{1, 2, \ldots, n\}\),

\[
\left| \bigcup_{j=1}^{m} N_0^D(x_{i_j}) \cup \{y_i: i = 1, \ldots, n\} \right| \geq r - 1 + m.
\]

Then in Part 2, analogously as above, we will set the embedding of \(F\) in \(D\) such that \(f(U_i) \subset Y, f(W_i) \subset X, i = 1, \ldots, 2n + 1\). \(Q.E.D.\)

In [8] we proved something more than Theorem 3. Especially we obtained:

**Lemma 6.** Let \(T = (U, W; A(T))\), \(1 \leq |U| < |W|\) be an oriented tree of order \(n\). Let \(L^+ = \{w \in W: d^+_T(w) = 1, d^-_T(w) = 0\}\). Let \(D = (X, Y; A(D))\) be a bipartite digraph with \(\delta^+_X(D) \geq n - 1\), \(\delta^-_X(D) \geq \lceil (n-1)/2 \rceil\) and \(\delta^+_Y(D) \geq \lceil (n-1)/2 \rceil, \delta^-_Y(D) \geq n - 1\). Let \(|W \setminus L^+| \leq \lceil (n-1)/2 \rceil\). Then \(D\) contains \(T\) such that \(W \subset X\) and \(U \subset Y\).

**Lemma 7.** Let \(T = (U, W; A(T))\), \(1 \leq |U| < |W|\) be an oriented tree of order \(n\). Let \(L^- = \{w \in W: d^+_T(w) = 0, d^-_T(w) = 1\}\). Let \(D = (X, Y; A(D))\) be a bipartite digraph with \(\delta^+_X(D) \geq n - 1\), \(\delta^-_X(D) \geq \lceil (n-1)/2 \rceil\) and \(\delta^+_Y(D) \geq \lceil (n-1)/2 \rceil, \delta^-_Y(D) \geq n - 1\). Let \(|W \setminus L^-| \leq \lceil (n-1)/2 \rceil\). Then \(D\) contains \(T\) such that \(U \subset X\) and \(W \subset Y\).

**Lemma 8.** Let \(T = (U, W; A(T))\), be an oriented balanced tree of order \(n\). Let \(l \in U, t \in W\) be leaves of \(T\) such that \(d^+_T(l) = 1, d^-_T(t) = 1\). Let \(D = (X, Y; A(D))\) be a bipartite digraph with \(\delta^+_X(D) \geq n - 1\), \(\delta^-_X(D) \geq \lceil (n-1)/2 \rceil\) and \(\delta^+_Y(D) \geq \lceil (n-1)/2 \rceil, \delta^-_Y(D) \geq n - 1\). Then \(D\) contains \(T\) such that \(U \subset X\) and \(W \subset Y\).
We go back to the proof of Theorem 6. Fix \( k > 0 \). Let \( F \) be an oriented forest of size \( k - 1 \), consisting of \( c \) (\( c \geq 1 \)) oriented trees \( T_1, \ldots, T_c \).

Let \( D = (X, Y; A(D)) \) be a bipartite digraph with \( \delta^+_X(D) \geq k - 1 \), \( \delta^-_X(D) \geq \lfloor (k - 1)/2 \rfloor \), \( \delta^+_Y(D) \geq \lfloor (k - 1)/2 \rfloor \), \( \delta^-_Y(D) \geq k - 1 \).

The proof is by induction on \( c \). By Theorem 3 we obtain a case when \( c = 1 \).

Let \( c > 1 \). Assume the property holds until \( c - 1 \). Let \( T \) be one of oriented trees \( T_i \), \( i = 1, \ldots, c \). By \( F - T \) we will denote the oriented forest obtained from \( F \) by removing the oriented tree \( T \).

**Case 1:** Let \( T \) be one of oriented trees \( T_i \), \( i = 1, \ldots, c \), such that \( T \) is an isolated vertex.

Then we use the same arguments as in Case 1 of the proof of Theorem 4.

**Case 2:** Let none of oriented trees \( T_i \), \( i = 1, \ldots, c \), be an isolated vertex. Let \( T \) be one of oriented trees \( T_i \), \( i = 1, \ldots, c \), such that \( T \) is not a balanced oriented tree.

Let \( T = (U, W; A(T)) \), \( u = |U| \), \( w = |W| \), \( n = u + w \), such that \( 1 \leq u < w \). Observe that \( u \leq \lfloor (n - 1)/2 \rfloor \). Set \( L^+ = \{ w \in W : d_T^+(w) = 1, d_T^-(w) = 0 \} \), \( L^- = \{ w \in W : d_T^+(w) = 0, d_T^-(w) = 1 \} \), \( l^+ = |L^+| \), \( l^- = |L^-| \) and \( w = \tilde{w} + l^+ + l^- \), where \( \tilde{w} \) is the number of vertices from \( W \) with at least two neighbours in \( T \).

If \( \tilde{w} = 0 \) then \( T \) is an oriented star and \( u \geq 1 \). Let \( \tilde{w} > 0 \). Then these \( \tilde{w} \) vertices have together at least \( \tilde{w} + 1 \) neighbours in \( T \). Exactly, these \( \tilde{w} \) vertices have at least \( \tilde{w} + 1 \) neighbours from \( U \). Hence \( u \geq \tilde{w} + 1 \) for \( \tilde{w} \geq 0 \).

If \( \tilde{w} + l^+ \geq \lfloor (n - 1)/2 \rfloor + 1 \) and \( \tilde{w} + l^- \geq \lfloor (n - 1)/2 \rfloor + 1 \), and consequently \( \tilde{w} + (l^+ + l^-)/2 \geq \lfloor (n - 1)/2 \rfloor + 1 \) then \( \lfloor (n - 1)/2 \rfloor + 2 \leq \tilde{w} + (l^+ + l^-)/2 + 1 \leq u + (l^+ + l^-)/2 = n - (\tilde{w} + (l^+ + l^-)/2) \leq n - (\lfloor (n - 1)/2 \rfloor + 1) = \lfloor (n - 1)/2 \rfloor \), a contradiction.

Thus \( \tilde{w} + l^+ \leq \lfloor (n - 1)/2 \rfloor \) or \( \tilde{w} + l^- \leq \lfloor (n - 1)/2 \rfloor \). Without loss of generality, we may assume that \( \tilde{w} + l^+ \leq \lfloor (n - 1)/2 \rfloor \). By Lemma 7, \( D \) contains \( T \) such that \( U \subset X \) and \( W \subset Y \). Let \( T - L^- \) denote the oriented tree obtained from \( T \) by removing vertices of \( L^- \) and let \( F - L^- \) denote the oriented forest obtained from \( F \) by removing vertices of \( L^- \). It is clear that there is the embedding \( f_1 \) of \( T - L^- \) in \( D \) such that \( f_1(U) \subset X \) and \( f_1(W \setminus L^-) \subset Y \).

Let \( D_1 = (X_1, Y_1; A(D_1)) \) be a bipartite digraph obtained from \( D \) by removing vertices of \( f_1(V(T - L^-)) \), \( X_1 \subset X \), \( Y_1 \subset Y \). Then \( \delta^+_X(D_1) \geq k - 1 - (\tilde{w} + l^+) \), \( \delta^-_X(D_1) \geq \lfloor (k - 1)/2 \rfloor - (\tilde{w} + l^+) \), \( \delta^+_Y(D_1) \geq \lfloor (k - 1)/2 \rfloor - u \), \( \delta^-_Y(D_1) \geq k - 1 - u \). The oriented forest \( F - T \) has \( k - (u + w) \) arcs and consists of \( c - 1 \) oriented trees. Since \( u \leq \lfloor (u + w - 1)/2 \rfloor \) and \( \tilde{w} + l^+ \leq \lfloor (u + w - 1)/2 \rfloor \), we have \( \lfloor (k - 1)/2 \rfloor - u \geq \lfloor (k - (u + w))/2 \rfloor \) and \( \lfloor (k - 1)/2 \rfloor - (\tilde{w} + l^+) \geq \lfloor (k - (u + w))/2 \rfloor \). By induction hypothesis, there is the embedding \( g_1 \) of \( F - T \) in \( D_1 \). Observe that the function \( f_2 \) defined in the following way: \( f_2(v) = f_1(v) \) for \( v \in V(T - L^-) \) and \( f_2(v) = g_1(v) \) for \( v \in V(F - T) \) is an embedding of \( F - L^- \) in \( D \). It is clear that \( f_2(U) \subset X \) and \( f_2(W \setminus L^-) \subset Y \). Set \( F - L^- = (V_X, V_Y; A(F - L^-)) \), \( V_X \cup V_Y = V(F - L^-) \), \( V_X \cap V_Y = \emptyset \) and \( f_2(V_X) \subset X \), \( f_2(V_Y) \subset Y \). It is clear that \( U \subset V_X \).

Since \( F \) has no isolated vertices, we have \( |V(F - L^-)| = k - 1 + c - l^- \), \( |V_X| \geq c \) and then \( |V_Y| \leq k - 1 - l^- \). Since \( \delta^+_X(D) \geq k - 1 \), we have \( |Y| \geq k - 1 \). Neighbours of vertices from \( L^- \) are mapped by \( f_2 \) on vertices of \( X \). Since \( \delta^+_X(D) - |V_Y| \geq |L^-| \), we obtain that \( f_2 \) can be extended to an embedding of \( F \) in \( D \).
Case 3: Let \( k \) be odd and let every oriented tree \( T_i, \ i = 1, \ldots, c \), be balanced.

Let \( T = (U, W; A(T)) \) such that \( 1 \leq u = |U| = |W| \) be one of oriented trees \( T_i, \ i = 1, \ldots, c \). By Theorem 3, there is the embedding \( f_i \) of \( T \) in \( D \). Let \( D_1 = (X_1, Y_1; A(D_1)), X_1 \subset X, Y_1 \subset Y \) be a bipartite digraph obtained from \( D \) by removing vertices of \( f_i(V(T)) \). Then \( \delta^+_Y(D_1) \geq k-1-u, \delta^-_X(D_1) \geq [(k-1)/2] - u, \delta^+_Y(D_1) \geq [(k-1)/2] - u, \) \( \delta^-_X(D_1) \geq k-1-u \). Observe that the oriented forest \( F - T \) has \( k-2u \) arcs and consists of \( c-1 \) oriented trees. Since \( k-1-u \geq k-2u \) and \( [(k-1)/2] - u \geq [(k-2u)/2] \), by induction hypothesis, \( D_1 \) contains \( F - T \), and then \( D \) contains \( F \).

Case 4: Let every oriented tree \( T_i, \ i = 1, \ldots, c \), be balanced. Let \( T \) be one of oriented trees \( T_i, \ i = 1, \ldots, c \), such that \( T \) is not leaves-uniform.

Let \( T = (U, W; A(T)) \), such that \( 1 \leq u = |U| = |W| \). By Observation 1 there is a leaf of \( T \) in \( U \) and there is a leaf of \( T \) in \( W \). Since \( T \) is not leaves-uniform there are two leaves of \( T \): \( l_1, l_2 \) such that \( d^+_T(l_1) = 1, d^-_T(l_2) = 1 \). If \( l_1, l_2 \in U \), there is the leaf \( l_3 \in W \) such that \( d^+_T(l_3) = 1 \) or \( d^-_T(l_3) = 1 \). Hence, without loss of generality, we may assume that there are two leaves of \( T \): \( l \in U, t \in W \) such that \( d^+_T(l) = 1, d^-_T(t) = 1 \). By Lemma 8, \( D \) contains \( T \) such that \( U \subset X \) and \( W \subset Y \). Let \( T' \) denote the oriented tree obtained from \( T \) by removing leaves \( l, t \). It is clear that there is an embedding \( f_1 \) of \( T' \) in \( D \) such that \( f_1(U \setminus \{l\}) \subset X, f_1(W \setminus \{t\}) \subset Y \). Let \( D_1 = (X_1, Y_1; A(D_1)), X_1 \subset X, Y_1 \subset Y \) be a bipartite digraph obtained from \( D \) by removing vertices of \( f_1(V(T')) \). Then \( \delta^+_Y(D_1) \geq k-1-(u-1), \delta^-_X(D_1) \geq [(k-1)/2] - (u-1), \delta^+_Y(D_1) \geq [(k-1)/2] - (u-1), \delta^-_X(D_1) \geq k-1-(u-1) \). The oriented forest \( F - T \) has \( k-2u \) arcs and consists of \( c-1 \) oriented trees. Since \( k-u \geq k-2u \) and \( [(k-1)/2] - (u-1) \geq [(k-2u)/2] \), by induction hypothesis, there is the embedding \( g_1 \) of \( F - T \) in \( D_1 \).

Let \( F' \) be the oriented forest obtained from \( F \) by removing vertices \( l, t \). Observe that the function \( f_2 \) such that \( f_2(v) = f_1(v) \) for \( v \in V(T') \) and \( f_2(v) = g_1(v) \) for \( v \in V(F - T) \) is an embedding of \( F' \) in \( D \). Then \( f_2(U \setminus \{l\}) \subset X, f_2(W \setminus \{t\}) \subset Y \). Set \( F' = (V_X, V_Y; A(F')) \), \( V_X \cup V_Y = V(F'), \) \( V_X \cap V_Y = \emptyset \) and \( f_2(V_X) \subset X, f_2(V_Y) \subset Y \). Since \( F \) has no isolated vertices we have \( |V(F')| = k-1+c-2, |V_X| \geq c-1, |V_Y| \geq c-1 \) and then \( |V_X| \leq k-2, |V_Y| \leq k-2 \). Since \( \delta^+_Y(D) \geq k-1, \delta^-_X(D) \geq k-1 \), we obtain that \( |X| \geq k-1, |Y| \geq k-1 \). Neighbours of vertices \( l, t \) are mapped by \( f_2 \) on vertices of \( X \) or \( Y \), respectively. Since \( \delta^+_Y(D) - |V_X| \geq 1 \) and \( \delta^-_X(D) - |V_Y| \geq 1 \), we obtain that \( f_2 \) can be extended to an embedding of \( F \) in \( D \).

If \( T \) is an oriented arc then \( V(T) = \{l, t\} \). In this case we do not consider the oriented tree \( T' \) and the embedding \( f_1 \). As \( f_2 \) we take an embedding of \( F' \) in \( D \). Finally, we conclude that there is the vertex \( x \in (X \setminus f_2(V_X)) \) and there is the vertex \( y \in (N^+_Y(x) \setminus f_2(V_Y)) \). Hence \( f_2 \) can be extended to an embedding of \( F \) in \( D \).

Case 5: Let every oriented tree \( T_i, \ i = 1, \ldots, c \), be balanced. Let \( k \) be even and every oriented tree \( T_i, \ i = 1, \ldots, c \), be leaves-uniform.

Set \( T_i = (U_i, W_i; A(T_i)), u_i = |U_i| = |W_i| \) for \( i = 1, \ldots, c \). Since \( k-1 = \sum_{i=1}^c (2u_i - 1) \), we obtain that \( c \) is an odd number. Consider \( T_{i_0}, i_0 \in \{1, \ldots, c\} \). By Observation 1 there is the leaf \( l_1 \in U_{i_0} \) and there is the leaf \( l_2 \in W_{i_0} \). Since \( T_{i_0} \) is leaves-uniform,
\[ d_{T_{l_0}}^+ (l_1) = d_{T_{l_0}}^+ (l_2), \quad d_{T_{l_0}}^- (l_1) = d_{T_{l_0}}^- (l_2). \] Hence \( l_1 \) and \( l_2 \) are not neighbours in \( T_{l_0} \). Then \( |U_{l_0}| \geq 2, \quad |W_{l_0}| \geq 2 \) and by Lemma 5, \( D \) contains \( F \).  

Changing a bit the proof of Lemma 5, we obtain its improved version.  

**Corollary 2.** Given \( k > 0 \), let \( F \) be an oriented forest with \( k - 1 \) arcs, consisting of \( c \) balanced leaves-uniform oriented trees, each of them of order at least 4. Let \( c \) be odd. Let \( D = (X, Y; A(D)) \) be a bipartite digraph with \( \delta_X^+(D) \geq k - 1, \quad \delta_X^-(D) \geq \lfloor k/2 \rfloor - \lceil c/2 \rceil \) and \( \delta_Y^+(D) \geq \lfloor k/2 \rfloor - \lfloor c/2 \rfloor, \quad \delta_Y^-(D) \geq k - 1 \). Then \( D \) contains \( F \).  

**Proof.** Under the same notation as in the proof of Lemma 5 we change a bit the method. First we take oriented trees such that their leaves have indegree equal to 1. For every of them we apply successively the following reasoning. Let \( T_{l_0}, \quad i_0 \in \{1, \ldots, c\} \) be the oriented tree with leaves with indegree equal to 1. Let \( F_0 \) be the oriented forest consisting of already considered oriented trees. Let \( f_0 \) be the embedding of \( F_0 \) in \( D \). By Theorem 3 there is the embedding in \( D \) of the first considered oriented tree. For \( T_{l_0} \) we will set a vertex from \( X \) and a set \( N_y \) \((y \in Y)\). To obtain it we will use the same method as in the proof of Lemma 5, Part 2, for finding a vertex of \( X \) and the set \( N_{n+1} \) associated with the oriented tree \( T_{n+1} \). Then, also in the same way as for \( T_{n+1} \), we extend \( f_0 \) to an embedding in \( D \) of the oriented subforest of \( F \) indicated by \( V(F_0) \cup V(T_{l_0}) \setminus \{l_0, t_0\} \). Next we consider oriented trees with leaves with outdegree equal to 1. For each of them we apply successively nearly the same method as above. It differs in fact that for such an oriented tree we find a vertex from \( Y \) and a set \( N_x \) \((x \in X, \quad N_x \subset N_D^-(x) \subset Y)\). In this way we obtain an embedding in \( D \) of the oriented forest obtained from \( F \) by removing leaves \( l_i, \quad t_i, \quad i = 1, \ldots, c \). As in the proof of Lemma 5, Part 2, by Theorem 5 (of a system of distinct representatives) we find vertices on which we can map leaves such that their indegree is equal to 1 and their neighbours are mapped on vertices from \( Y \) and find vertices on which we can map leaves such that their outdegree is equal to 1 and their neighbours are mapped on vertices from \( X \). Then, since \( \delta_X^+ \geq k - 1 \) and \( \delta_Y^-(D) \geq k - 1 \), we can map remaining leaves.  

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**References**