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Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces ☆

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Abstract

The iteration scheme for families of nonexpansive mappings, essentially due to Halpern [Bull. Amer. Math. Soc. 73 (1967) 957–961], is established in a Banach space. The main theorem extends a recent result of O'Hara et al. [Nonlinear Anal. 54 (2003) 1417–1426] to a Banach space setting. For the same iteration scheme, with finitely many mappings, a complementary result to a result of Jung and Kim [Bull. Korean Math. Soc. 34 (1997) 93–102] (also Bauschke [J. Math. Anal. Appl. 202 (1996) 150–159]) is obtained by imposing other condition on the sequence of parameters. Our results also improve results in [C. R. Acad. Sci. Sér A–B Paris 284 (1977) 1357–1359; J. Math. Anal. Appl. 211 (1997) 71–83; Arch. Math. 59 (1992) 486–491] in framework of a Hilbert space.

Keywords: Nonexpansive mapping; Common fixed points; Iteration scheme; Sunny and nonexpansive retraction

1. Introduction

Let C be a nonempty closed convex subset of a Banach space E and let $T_1, \dots, T_N$ be nonexpansive mappings from C into itself (recall that a mapping $T : C \to C$ is *nonexpansive* if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$).

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We consider the iteration scheme: for N, nonexpansive mappings $T_1, T_2, \dots, T_N$ and $a, x_0 \in C$,

$$x_{n+1} = \lambda_{n+1} a + (1 - \lambda_{n+1}) T_{n+1} x_n, \quad n \geqslant 0. \tag{1}$$

In 1967, Halpern [9] firstly introduced the iteration scheme (1) for $a = 0$, $N = 1$ (that is, he considered only one mapping T); see also Browder [3]. He pointed out that the conditions $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$ are necessary in the sense that, if the iteration scheme (1) converges to a fixed point of T, then these conditions must be satisfied. Ten years later, Lions [12] investigated the general case in Hilbert space under the conditions $\lim_{n\to\infty} \lambda_n = 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\lim_{n\to\infty} (\lambda_n - \lambda_{n+1})/\lambda_{n+1}^2 = 0$ on the parameters. However, Lions' conditions on the parameters were more restrictive and did not include the natural candidate $\lambda_n = 1/n + 1$. In 1980, Reich [16] gave the iteration scheme (1) for $N = 1$ in the case when E is uniformly smooth and $\lambda_n = n^{-a}$ with $0 < a < 1$.

In 1992, Wittmann [20] studied the iteration scheme (1) for $N = 1$ in the case when E is a Hilbert space and $\{\lambda_n\}$ satisfies

$$0 \leqslant \lambda_n \leqslant 1, \quad \lim_{n\to\infty} \lambda_n = 0, \quad \sum_{n=1}^{\infty} \lambda_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$$

In 1994, Reich [17] obtained a strong convergence of the iterates (1) for $N = 1$ with two necessary and decreasing conditions on parameters for convergence in the case when E is uniformly smooth with a weakly continuous duality mapping. In 1996, Bauschke [2] improves results of Wittmann to finitely many mappings with additional condition on the parameters $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}| < \infty$, where $T_n =: T_{n \bmod N}$, $N > 1$. He also provided an algorithmic proof which has been used successfully, with modifications, by many authors [5,13,18,21,22]. In 1997, Jung and Kim [10] extended Bauschke's result to a Banach space and Shioji and Takahashi [19] improved Wittmann's result to a Banach space. Shimizu and Takahashi [18], in 1997, dealt with the above iteration scheme with the necessary conditions on the parameters and some additional conditions imposed on the mappings in a Hilbert space.

Very recently, O'Hara et al. [13] generalized the result of Shimizu and Takahashi [18] and proved a result of Bauschke [1] by imposing a new condition on the parameters, $\lim_{n\to\infty} \lambda_n/\lambda_{n+N} = 1$, in the framework of a Hilbert space, which is not comparable with Bauschke's condition $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}| < \infty$.

In this paper, we establish the strong convergence of the iteration scheme $\{x_n\}$ defined by (1) for infinitely many nonexpansive mappings in a uniformly smooth Banach space with a weakly sequentially continuous duality mapping. The results extend results of O'Hara et al. [13] to a Banach space setting. Then we obtain a complementary result to a result of Jung and Kim [10] (also Bauschke [2]) for the same iteration scheme, with finitely many mappings. Our main results also improve and unify results in [12,18,20] in Hilbert spaces.

2. Preliminaries and lemmas

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f \rangle$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$

(respectively $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (respectively weak, weak*) convergence of the sequence $\{x_n\}$ to x.

The norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth)* if

$$\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t} \tag{2}$$

exists for each x, y in its unit sphere $U = \{x \in E\colon \|x\| = 1\}$. It is said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth)* if the limit in (2) is attained uniformly for $(x, y) \in U \times U$.

The (normalized) *duality* mapping J from E into the family of nonempty (by Hahn–Banach theorem) weak-star compact subsets of its dual E^* is defined by

$$J(x) = \{f \in E^*\colon \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$$

for each $x \in E$. It is single valued if and only if E is smooth. It is also well known that if E has a uniformly Fréchet differentiable norm, J is uniformly continuous on bounded subsets of E (cf. [4,6]). Suppose that J is single valued. Then J is said to be *weakly sequentially continuous* if for each $\{x_n\} \in E$ with $x_n \rightharpoonup x$, $J(x_n) \overset{*}{\rightharpoonup} J(x)$.

We need the following lemma for the proof of our main results, which was given in Jung and Morales [11]. It is actually Lemma 1 of Petryshyn [15] (also see Asplund [1]).

Lemma 1. *Let X be a real Banach space and let J be the normalized duality mapping. Then for any given $x, y \in X$, we have*

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x+y)\rangle \tag{3}$$

for all $j(x+y) \in J(x+y)$.

A Banach space E is said to satisfy *Opial's condition* [14] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$. We know that if E admits a weakly sequentially continuous duality mapping, then E satisfies Opial's condition; see [8].

Recall that a mapping T defined on a subset C of a Banach space E (and taking values in E) is said to be *demiclosed* if for any sequence $\{u_n\}$ in C the following implication holds:

$$u_n \rightharpoonup u \quad \text{and} \quad \lim_{n\to\infty} \|Tu_n - w\| = 0$$

implies

$$u \in C \quad \text{and} \quad Tu = w.$$

The following lemma can be found in [7, p. 108].

Lemma 2. *Let E be a reflexive Banach space which satisfies Opial's condition, let C be a nonempty closed convex subset of E, and suppose $T : C \to E$ is nonexpansive. Then the mapping $I - T$ is demiclosed on C, where I is the identity mapping.*

Let C be a nonempty closed convex subset of E. A mapping Q of C into C is said to be a *retraction* if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Qz = z$ for every $z \in R(Q)$, where $R(Q)$ is range of Q. Let D be a subset of C and let Q be a mapping of C into D. Then Q is said to be *sunny* if each point on the ray $\{Qx + t(x - Qx)\colon\ t > 0\}$ is mapped by Q back onto Qx, in other words,

$$Q\big(Qx + t(x - Qx)\big) = Qx$$

for all $t \geqslant 0$ and $x \in C$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D; for more details, see [6].

The following lemma is well known (cf. [6, p. 48]).

Lemma 3. *Let C be a nonempty closed convex subset of a smooth Banach space E, D a subset of C, $J : E \to E^*$ the duality mapping of E, and $Q : C \to D$ a retraction. Then the following are equivalent*:

(a) $\langle x - Qx, J(y - Qx)\rangle \leqslant 0$ *for all $x \in C$ and $y \in D$*;
(b) $\|Qz - Qw\|^2 \leqslant \langle z - w, J(Qz - Qw)\rangle$ *for all z and w in C*;
(c) *Q is both sunny and nonexpansive.*

Finally, we need the following lemma.

Lemma 4 (Xu [21]). *Let $\{\lambda_n\}$ be a sequence in $(0, 1)$ that satisfies $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Let $\{a_n\}$ be a sequence of nonnegative real numbers that satisfies any one of the following conditions*:

(a) *For all $\varepsilon > 0$, there exists an integer $N \geqslant 1$ such that for all $n \geqslant N$,*

$$a_{n+1} \leqslant (1 - \lambda_n)a_n + \lambda_n\varepsilon;$$

(b) $a_{n+1} \leqslant (1 - \lambda_n)a_n + \mu_n$, $n \geqslant 0$, *where $\mu_n \geqslant 0$ satisfies* $\lim_{n\to\infty} \mu_n/\lambda_n = 0$;
(c) $a_{n+1} \leqslant (1 - \lambda_n)a_n + \lambda_n c_n$, *where* $\limsup_{n\to\infty} c_n \leqslant 0$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

First, we study the strong convergence result in a Banach space which generalizes Theorem 3.3 of O'Hara et al. [13].

Theorem 5. *Let E be a uniformly smooth Banach space with a weakly sequentially continuous duality mapping $J : E \to E^*$, C a nonempty closed convex subset of E, and $T_n : C \to C$ $(n = 1, 2, 3, \ldots)$ nonexpansive mappings such that*

$$F := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset.$$

Assume that $V_1, \dots, V_N : C \to C$ are nonexpansive mappings with the property: for all $k = 1, 2, \dots, N$ and for any bounded subset $\tilde{C}$ of C, there holds

$$\lim_{n\to\infty} \sup_{x\in\tilde{C}} \big\| T_n x - V_k(T_n x)\big\| = 0. \tag{4}$$

Let $\{\lambda_n\}$ be a sequence in $(0,1)$ which satisfies $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. For any a and x_0 in C, define

$$x_{n+1} = \lambda_{n+1} a + (1-\lambda_{n+1}) T_{n+1} x_n, \quad n \geqslant 0.$$

Then the sequence $\{x_n\}$ converges strongly to $Q_{F(V)} a$, where Q is a sunny nonexpansive retraction of C onto $F(V) := \bigcap_{k=1}^{N} \mathrm{Fix}(V_k)$.

Proof. First, we note that assumption (4) implies that $\bigcap_{k=1}^{N} \mathrm{Fix}(V_k) \supset F$. Note that $\{x_n\}$ is bounded since $F \neq \emptyset$. In fact, by induction, we show that $\|x_n - z\| \leqslant \max\{\|x_0 - z\|, \|a - z\|\}$ for all $n \geqslant 0$ and all $z \in F$. The result is clearly true for $n = 0$. Suppose the result is true for n. Let $z \in F$. Then by the nonexpansivity of T_{n+1},

$$\begin{aligned}
\|x_{n+1} - z\| &= \big\|\lambda_{n+1} a + (1-\lambda_{n+1}) T_{n+1} x_n - z\big\| \\
&\leqslant \lambda_{n+1}\|a - z\| + (1-\lambda_{n+1})\|T_{n+1} x_n - z\| \\
&\leqslant \lambda_{n+1}\|a - z\| + (1-\lambda_{n+1})\|x_n - z\| \\
&\leqslant \lambda_{n+1} \max\big\{\|x_0 - z\|, \|a - z\|\big\} \\
&\quad + (1-\lambda_{n+1}) \max\big\{\|x_0 - z\|, \|a - z\|\big\} \\
&= \max\big\{\|x_0 - z\|, \|a - z\|\big\}.
\end{aligned}$$

Moreover, since for all $n \geqslant 0$ and for any $z \in F$,

$$\begin{aligned}
\|T_{n+1} x_n\| &\leqslant \|T_{n+1} x_n - z\| + \|z\| \leqslant \|x_n - z\| + \|z\| \\
&\leqslant \max\big\{\|x_0 - z\|, \|a - z\|\big\} + \|z\|,
\end{aligned}$$

it follows that $\{T_{n+1} x_n\}$ is bounded. Since

$$\|x_{n+1} - T_{n+1} x_n\| = \lambda_{n+1}\|a - T_{n+1} x_n\| \leqslant \lambda_{n+1}\big(\|a\| + \|T_{n+1} x_n\|\big) \leqslant \lambda_{n+1} M$$

for some M, we also have

$$\lim_{n\to\infty} \|x_{n+1} - T_{n+1} x_n\| = 0. \tag{5}$$

Let a subsequence $\{T_{n_j+1} x_{n_j}\}$ of $\{T_{n+1} x_n\}$ be such that

$$\begin{aligned}
&\lim_{j\to\infty} \big\langle a - Q_{F(V)} a, J(x_{n_j+1} - Q_{F(V)} a)\big\rangle \\
&\quad = \limsup_{n\to\infty} \big\langle a - Q_{F(V)} a, J(x_{n+1} - Q_{F(V)} a)\big\rangle
\end{aligned}$$

and $T_{n_j+1} x_{n_j} \rightharpoonup p$ for some $p \in C$. By assumption, we have for any $k = 1, 2, \dots, N$ and for $\tilde{C} = \{x_n\}$,

$$0 = \lim_{n\to\infty} \sup_{x\in\tilde{C}} \left\| T_{n+1}x - V_k(T_{n+1}x) \right\| \geqslant \limsup_{n\to\infty} \left\| T_{n+1}x_n - V_k(T_{n+1}x_n) \right\|$$
$$\geqslant \limsup_{j\to\infty} \left\| T_{n_j+1}x_{n_j} - V_k(T_{n_j+1}x_{n_j}) \right\|,$$

and so

$$\lim_{j\to\infty} \left\| T_{n_j+1}x_{n_j} - V_k(T_{n_j+1}x_{n_j}) \right\| = 0 \quad \text{for all } k = 1, 2, \ldots, N.$$

Thus, by Lemma 2, we have $p \in \mathrm{Fix}(V_k)$ for $k = 1, 2, \ldots, N$, that is, $p \in \bigcap_{k=1}^{N} \mathrm{Fix}(V_k)$.

On the other hand, since E is uniformly smooth, F is a sunny nonexpansive retract of C (cf. [6, p. 49]). Thus, by weakly sequentially continuity of duality mapping J and Lemma 3, we have

$$\begin{aligned} &\limsup_{n\to\infty} \left\langle a - Q_{F(V)}a, J(T_{n+1}x_n - Q_{F(V)}a) \right\rangle \\ &\quad = \lim_{j\to\infty} \left\langle a - Q_{F(V)}a, J(T_{n_j+1}x_{n_j} - Q_{F(V)}a) \right\rangle \\ &\quad = \left\langle a - Q_{F(V)}a, J(p - Q_{F(V)}a) \right\rangle \leqslant 0. \end{aligned} \tag{6}$$

This together with (5) implies that

$$\limsup_{n\to\infty} \left\langle a - Q_{F(V)}a, J(x_{n+1} - Q_{F(V)}a) \right\rangle \leqslant 0. \tag{7}$$

Since $(1 - \lambda_{n+1})(T_{n+1}x_n - Q_{F(V)}a) = (x_{n+1} - Q_{F(V)}a) - \lambda_{n+1}(a - Q_{F(V)}a)$, by using the inequality (3) in Lemma 1, we have

$$\begin{aligned} \|x_{n+1} - Q_{F(V)}a\|^2 &= \left\| (1 - \lambda_{n+1})(T_{n+1}x_n - Q_{F(V)}a) + \lambda_{n+1}(a - Q_{F(V)}a) \right\|^2 \\ &\leqslant (1 - \lambda_{n+1})^2 \|T_{n+1}x_n - Q_{F(V)}a\|^2 \\ &\quad + 2\lambda_{n+1} \left\langle a - Q_{F(V)}a, J(x_{n+1} - Q_{F(V)}a) \right\rangle \\ &\leqslant (1 - \lambda_{n+1}) \|x_n - Q_{F(V)}a\|^2 \\ &\quad + 2\lambda_{n+1} \left\langle a - Q_{F(V)}a, J(x_{n+1} - Q_{F(V)}a) \right\rangle. \end{aligned} \tag{8}$$

Now, let $\varepsilon > 0$ be arbitrary. Then by (7), there exists N_ε such that

$$\left\langle a - Q_{F(V)}a, J(x_{n+1} - Q_{F(V)}a) \right\rangle \leqslant \frac{\varepsilon}{2} \quad \text{for all } n \geqslant N_\varepsilon.$$

Thus, from (8), we have

$$\|x_{n+1} - Q_{F(V)}a\|^2 \leqslant (1 - \lambda_{n+1}) \|x_n - Q_{F(V)}a\|^2 + \lambda_{n+1}\varepsilon. \tag{9}$$

Putting $a_n = \|x_n - Q_{F(V)}a\|^2$, we have from (9),

$$a_{n+1} \leqslant (1 - \lambda_{n+1}) a_n + \lambda_{n+1}\varepsilon.$$

It follows from Lemma 4 that $a_n \to 0$ and hence $\{x_n\}$ converges strongly to $Q_{F(V)}a$ This completes the proof. □

As a direct consequence, we have the following

Corollary 6 (O'Hara et al. [13, Theorem 3.3]). *Let H be a Hilbert space, C a nonempty closed convex subset of H, and $T_n : C \to C$ $(n = 1, 2, 3, \ldots)$ nonexpansive mappings such that*

$$F := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset.$$

Assume that $V_1, \ldots, V_N : C \to C$ are nonexpansive mappings with the property: for all $k = 1, 2, \ldots, N$ and for any bounded subset $\tilde{C}$ of C, there holds

$$\lim_{n\to\infty} \sup_{x\in\tilde{C}} \left\| T_n x - V_k(T_n x) \right\| = 0.$$

Let $\{\lambda_n\}$ be a sequence in $(0, 1)$ which satisfies $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} = \lambda_n = \infty$. For any a and x_0 in C, define

$$x_{n+1} = \lambda_{n+1} a + (1 - \lambda_{n+1}) T_{n+1} x_n, \quad n \geqslant 0.$$

Then the sequence $\{x_n\}$ converges strongly to $P_{F(V)} a$, where P is the nearest point projection of C onto $F(V) := \bigcap_{k=1}^{N} \mathrm{Fix}(V_k)$.

Proof. Note that the nearest point projection P of C onto F is a sunny nonexpansive retraction. Thus the result follows from Theorem 5. □

As in [13], by using Theorem 5 together with Lemmas 3.1 and 3.2 of [13] (Lemma 1 of [18]), we can also obtain the following result.

Corollary 7 (O'Hara et al. [13, Corollary 3.4]). *Let E be a Banach space, C a nonempty closed convex subset of E, and $T, S : C \to C$ nonexpansive mappings with fixed points.*

(a) *Set $T_n(x) = \frac{1}{n} \sum_{j=0}^{n-1} T^j x$ for $n \geqslant 1$ and $x \in C$. For $x_0, a \in C$, define*

$$x_{n+1} = \lambda_{n+1} a + (1 - \lambda_{n+1}) T_{n+1} x_n, \quad n \geqslant 0.$$

If E is a uniformly convex and uniformly smooth Banach space with a weakly sequentially continuous duality mapping, then the sequence $\{x_n\}$ converges strongly to $Q_F a$, where Q is a sunny nonexpansive retraction of C onto $F := \mathrm{Fix}(T)$.

(b) *Set $T_n(x) = \frac{2}{n(n+1)} \sum_{k=0}^{n-1} \sum_{i+j=k} S^i T^j(x)$ for $n \geqslant$ and $c \in C$, define*

$$x_{n+1} = \lambda_{n+1} a + (1 - \lambda_{n+1}) T_{n+1} x_n, \quad n \geqslant 0.$$

Suppose that $ST = TS$ and $\mathrm{Fix}(S) \cap \mathrm{Fix}(T) \neq \emptyset$. If E is a Hilbert space H, then the sequence $\{x_n\}$ converges strongly to $P_{F(ST)} a$, where P is the nearest point projection of C onto $F(ST) := \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$.

Remark 8. (1) Corollary 7(a) extends Corollary 3.4(a) in [13] to a Banach space setting.

(2) Theorem 1 of Shimizu and Takahashi [18] is just Corollary 7(b).

Now we consider the results developed by Bauschke [2] (also Jung and Kim [10]), in which he defined the following control conditions on the parameters $\{\lambda_n\}$:

(B1) $\lim_{n\to\infty} \lambda_n = 0$;

(B2) $\sum_{n=1}^{\infty} \lambda_n = \infty$; equivalently $\prod_{n=1}^{\infty}(1-\lambda_n) = 0$;

(B3) $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}| < \infty$.

We will replace (B3) by the condition

(N3) $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+N}} = 1$.

This condition also improves Lions' condition [12],

(L3) $\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}^2} = 0$.

Remark 9. Both (N3) and (B3) cover the natural candidate of $\lambda_n = \frac{1}{n+1}$, but (L3) does not. However, (B3) and (N3) are independent of each other. For more details, see [21].

We will give a complementary result to Theorem 1 of Jung and Kim [10] (also Theorem 3.1 of Bauschke [2]) with condition (B3) replaced by condition (N3).

We consider N mappings $T_1, T_2, \dots, T_N$. For $n > N$, set $T_n := T_{n \bmod N}$, where $n \bmod N$ is defined as follows: if $n = kN + l$, $0 \leqslant l < N$, then

$$n \bmod N := \begin{cases} l & \text{if } l \neq 0, \\ N & \text{if } l = 0. \end{cases}$$

Theorem 10. *Let E be a uniformly smooth Banach space with a weakly sequentially continuous duality mapping $J : E \to E^*$ and C a nonempty closed convex subset of E. Let $T_1, \dots, T_N$ be nonexpansive mappings from C into itself with $F := \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$ nonempty and*

$$F = \mathrm{Fix}(T_N \dots T_1) = \mathrm{Fix}(T_1 T_N \dots T_3 T_2) = \dots = \mathrm{Fix}(T_{N-1} T_{N-2} \dots T_1 T_N).$$

Let $\{\lambda_n\}$ be a sequence in $(0,1)$ which satisfies

(N1) $\lim_{n\to\infty} \lambda_n = 0$;

(N2) $\sum_{n=1}^{\infty} \lambda_n = \infty$;

(N3) $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+N}} = 1$.

For any a and x_0 in C, define

$$x_{n+1} = \lambda_{n+1} a + (1 - \lambda_{n+1}) T_{n+1} x_n, \quad n \geqslant 0.$$

Then the sequence $\{x_n\}$ converges strongly to $Q_F a$, where Q is a sunny nonexpansive retraction of C onto F.

Proof. We follows the same idea as in the proof of Theorem 1 in [10]. So we just sketch it. As in proof of Theorem 5, we can obtain the following facts and so the proofs are omitted:

(1) $\|x_n - z\| \leqslant \max\{\|x_0 - z\|, \|a - z\|\}$ for all $n \geqslant 0$ and for all $z \in F$;
(2) $\{x_n\}$ is bounded;
(3) $\{T_{n+1}x_n\}$ is bounded;
(4) $x_{n+1} - T_{n+1}x_n \to 0$.

Since (N3) is different from the condition (A3) in [2] (that is, (B3) above), we give the details of proof for $x_{n+N} - x_n \to 0$ as in [13]. By (3) above, there exists a constant $L > 0$ such that for all $n \geqslant 1$,

$$\|z - T_{n+1}x_n\| \leqslant L.$$

Since for all $n \geqslant 1$, $T_{n+N} = T_n$, we have

$$\begin{aligned}\|x_{n+N} - x_n\| &= \big\|(\lambda_{n+N} - \lambda_n)(z - T_{n+N}x_{n+N-1}) \\ &\quad + (1 - \lambda_{n+N})(T_n x_{n+N-1} - T_n x_{n-1})\big\| \\ &\leqslant L|\lambda_{n+N} - \lambda_n| + (1 - \lambda_{n+N})\|x_{n+N-1} - x_{n-1}\| \\ &= (1 - \lambda_{n+N})\|x_{n+N-1} - x_{n-1}\| + \lambda_{n+N} L\left|1 - \frac{\lambda_n}{\lambda_{n+N}}\right|.\end{aligned}$$

By (N3), we have $\lim_{n\to\infty} L\left|1 - \frac{\lambda_n}{\lambda_{n+N}}\right| = 0$, and so by Lemma 4,

$$x_{n+N} - x_n \to 0.$$

By the proof in [2], we also have

$$x_n - T_{n+N} \dots T_{n+1}x_n \to 0. \tag{10}$$

Finally we prove the strong convergence of $\{x_n\}$. Let a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ be such that

$$\lim_{j\to\infty} \langle a - Q_F a, J(x_{n_j+1} - Q_F a)\rangle = \limsup_{n\to\infty} \langle a - Q_F a, J(x_{n+1} - Q_F a)\rangle.$$

We assume (after passing to another subsequence if necessary) that $n_j + 1 \bmod N = i$ for some $i \in \{1, \dots, N\}$ and that $x_{n_j+1} \rightharpoonup x$. From (10), it follows that $\lim_{j\to\infty} \|x_{n_j+1} - T_{i+N} \dots T_{i+1}x_{n_j+1}\| = 0$. Hence, by Lemma 2, we have $x \in \mathrm{Fix}(T_{i+N} \dots T_{i+1}) = F$.

On the other hand, since E is uniformly smooth, F is a sunny nonexpansive retract of C (cf. [6, p. 49]). Thus, by weakly sequentially continuity of duality mapping J and Lemma 3, we have

$$\begin{aligned}\limsup_{n\to\infty} \langle a - Q_F a, J(x_{n+1} - Q_F a)\rangle &= \lim_{j\to\infty} \langle a - Q_F a, J(x_{n_j+1} - Q_F a)\rangle \\ &= \langle a - Q_F a, J(x - Q_F a)\rangle \leqslant 0.\end{aligned} \tag{11}$$

Since $(1 - \lambda_{n+1})(T_{n+1}x_n - Q_F a) = (x_{n+1} - Q_F a) - \lambda_{n+1}(a - Q_F a)$, by Lemma 1, we have

$$\begin{aligned}\|x_{n+1} - Q_F a\|^2 &\leqslant (1 - \lambda_{n+1})^2\|T_{n+1}x_n - Q_F a\|^2 \\ &\quad + 2\lambda_{n+1}\langle a - Q_F a, J(x_{n+1} - Q_F a)\rangle \\ &\leqslant (1 - \lambda_{n+1})\|x_n - Q_F a\|^2 \\ &\quad + 2\lambda_{n+1}\langle a - Q_F a, J(x_{n+1} - Q_F a)\rangle.\end{aligned} \tag{12}$$

Now, let $\varepsilon > 0$ be arbitrary. Then by (11), there exists N_ε such that

$$\langle a - Q_F a, J(x_{n+1} - Q_F a)\rangle \leqslant \frac{\varepsilon}{2} \quad \text{for all } n \geqslant N_\varepsilon.$$

Thus, from (12), we have

$$\|x_{n+1} - Q_F a\|^2 \leqslant (1 - \lambda_{n+1})\|x_n - Q_F a\|^2 + \lambda_{n+1}\varepsilon.$$

Thus, it follows from Lemma 4 that $\{x_n\}$ converges strongly to $Q_F a$. This completes the proof. □

As an immediate consequence, we have the following

Corollary 11 (O'Hara et al. [13, Theorem 4.1]). *Let H be a Hilbert space, C a nonempty closed convex subset of H, and $T_1, \ldots, T_N$ nonexpansive mappings from C into itself with $F := \bigcap_{i=1}^N \mathrm{Fix}(T_i)$ nonempty and*

$$F = \mathrm{Fix}(T_N \ldots T_1) = \mathrm{Fix}(T_1 T_N \ldots T_3 T_2) = \cdots = \mathrm{Fix}(T_{N-1} T_{N-2} - T_1 T_N).$$

Let $\{\lambda_n\}$ be a sequence in $(0, 1)$ which satisfies (N1)–(N3) in Theorem 2. For any a and x_0 in C, define

$$x_{n+1} = \lambda_{n+1} a + (1 - \lambda_{n+1}) T_{n+1} x_n, \quad n > 0.$$

Then the sequence $\{x_n\}$ converges strongly to $P_F a$, where P is the nearest point projection of C onto F.

The following is a complementary result of the result of Wittmann [20].

Corollary 12. *Let H be a Hilbert space, C a nonempty closed convex subset of H, and T a nonexpansive mapping from C into itself with $\mathrm{Fix}(T) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence in $(0, 1)$ which satisfies (N1)–(N3) in Theorem 10. For any a and x_0 in C, define (with $N = 1$)*

$$x_{n+1} = \lambda_{n+1} a + (1 - \lambda_{n+1}) T x_n, \quad n \geqslant 0.$$

Then the sequence $\{x_n\}$ converges strongly to $P_F a$, where P is the nearest point projection of C onto F.

Let D be a subset of a Banach space E. Recall that a mapping $T : D \to E$ is said to be firmly nonexpansive if for each x and y in D, the convex function $\phi : [0, 1] \to [0, \infty)$ defined by

$$\phi(s) = \big\|(1 - s)x + sTx - \big((1 - s)y + sTy\big)\big\|$$

is nonincreasing. Since ϕ is convex, it is easy to check that a mapping $T : D \to E$ is firmly nonexpansive if and only if

$$\|Tx - Ty\| \leqslant \big\|(1 - t)(x - y) + t(Tx - Ty)\big\|$$

for each x and y in D and $t \in [0, 1]$. It is clear that every firmly nonexpansive mapping is nonexpansive (cf. [6,7]).

The following result extends a Lions-type iteration scheme [12] with the condition (N3) to a Banach space setting.

Corollary 13. *Let E be a uniformly smooth Banach space with a weakly sequentially continuous duality mapping $J : E \to E^*$ and C a nonempty closed convex subset of E. Let $T_1, \dots, T_N$ be firmly nonexpansive mappings from C into itself with $F := \bigcap_{i=1}^N \mathrm{Fix}(T_i)$ nonempty and*

$$F = \mathrm{Fix}(T_N \dots T_1) = \mathrm{Fix}(T_1 T_N \dots T_3 T_2) = \dots = \mathrm{Fix}(T_{N-1} T_{N-2} \dots T_1 T_N).$$

Let $\{\lambda_n\}$ be a sequence in $[0, 1)$ which satisfies (N1)–(N3) *in Theorem* 10. *For any a and x_0 in C, define*

$$x_{n+i} = \lambda_{n+1} a + (1 - \lambda_{n+1}) T_{n+1} x_n, \quad n \geqslant 0.$$

Then the sequence $\{x_n\}$ converges strongly to $Q_F a$, where Q is a sunny nonexpansive retraction of C onto F.

Remark 14. (1) In Hilbert space, Lions [12, Théorèm 4] had used

(L1) $\lim_{n\to\infty} \lambda_n = 0$;

(L2) $\sum_{k=1}^\infty \lambda_{kN+i} = \infty$ for all $i = 0, \dots, N-1$, which is more restrictive than (N2); and

(L3)′ $\lim_{k\to\infty} \frac{\sum_{i=1}^N |\lambda_{kN+i} - \lambda_{(k-1)N+i}|}{(\sum_i^N \lambda_{kN+i})^2} = 0$ in place of (B3).

(2) In general, (B3) and (L3)′ are independent, even when $N = 1$. For more details, see [2].

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References

[1] E. Asplund, Positivity of duality mappings, Bull. Amer. Math. Soc. 73 (1967) 200–203.
[2] H.H. Bauschke, The approximation of fixed points of compositions of nonexpansive mappings in Hilbert space, J. Math. Anal. Appl. 202 (1996) 150–159.
[3] F.E. Browder, Convergence of approximations to fixed points of nonexpansive mappings in Banach spaces, Arch. Rational Mech. Anal. 24 (1967) 82–90.
[4] J. Diestel, Geometry of Banach Spaces, Lecture Notes in Math., vol. 485, Springer-Verlag, Berlin, 1975.
[5] F. Deutsch, I. Yamada, Minimizing certain convex functions over the intersection of the fixed point sets of nonexpansive mappings, Numer. Funct. Anal. Optim. 19 (1998) 33–56.
[6] K. Goebel, S. Reich, Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings, Dekker, New York, 1984.
[7] K. Goebel, W.A. Kirk, Topics in Metric Fixed Point Theory, Cambridge Stud. Adv. Math., vol. 28, Cambridge Univ. Press, Cambridge, UK, 1990.

[8] J.P. Gossez, E.L. Dozo, Some geometric properties related to the fixed point theory for nonexpansive mappings, Pacific J. Math. 40 (1972) 565–573.
[9] B. Halpern, Fixed points of nonexpansive maps, Bull. Amer. Math. Soc. 73 (1967) 957–961.
[10] J.S. Jung, T.H. Kim, Convergence of approximate sequences for compositions of nonexpansive mappings in Banach spaces, Bull. Korean Math. Soc. 34 (1997) 93–102.
[11] J.S. Jung, C. Morales, The Mann process for perturbed m-accretive operators in Banach spaces, Nonlinear Anal. 46 (2001) 231–243.
[12] P.L. Lions, Approximation de points fixes de contractions, C. R. Acad. Sci. Sér A–B Paris 284 (1977) 1357–1359.
[13] J.G. O'Hara, P. Pillay, H.K. Xu, Iterative approaches to finding nearest common fixed points of nonexpansive mappings in Hilbert spaces, Nonlinear Anal. 54 (2003) 1417–1426.
[14] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967) 591–597.
[15] W.V. Petryshn, A characterization of strictly convexity of Banach spaces and other uses of duality mappings, J. Funct. Anal. 6 (1970) 282–291.
[16] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, J. Math. Anal. Appl. 75 (1980) 287–292.
[17] S. Reich, Approximating fixed points of nonexpansive mappings, Panamer. Math. J. 4 (1994) 486–491.
[18] T. Shimizu, W. Takahashi, Strong convergence to common fixed points of families of nonexpansive mappings, J. Math. Anal. Appl. 211 (1997) 71–83.
[19] N. Shioji, W. Takahashi, Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces, Proc. Amer. Math. Soc. 125 (1997) 3641–3645.
[20] R. Wittmann, Approximation of fixed points of nonexpansive mappings, Arch. Math. 59 (1992) 486–491.
[21] H.K. Xu, An iterative approach to quadratic optimization, J. Optim. Theory Appl. 116 (2003) 659–678.
[22] I. Yamada, N. Ogura, Y. Yamashita, K. Sakaniwa, Quadratic approximation of fixed points of nonexpansive mappings in Hilbert spaces, Numer. Funct. Anal. Optim. 19 (1998) 165–190.