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NOTE

THE CHROMATICITY OF WHEELS WITH A MISSING SPOKE

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We show that the graph U_{n+1} obtained from the Wheel W_{n+1} by deleting one spoke is chromatically unique if $n \geqslant 3$ is odd.

Let G be a finite undirected graph without loops or multiple edges. Let $P(G; \lambda)$ denote its chromatic polynomial. Two graphs are said to be *chromatically equivalent* if they have the same chromatic polynomial. G is said to be *chromatically unique* if it is chromatically equivalent only to itself. A *wheel* is a graph obtained by taking the join of K_1 and the cycle C_n on n vertices. It is denoted by W_{n+1}. The edges which join K_1 to the vertices of C_n are called the *spokes* of the wheel. In [8], Xu and Li, in an attempt to answer a question raised by Chao and Whitehead Jr. [1], showed that W_{n+1} is chromatically unique if $n \geqslant 4$ is even. Further, they showed that W_8 is not chromatically unique and went on to conjecture that W_{n+1} is not chromatically unique if $n \geqslant 9$ is odd. Earlier than this it was known that W_4 and W_5 are chromatically unique while W_6 is not (see [2]). In [6], Read showed that W_{10} is chromatically unique. The conjecture remains to be settled for odd $n \geqslant 11$. Here we shall show that the graph U_{n+1} obtained by deleting a spoke from W_{n+1} is chromatically unique if $n \geqslant 3$ is odd. Note that U_4, U_5 and U_6 are all chromatically unique while U_7 is not (see [2]). Note also that those wheels with $n-3$ and $n-4$ consecutive missing spokes have been shown to be chromatically unique for $n \geqslant 5$ and $n \geqslant 6$ respectively (see [2, Theorems 2, 3]).

Suppose G has p vertices and q edges. If $P(G; \lambda) = \sum_{i=1}^{p} a_i \lambda^i$, then it is well-known that the coefficients a_p, a_{p-1} and a_{p-2} are equal to 1, $-q$ and $\binom{q}{2} - A_G$, respectively, where A_G is the number of triangles in G. A *pure quadrilateral* is a quadrilateral having no diagonals. Let B_G and C_G denote respectively the numbers of pure quadrilaterals and complete subgraphs on 4 vertices in G. Then $a_{p-3} = -\binom{q}{3} + (q-2)A_G + B_G - 2C_G$ (see [3, Theorem 1]). It follows that if G and H are chromatically equivalent, then it is necessary that they have the same numbers of vertices, edges and triangles. Further, $B_G - 2C_G = B_H - 2C_H$.

In computing the chromatic polynomial of U_{n+1}, we make use of Theorem 1 of [5]. This states that $P(G; \lambda) = P(G'; \lambda) + P(G''; \lambda)$ where G' is obtained from G

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by adding an edge while G'' is obtained from G by identifying the end vertices of this edge. Here we let $G = U_{n+1}$ and $G' = W_{n+1}$ in which case G'' is the join of K_1 and the path on $n-1$ vertices. So $P(G'; \lambda) = \lambda[(\lambda-2)^n + (-1)^n(\lambda-2)]$ and $p(G''; \lambda) = \lambda(\lambda-1)(\lambda-2)^{n-2}$. Consequently, when n is odd

$$\begin{aligned}P(U_{n+1}; \lambda) &= \lambda[(\lambda-2)^n - (\lambda-2)] + \lambda(\lambda-1)(\lambda-2)^{n-2}\\ &= \lambda(\lambda-1)(\lambda-2)f(\lambda),\end{aligned}$$

where $f(\lambda) = (\lambda-2)^{n-2} + (\lambda-2)^{n-4} - (\lambda-2)^{n-5} + \cdots - (\lambda-2)^2 + (\lambda-2) - 1$.

In proving the theorem, we are making use of the facts that U_{n+1} is uniquely 3-colorable if $n \geqslant 3$ is odd, and that the subgraph induced by any two color classes of the unique 3-coloring is a tree. Further, U_{n+1} contains $n-2$ triangles and exactly one pure quadrilateral.

Throughout we let $N(x)$ denote the neighbourhood of x. We may sometimes use it to denote the graph induced by $N(x)$.

Lemma. *Let T be a tree and let a new vertex x be adjacent to some vertices of T. Then the number of triangles containing x is $d(x) - k$ where k is the number of components in $N(x)$.*

Proof. By using Lemma 2 of [8] and by induction on k, we obtain the conclusion immediately. □

Theorem. *U_{n+1} is chromatically unique if $n \geqslant 3$ is odd.*

Proof. U_4 is the degenerate case and since it is chromatically unique, we may assume that $n \geqslant 5$.

Suppose there is a graph Y which is chromatically equivalent to U_{n+1}. Then it is clear that Y is connected and has $n+1$ vertices and $2n-1$ edges. Moreover, Y has $n-2$ triangles.

Since $(\lambda-1)^2$ does not divide $P(Y; \lambda)$, Y has no cut vertex (see [7, Theorem 1]). Also, since $(\lambda-2)^2$ does not divide $P(Y; \lambda)$, the degree of any vertex of every triangle in Y is at least 3 (see [2, Lemma 1]). Since $\chi(Y) = 3$, Y has no complete subgraph on 4 vertices. This means that Y has exactly one pure quadrilateral.

Since $P(Y; 3) = 6$, Y is uniquely 3-colorable. Let the color classes of the unique 3-coloring of Y be V_0, V_1 and V_2. For each $i = 0, 1, 2$, let $|V_i| = n_i$. So $n_0 + n_1 + n_2 = n + 1$.

Let the subgraph induced by $V_i \cup V_{i+1}$ be denoted by G_{i+2} where the subscripts are reduced modulo 3. By Theorem 12.16 of [4], each G_i is connected. So the number of edges in G_i is at least $n_{i+1} + n_{i+2} - 1$. Since

$$2n - 1 = \sum_{i=0}^{2} |E(G_i)| \geqslant \sum_{i=0}^{2} (n_{i+1} + n_{i+2} - 1) = 2n - 1,$$

we see that each G_i is a tree. Now the number of edges joining V_0 to G_0 is $2n-1-(n_1+n_2-1)=2n-1-(n-n_0)=n+n_0-1$.

We claim that $n_i \geqslant 2$ for any i. To see this, suppose $n_0=1$. Since V_0 has only one vertex, it is necessary that this vertex be adjacent to every vertex of G_0. By the lemma, Y has $n_1+n_2-1=n-1$ triangles, which is a contradiction.

Let $Q=\{x, y, z, w\}$ be the pure quadrilateral of Y. Since each G_i is a tree, it is not possible for Q to be contained in some G_i. Without loss of generality, we may assume that $x, y \in V_0$, $z \in V_1$ and $w \in V_2$ with x and y both adjacent to z and w.

We observe that if x is adjacent to a vertex u in $V(G_0)-\{z, w\}$, then y is not adjacent to u for otherwise $\{u, x, y, z\}$ or $\{u, x, y, w\}$ forms a pure quadrilateral in Y different from Q. In view of this observation, we see that $N(x)$ and $N(y)$ are not both trees. So by the lemma, the number of triangles containing x and y is at most $d(x)+d(y)-3$. So Y has at most

$$\sum_{v \in V_0} (d(v)-1)-1=(n+n_0-1)-n_0-1=n-2$$

triangles. According to the lemma, equality holds if and only if for each $v \in V_0$, $N(v)$ is a tree in G_0 with exactly one of $N(x)$ or $N(y)$ being a forest having exactly 2 connected components.

We assert that either x or y is adjacent to all the vertices of G_0. To see this, we assume that this is not the case and get a contradiction. We may assume that $N(x)$ is a tree and $N(y)$ is a forest with exactly 2 connected components. We consider two cases:

Case (i). $d(y)=2$. Let u be an end vertex of $N(x)$ in G_0. If $d(u)=1$ in G_0, then $u \notin \{z, w\}$ and u is adjacent to some $v \in V_0-\{x, y\}$ because $d(u) \geqslant 3$ in Y. Since $N(v)$ is a tree in G_0, and $|N(v)| \geqslant 3$, we must have $|N(x) \cap N(v)| \geqslant 3$ in which case there exists $t \in N(x) \cap N(v)$ such that $\{x, u, v, t\}$ forms a pure quadrilateral different from Q (see *Fig. 1*), and this is a contradiction.

If $d(u)>1$ in G_0, then since u is not a cut vertex in Y, there exists $v \in V_0-\{x, y\}$ such that $N(v)$ contains u and $|N(x) \cap N(v)| \geqslant 2$. Now $|N(v)| \geqslant 3$ and $N(v)$ is a tree. Again, if $|N(x) \cap N(v)| \geqslant 3$, there is a pure quadrilateral in Y different from Q. So $|N(x) \cap N(v)|=2$. Let u_1 be an end vertex of $N(v)$ in G_0. If $d(u_1)=1$ in G_0, then as before we get a pure quadrilateral in Y different from Q.

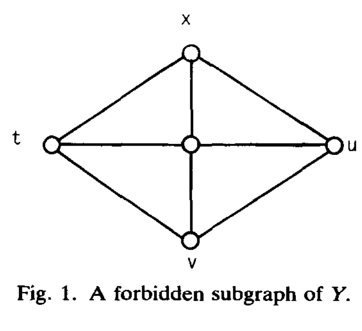

Fig. 1. A forbidden subgraph of Y.

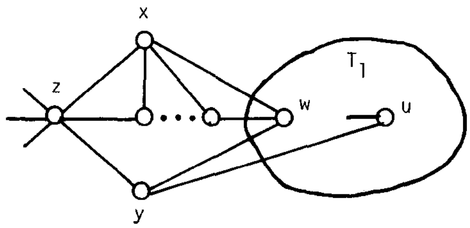

Fig. 2.

If $d(u_1) > 1$ in G_0, continue the same argument as before and in a finite number of steps we get an end vertex u_i of $N(v_i)$ where $v_i \in V_0 - \{x, y\}$ with $d(u_i) = 1$ in G_0 which will then lead to a contradiction.

Case (ii). $d(y) > 2$. Let $N(y) = T_1 \cup T_2$ where each T_i is a tree. Suppose $|V(T_1)| \geqslant 2$. Let u be an end vertex of T_1 and $u \notin \{w, z\}$. (See *Fig. 2.*) If $d(u) = 1$ in G_0, then Y has a pure quadrilateral different from Q. If $d(u) > 1$ in G_0, then argue as before, we eventually get an end vertex u_i of $N(v)$ for some $v \in V_0 - \{x, y\}$ with $d(u_i) = 1$ in G_0, and so there is a pure quadrilateral in Y different from Q. Hence $|V(T_i)| = 1$ for all i.

Since both cases lead to contradiction, we have the assertion. Further, $d(y) = 2$. Now if there is a $v \in V_0 - \{x, y\}$, then since $N(v)$ is a tree and $|N(v)| \geqslant 3$ we will get a pure quadrilaterial in Y different from Q which is not possible. Consequently $V_0 = \{x, y\}$.

Since $N(x)$ is G_0 and $N(y) = \{z, w\}$ and since the degree of any vertex of every triangle in Y is at least 3, we see that z and w are the only two end vertices of G_0. Consequently G_0 is a path on $n - 1$ vertices and so Y is isomorphic to U_{n+1}. □

Acknowledgments

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