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Solutions to Nonlinear Partial Differential Equations from Symmetry-Enhancing and Symmetry-Preserving Constraints

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We show how solutions to practical partial differential equations can be found by classical symmetry reductions of a larger system of equations including constraints that result in the enlarged system having a larger symmetry group or an identical symmetry group compared to the original target equation. Symmetry-enhancing constraints provide additional similarity solutions beyond those of the standard Lie algorithm for scalar equations. Symmetry-preserving constraints enable solutions to be found more easily, after reduction of variables. Examples are given for the cylindrical boundary layer equations and for a Navier–Stokes formulation of Schrödinger wave mechanics. © 1999 Academic Press

1. INTRODUCTION

Lie's classical algorithm [12] for finding continuous symmetry groups has been implemented on several computer algebra systems [10]. Hence there are advantages in incorporating *ad hoc* solution methods in the Lie symmetry algorithm. This has largely been achieved for the method of separation of variables [14], for convolution by fundamental solutions of linear initial value problems (see, e.g., [12]), and for nonlinear superposition principles [9].

Many solutions of a target partial differential equation (PDE), obtained by *ad hoc* methods, can in fact be interpreted as classical symmetry reductions following the addition of *symmetry-enhancing constraints*. We define these constraints as equations which when added to a target equation, result in the enlarged system having at least one additional symmetry not possessed by the target equation on its own. In Section 2.1,

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we illustrate this concept in terms of the *ad hoc method* of equation-splitting, whereby we "split" single PDEs into a larger system of PDEs that are chosen so that they have a larger classical Lie symmetry group than the original single equation. We show that this technique can lead to new symmetry reductions and new solutions. The cylindrical boundary layer equations are used as a practical example. In Section 2.2 we consider, as a further example of potential symmetry-enhancing constraints, the zero-vorticity condition added to the Navier–Stokes equations in 3 + 1 dimensions. For irrotational flow, the Hopf–Cole transformation achieves a linear canonical form, just as for the 1 + 1-dimensional Burgers' equation. This adds nothing new to fluid mechanics but it provides a new perspective on the classical formulation of quantum mechanics. If we replace the viscosity of externally forced, incompressible Navier–Stokes equations by a pure imaginary quantity, then we see that Schrödinger's equation follows from an order-$\hbar$ correction to Newton's second law.

As well, it may be possible to find new solutions to PDEs following the addition of *symmetry-preserving constraints*. We define these constraints as equations which when added to a target equation, result in the enlarged system having the same nontrivial symmetry group as that of the target equation on its own. In Section 3 we show how such constraints can lead to simpler quotient equations to solve, thus providing solutions that are otherwise unobtainable.

2. SYMMETRY-ENHANCING CONSTRAINTS

2.1. *Equation-Splitting*

In [1], Ames split the boundary layer equation

$$u_y u_{xy} - u_x u_{yy} = \nu u_{yyy}, \tag{2.1}$$

into the system

$$u_{yyy} = 0, \tag{2.2a}$$

$$u_y u_{xy} - u_x u_{yy} = 0, \tag{2.2b}$$

so that at least one of the equations in the system, could be solved in general. He substituted the solution to (2.2b), namely, $u = F(y + G(x))$; F, G arbitrary, into (2.2a) to find, if possible, a common solution. He thus found the solution

$$u = a(y + G(x))^2 + b(y + G(x)) + c,$$

where a, b, c are constants and G is an arbitrary function of x, to the system (2.2), which is also a solution to (2.1).

The approach of splitting a single PDE into a system of PDEs may be regarded as embedding the original equation in a larger system of PDEs by adding a side condition with a special property. In the general scheme of solution via differential constraints ([17, 16]) one merely requires the side condition to be compatible with the original equation. If we choose this side condition so that the enlarged system of PDEs has a larger Lie symmetry group than the original PDE, then this can be viewed as adding a symmetry-enhancing constraint.

As an example, the PDE

$$u_{xx} - u_t + uu_x + u_{xt} = u_t\left(\frac{u_x}{u} - \frac{u}{2}\right) \tag{2.3}$$

has only the classical translation symmetries generated by the vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t}$. However the system

$$\begin{aligned} u_{xx} &= u_t - uu_x, \\ u_{xt} &= u_t\left(\frac{u_x}{u} - \frac{u}{2}\right) \end{aligned} \tag{2.4}$$

has the added symmetry with generator

$$\Gamma = -x\frac{\partial}{\partial x} - 2t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \tag{2.5}$$

with which we find the solutions

$$u = \frac{4x}{x^2 + 2t} \quad \text{and} \quad u = \frac{2}{x},$$

to the system (2.4) and hence also to the PDE (2.3). We now concentrate on applying this same method to the axisymmetric boundary layer equation, and show that by using this method, we can recover the solutions to this PDE that were found by Burde [6].

In [6], Burde outlines a new ad hoc technique for finding explicit similarity solutions of PDEs. The technique is a variation of the direct method developed by Clarkson and Kruskal [8], whereby a similarity form of the solution is substituted into the PDE. However, whereas with the usual direct method, one imposes the requirement that the substitution leads to an ordinary differential equation, Burde instead imposes the weaker condition that the substitution of the similarity form be reduced to an overdetermined system of ODEs which can be solved in closed form (see also Hood [11]). Using this new technique, Burde finds interesting new solutions to the boundary layer problem which cannot be recovered by using the standard Lie group method of infinitesimal transformations or its nonclassical generalisations. However, we show that Burde's solutions and some others, can be recovered by splitting a single PDE into a system of PDEs with a large Lie classical symmetry group and then using the classical symmetries of this system.

We also show that two other solutions thus found can again not be recovered by using the classical and nonclassical methods of group invariant solutions.

The axisymmetric boundary layer equations take the form

$$\begin{gathered} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = UU' + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0, \end{gathered} \tag{2.6}$$

where $U = U(x)$.

Defining new variables $u = \frac{1}{r}\frac{\partial \psi}{\partial r}$, $v = -\frac{1}{r}\frac{\partial \psi}{\partial x}$, and $\mu = r^2/4$, Eq. (2.6) becomes

$$\psi_\mu \psi_{\mu x} - \psi_x \psi_{\mu\mu} - 4UU' = 2\nu(\mu\psi_{\mu\mu\mu} + \psi_{\mu\mu}). \tag{2.7}$$

A solution to (2.7) given in [6] is

$$\psi = 2\nu(2x + \phi(x)F(\eta)), \tag{2.8}$$

where

$$\eta = \frac{\mu}{(\phi(x))^2} + \phi'(x),$$

$$F(\eta) = c\eta - \frac{2}{\eta},$$

and

$$U(x) = \frac{\nu c}{\phi(x)}.$$

In [6], Burde shows that this similarity solution cannot be obtained by considering classical and nonclassical symmetries of (2.7). The special case of $\phi(x) = x^n$ is used to illustrate this point.

Here, we show that the similarity solution (2.8) can be obtained by considering the classical symmetries of the system

$$\begin{aligned} \psi_\mu \psi_{x\mu} - \psi_x \psi_{\mu\mu} - 4UU' &= j(x)\psi_{\mu\mu\mu} + h(x)\psi_{\mu\mu}, \\ 2\nu(\psi_{\mu\mu} + \mu\psi_{\mu\mu\mu}) &= j(x)\psi_{\mu\mu\mu} + h(x)\psi_{\mu\mu}, \end{aligned} \tag{2.9}$$

with

$$U(x) = \frac{\nu c}{\phi(x)}.$$

We note that all the solutions of the system (2.9) are solutions of (2.7).

For particular functions j and h of x, the system (2.9) has more classical symmetries than those of the original equation (2.7). Here we list some examples and give the corresponding solutions to (2.7).

(a) $j(x) = A\phi'(x)(\phi(x))^m$ and $h(x) = d$ with A, m, and d nonzero constants. The system (2.9) then has the additional symmetry generator,

$$\begin{aligned} \Gamma = &\left(\frac{2c_2\phi}{\phi' m}\right)\frac{\partial}{\partial x} \\ &+ \left(c_2\left(1 + \frac{2}{m}\right)\mu + \frac{Ac_2\phi'\phi^m}{2\nu}\left[\frac{2\phi\phi''}{(\phi')^2 m} + 1 - \frac{2}{m}\right]\right)\frac{\partial}{\partial\mu} \\ &+ \left(c_2\psi + c_2 d\left(x - \frac{2}{m}\frac{\phi}{\phi'}\right) + \alpha\right)\frac{\partial}{\partial\psi}, \end{aligned} \tag{2.10}$$

where c_2 and α are constants.

The corresponding similarity solution is

$$\psi = -dx - \frac{\alpha}{c_2} + \phi^{m/2}(x) f(\xi), \tag{2.11}$$

where

$$\xi = \phi^{-m/2-1}\mu - \frac{A}{2\nu}\phi^{m/2-1}\phi'.$$

Substitution of (2.11) into (2.9) gives the following ordinary differential equations for f,

$$2Af''' + mff'' + 2(f')^2 = 8\nu^2 c^2, \tag{2.12a}$$

$$\left(1 - \frac{d}{2\nu}\right)f'' + \xi f''' = 0. \tag{2.12b}$$

The general solution of (2.12b) is

$$f(\xi) = p\xi^{d/2\nu+1} + b\xi + q; \qquad d \neq -2\nu, \tag{2.13a}$$

$$= -p\log\xi + b\xi + q; \qquad d = -2\nu, \tag{2.13b}$$

where p, b, q are constants. Substitution of (2.13a) into (2.12a) gives some particular restrictions on the constants involved. Using these together with (2.13a) and (2.11), we obtain the following solutions to (2.7):

(i) $d = -4\nu$, $m = 2$, $b = \pm 2\nu c$, $q = 0$, $p = 2A$. In this case

$$\psi = 4\nu x - \frac{\alpha}{c_2} + 2\phi(x)\left[\frac{A}{\xi} \pm \nu c\xi\right], \tag{2.14}$$

where

$$\xi = \frac{\mu}{\phi^2} - \frac{A}{2\nu}\phi'.$$

When $A = -2\nu$, then this is the solution (2.8) given in [6].

(ii) $d = 2\nu$, $m = -4$, and $b^2 = 4\nu^2c^2 + 4pq$. Then we obtain the solution

$$\psi = -2\nu x - \frac{\alpha}{c_2} + (\phi(x))^{-2} f(\xi), \tag{2.15}$$

where

$$\xi = \phi\mu - \frac{A}{2\nu}\phi^{-3}\phi'$$

and

$$f(\xi) = p\xi^2 + \sqrt{4\nu^2c^2 + 4pq}\,\xi + q.$$

(iii) $d = 4\nu$, $m = -3$, $b = 0$, $q = 0$, and $p = 2\nu^2c^2/3A$. In this case

$$\psi = -4\nu x - \frac{\alpha}{c_2} + \frac{2\nu^2c^2}{3A}\left(\mu - \frac{A}{2\nu}\frac{\phi'}{\phi^3}\right)^3. \tag{2.16}$$

Also, using (2.13b), we obtain the solution

$$\psi = 2\nu x - \frac{\alpha}{c_2} + \phi^{m/2}(x)[\pm 2\nu c\xi + q], \tag{2.17}$$

where

$$\xi = \phi^{-m/2-1}\mu - \frac{A}{2\nu}\phi^{m/2-1}\phi'.$$

This is a particular case of the more general solution (2.21).

(b) $j(x) = A\phi'(x)(\phi(x))^m$ and $h(x) = 2\nu$ with A and m nonzero constants. The system (2.9) then has the additional symmetry generator

$$\Gamma = \left(\frac{2c_2\phi}{\phi' m}\right)\frac{\partial}{\partial x} + \left(c_2\mu + \frac{2c_2}{m}\mu + k(x)\right)\frac{\partial}{\partial \mu} + \left(c_2\psi + 2c_2\nu\left(x - \frac{2}{m}\frac{\phi}{\phi'}\right) + \alpha\right)\frac{\partial}{\partial \psi}, \tag{2.18}$$

where k is an arbitrary function of x.

Using (2.18), we find the following solution to (2.7),

$$\psi = -2\nu x - \frac{\alpha}{c_2} + (\phi(x))^{-2} f(\xi), \tag{2.19}$$

where

$$\xi = \phi\mu - q(x),$$

and $f(\xi) = p\xi^2 + \sqrt{4\nu^2c^2 + 4pd}\,\xi + d$ with q an arbitrary function of x, and with p and d constants.

(c) $j(x)$ arbitrary and $h(x) = 2\nu$. The system (2.9) then has the additional symmetry generator

$$\Gamma = k(x)\frac{\partial}{\partial \mu} + \frac{\partial}{\partial \psi}, \tag{2.20}$$

where k is an arbitrary function of x.

Using (2.20), we find the solution to (2.7),

$$\begin{aligned}\psi &= \left(\pm\sqrt{4(U(x))^2 + \alpha}\right)\mu + f(x) \\ &= \left(\pm\sqrt{\frac{4\nu^2c^2}{(\phi(x))^2} + \alpha}\right)\mu + f(x),\end{aligned} \tag{2.21}$$

where f is an arbitrary function of x.

(d) $j(x) = A\phi'(x)$ and $h(x) = d$ with A and d nonzero constants. The system (2.9) then has the additional symmetry generator

$$\Gamma = \left(c_4 \frac{\phi}{\phi'}\right)\frac{\partial}{\partial x} + \left(c_4 \mu + \left(\frac{c_4 \phi\phi''}{(\phi')^2} - c_4\right)\frac{A\phi'}{2\nu}\right)\frac{\partial}{\partial \mu} + \left(-c_4 d \frac{\phi}{\phi'} + \alpha\right)\frac{\partial}{\partial \psi}. \tag{2.22}$$

However, using (2.22) we simply get a particular case of the more general solution (2.21), namely,

$$\psi = 2\nu x + \frac{\alpha}{c_4}\log\phi \pm 2\nu c\left(\frac{\mu}{\phi} - \frac{A}{2\nu}\frac{\phi'}{\phi}\right) + q, \tag{2.23}$$

where q is constant.

Equation (2.7) can obviously be split in an infinite number of ways. In this section we have considered only the particular split of the type (2.9). In fact classical and nonclassical symmetry analysis on different splits will produce different solutions. For example, consider the split

$$\begin{aligned} \psi_x \psi_{x\mu} - \psi_x \psi_{\mu\mu} - 4UU' &= g(x,\mu)\psi_{\mu\mu\mu} + k(x)\psi_{\mu\mu}, \\ 2\nu(\mu\psi_{\mu\mu\mu} + \psi_{\mu\mu}) &= g(x,\mu)\psi_{\mu\mu\mu} + k(x)\psi_{\mu\mu}, \end{aligned} \tag{2.24}$$

where $U(x) = \frac{\nu\lambda}{L}(x - c)$.

We can show that when

$$g(x,\mu) = 2\nu\mu + \frac{2\nu L}{\lambda}\phi'(x),$$

and

$$k(x) = 2\nu + 2\nu\phi'(x),$$

both equations in (2.24) have the additional symmetry generator

$$\Gamma = c_2(x - c)\frac{\partial}{\partial x} + \left(\frac{c_2 L}{\lambda}\phi''(x)(c - x)\right)\frac{\partial}{\partial \mu} + (c_2\psi + 2c_2\nu[(c - x)\phi'(x) + \phi(x)] + \alpha)\frac{\partial}{\partial \psi},$$

from which we get the solution to (2.7),

$$\psi = -2\nu\phi(x) + (x - c)f(\xi),$$

where

$$\xi = \frac{\mu}{L} + \frac{\phi'(x)}{\lambda}, \tag{2.25}$$

and

$$f(\xi) = 2\nu(Ae^{-\lambda\xi} + \lambda\xi - 3).$$

This is the second solution found in [6].

These are Not Classical or Nonclassical Symmetries

In [6], Burde showed that the new similarity solution (2.14) found for Eq. (2.7), cannot be obtained by considering similarity reductions of (2.7) by the Lie group method of transformation or by the nonclassical method of group invariant solutions. We now show that the solutions (2.15), (2.16) also cannot be found by considering similarity reductions of (2.7) by the classical and nonclassical method of group-invariant solutions.

From (2.10) we find the global form of the 1-parameter group of transformations associated with the similarity reductions of the solutions (2.14), (2.15), and (2.16), namely,

$$\begin{aligned} x^* &= \phi^{-1}\{\phi(x)e^{(2c_2/m)\varepsilon}\}, \\ \mu^* &= \frac{A}{2\nu}\left[\phi'\{\phi^{-1}(\phi(x)e^{(2c_2/m)\varepsilon})\}\right]\phi^m(x)e^{2c_2\varepsilon} \\ &\quad + \left(\mu - \frac{A}{2\nu}\phi'\phi^m\right)e^{c_2(1+2/m)\varepsilon}, \\ \psi^* &= -d\phi^{-1}\left[\phi(x)e^{(2c_2/m)\varepsilon}\right] - \frac{\alpha}{c_2} + \left(\psi + dx + \frac{\alpha}{c_2}\right)e^{c_2\varepsilon}. \end{aligned} \tag{2.26}$$

Substitution of (2.26) into

$$\psi^*_{\mu^*}\psi^*_{x^*\mu^*} - \psi^*_{x^*}\psi^*_{\mu^*\mu^*} - 4U(x^*)U'(x^*) - 2\nu(\mu^*\psi^*_{\mu^*\mu^*\mu^*} + \psi^*_{\mu^*\mu^*}) = 0, \tag{2.27}$$

where

$$U(x) = \frac{\nu c}{\phi(x)}$$

gives

$$\begin{aligned} &\psi_\mu\psi_{x\mu} - \psi_x\psi_{\mu\mu} - 4UU' - 2\nu(\mu\psi_{\mu\mu\mu} + \psi_{\mu\mu}) \\ &\quad + \left(1 - \frac{\phi'(x)}{\phi'(\phi^{-1}[\phi(x)e^{(2c_2/m)\varepsilon}])}e^{-c_2\varepsilon + 2c_2/m\,\varepsilon}\right)P = 0, \end{aligned} \tag{2.28}$$

where $P = \psi_{\mu\mu\mu}(2\nu\mu - A(\phi(x))^m\phi'(x)) + (2\nu - d)\psi_{\mu\mu}$.

For the solution (2.14), (2.28) becomes

$$P = \psi_{\mu\mu\mu}\Big(2\nu\mu + 2\nu\big(\phi(x)\big)^2\phi'(x)\Big) + 6\nu\psi_{\mu\mu} \tag{2.29}$$

(which agrees with the result in [6] for $\phi(x) = x^n$).

For the solution (2.15), (2.28) becomes

$$P = \psi_{\mu\mu\mu}\big(2\nu\mu - A\phi^{-4}(x)\phi'(x)\big), \tag{2.30}$$

and for the solution (2.16), (2.28) becomes

$$P = \psi_{\mu\mu\mu}\big(2\nu\mu - A\phi^{-3}(x)\phi'(x)\big) - 2\nu\psi_{\mu\mu}. \tag{2.31}$$

Thus the group (2.26) does not map solutions of (2.7) to itself and so the similarity reduction in (2.14), (2.15), and (2.16) cannot be obtained by the classical Lie group method of transformations.

It is also obvious that the auxiliary equation $P = 0$ cannot be obtained from the invariant surface condition, i.e.,

$$\frac{2c_2\phi}{\phi' m}\psi_x + \left\{c_2\,\mu\left(1 + \frac{2}{m}\right) + \frac{Ac_2\phi'\phi^m}{2\nu}\left[\frac{2\phi\phi''}{(\phi')^2 m} + 1 - \frac{2}{m}\right]\right\}\psi_\mu$$
$$= c_2\psi + c_2 d\left(x - \frac{2}{m}\frac{\phi}{\phi'}\right) + \alpha, \tag{2.32}$$

where for the solution (2.14), we have $m = 2$, $d = -4\nu$, $A = -2\nu$; for the solution (2.15), $m = -4$, $d = 2\nu$, and for the solution (2.16), $m = -3$, $d = 4\nu$.

Thus our solutions are unobtainable by the nonclassical method [3] or by the direct method [8] whose solutions are a subset of the nonclassical solutions (see [15, 2, 18]). In a subsequent paper [7], Burde gave a group theoretic interpretation of his new solutions, that is more complicated than that given here.

2.2. *Navier–Stokes and Schrödinger Equations*

As a further example, consider the equation

$$\frac{\partial v^i}{\partial t} + v^j\frac{\partial v^i}{\partial x^j} = -\frac{1}{m}\frac{\partial U}{\partial x^i} + \frac{i\hbar}{2m}\nabla^2 v^i \quad \text{for } i = 1, 2, 3. \tag{2.33}$$

This equation is analogous to the incompressible Navier–Stokes momentum equation, except that the "viscosity" is replaced here by a pure imaginary number $\frac{i\hbar}{2m}$, with m and $\hbar$ positive constants. Hence, its symmetries are in one-to-one correspondence with those of the standard Navier–Stokes equation. The symmetry group is infinite dimensional, including a free function of t (see Boisvert, Ames, and Srivastava [5]). However, if we add the constraint of irrotational flow,

$$\varepsilon^{ijk}\frac{\partial}{\partial x^j}v^k = 0 \quad \text{or} \quad \nabla \times \mathbf{v} = \mathbf{0}, \tag{2.34}$$

then the Lie group of potential symmetries is, in a sense, even larger, because it now includes a general solution of the linear heat equation in 3 + 1 dimensions. In order to see this, we explicitly write the linearising transformation

$$\mathbf{v} = -i\frac{\hbar}{m}\nabla \ln \psi(\mathbf{r}, t), \tag{2.35}$$

which is valid only for irrotational flow. This is essentially a three-dimensional extension of the Hopf–Cole transformation that relates Burgers' equation to the linear heat equation. Since the Navier–Stokes equations are inherently nonlinear, they cannot in general be linearised. However, when the irrotational flow constraint is assumed, so that (2.35) is valid, we obtain, up to a gauge transformation, the factorisation

$$\left[\psi^{-2}\frac{\partial\psi}{\partial x^l} - \psi^{-1}\frac{\partial}{\partial x^l}\right]\left[\psi_t - \frac{i\hbar}{2m}\nabla^2\psi + \frac{i}{\hbar}U\psi\right] = 0. \tag{2.36}$$

If we define a complex-valued vector field $\mathbf{p} = m\mathbf{v}$, with $\mathbf{v}$ as in (2.35), then (2.33) may be viewed as a quantum mechanical order-$\hbar$ (with $\hbar$ identified as Planck's constant divided by 2π) extension of Newton's second law,

$$\frac{D\mathbf{p}}{Dt} = \mathbf{f} + \frac{i\hbar}{2m}\nabla^2\mathbf{p}. \tag{2.37}$$

Here, $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}$ and $\mathbf{f} = -\boldsymbol{\nabla}U$, having the physical interpretation of a conservative force field.

We now see that this quantum continuum extension (2.37) of Newton's second law follows directly, via (2.36), from the linear Schrödinger equation

$$\psi_t = \frac{i\hbar}{2m}\nabla^2\psi - \frac{i}{\hbar}U\psi. \tag{2.38}$$

The possibility of extending the Hopf–Cole transformation to three-dimensional irrotational incompressible fluid flow ws mentioned by Ames (p. 24 of [1]). However this possibility is not widely known because such classical fluids are readily analysed by conventional methods of potential theory (e.g., see Rutherford [19]).

In a related classical reformulation of Schrödinger's wave mechanics by Bohm [4], the classical velocity field $\mathbf{v}(\mathbf{r}, t)$ is represented by

$$\mathbf{v} = \frac{\hbar}{m} \operatorname{Im} \nabla \ln \psi .$$

3. SYMMETRY-PRESERVING CONSTRAINTS

Some PDEs may have nontrivial symmetries, which although reduce the PDE, result in a quotient PDE or ODE that is still difficult to solve. We now consider some examples of such PDEs of two independent variables possessing a nontrivial symmetry generator Γ. We split the quotient ODE

$$\sum_i \alpha_i T_i(z, \phi, \phi', \phi'', \dots) = 0, \tag{2.39}$$

where z and ϕ are the invariants corresponding to Γ, into the system

$$\begin{aligned} \sum_i \beta_i T_i(z, \phi, \phi', \phi'', \dots) &= 0, \\ \sum_i (\alpha_i - \beta_i) T_i(z, \phi, \phi', \phi'', \dots) &= 0, \end{aligned} \tag{2.40}$$

where $\boldsymbol{\beta}$ is not a multiple of $\boldsymbol{\alpha}$, and T_i are linearly independent functions (e.g., monomials).

We then use the package DIFFGROB2 [13], to determine constants β_i such that the system (2.40) is compatible and also to give the integrability and compatibility conditions of the system.

Note that this procedure is equivalent to splitting the original PDE into a compatible system which maintains a symmetry that was present in the original target PDE; i.e., it is equivalent to adding a symmetry-preserving constraint to the original PDE.

EXAMPLE. Consider the boundary layer equation

$$u_y u_{xy} - u_x u_{yy} = u_{yyy}, \tag{2.41}$$

which admits the classical symmetry generator

$$\Gamma = u \frac{\partial}{\partial u} + x \frac{\partial}{\partial x}. \tag{2.42}$$

Using (2.42) the PDE (2.41) has a similarity solution

$$u = x\phi(y), \tag{2.43}$$

where ϕ satisfies

$$\phi'''(y) = (\phi'(y))^2 - \phi(y)\phi''(y). \tag{2.44}$$

We now look for a compatible split of (2.44) of the type

$$\begin{aligned} a\phi'''(y) + b(\phi'(y))^2 + k\phi(y)\phi''(y) &= 0, \\ (1-a)\phi'''(y) + (-1-b)(\phi'(y))^2 & \\ +(1-k)\phi(y)\phi''(y) &= 0, \end{aligned} \tag{2.45}$$

where a, b, and k are constants.

Use of DIFFGROB2 indicates compatibility for

$$a = 1, \qquad b = -3 \quad \text{and} \quad k = 2, \tag{2.46}$$

and for this choice of constants gives the further conditions

$$\begin{aligned} -\phi^2(\phi')^2 - 6(\phi')^3 &= 0, \\ 2(\phi')^2 - \phi\phi'' &= 0, \\ \phi''' + (\phi')^2 &= 0. \end{aligned} \tag{2.47}$$

Solving (2.47) we find $\phi(y) = 6/(y + \gamma)$, and hence from (2.43) we find the solution

$$u = \frac{6x}{y + \gamma} \tag{2.48}$$

to (2.41).

We note that the system (2.45), with choice of constants as in (2.46), corresponds to a split of the PDE (2.41), which might, for example, have been

$$\begin{aligned} u_{yyy} - 3u_y u_{xy} + 2u_x u_{yy} &= 0, \\ 2u_y u_{xy} - u_x u_{yy} &= 0. \end{aligned}$$

DIFFGROB2 also indicated that the choice of constants $a = 2$, $k = 1$, $b = 0$ leads to a compatible system (2.45). However this choice of constants leads to the same solution (2.48).

EXAMPLE. The PDE

$$u_{xx} = u_t - u^2\left(\frac{u_{xt}}{u_t} + \frac{u}{2}\right) \tag{2.49}$$

has a classical symmetry with generator

$$\Gamma = -x\frac{\partial}{\partial x} - 2t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \tag{2.50}$$

and thus a similarity solution

$$u = \frac{1}{\sqrt{t}}\phi(z); \qquad z = \frac{x}{\sqrt{t}}, \tag{2.51}$$

where $\phi(z)$ satisfies

$$\phi\phi'' + z\phi'\phi'' + \frac{z^2}{2}(\phi')^2 + z\phi'\phi + \frac{\phi^2}{2} + 2\phi^2\phi' + z\phi^2\phi'' + \frac{z}{2}\phi^3\phi' + \frac{\phi^4}{2} = 0. \tag{2.52}$$

We look for a compatible split of (2.52) of the type

$$\begin{aligned} &c_1\phi\phi'' + c_2 z\phi'\phi'' + c_3 z^2(\phi')^2 + c_4 z\phi'\phi + c_5\phi^2 + c_6\phi^2\phi' \\ &\quad + c_7 z\phi^2\phi'' + c_8 z\phi^3\phi' + c_9\phi^4 = 0, \\ &(1 - c_1)\phi\phi'' + (1 - c_2)z\phi'\phi'' + \left(\tfrac{1}{2} - c_3\right)z^2(\phi')^2 \\ &\quad + (1 - c_4)z\phi'\phi + \left(\tfrac{1}{2} - c_5\right)\phi^2 \\ &\quad + (2 - c_6)\phi^2\phi' + (1 - c_7)z\phi^2\phi'' + \left(\tfrac{1}{2} - c_8\right)z\phi^3\phi' \\ &\quad + \left(\tfrac{1}{2} - c_9\right)\phi^4 = 0, \end{aligned} \tag{2.53}$$

where c_i, $i = 1 \cdots 9$ are constants.

Use of DIFFGROB2 indicates compatibility when

$$\begin{aligned} &c_1 = 2, \quad c_2 = 0, \quad c_3 = 0, \quad c_4 = 1, \quad c_5 = 1, \\ &c_6 = 2, \quad c_7 = 0, \quad c_8 = 0 \quad \text{and} \quad c_9 = 0, \end{aligned} \tag{2.54}$$

i.e., the system

$$\begin{aligned} &2\phi\phi'' + z\phi'\phi + \phi^2 + 2\phi^2\phi' = 0, \\ &-\phi\phi'' + z\phi'\phi'' + \frac{z^2}{2}(\phi')^2 - \frac{\phi^2}{2} \\ &\quad + z\phi^2\phi'' + \frac{z\phi^3}{2}\phi' + \frac{\phi^4}{2} = 0. \end{aligned} \tag{2.55}$$

For this system, DIFFGROB2 gives the extra compatibility condition

$$(2z + z^3)\phi^2 - (6z^2 + 4)\phi + 8z = 0. \tag{2.56}$$

Solving (2.56) we find

$$\phi(z) = \frac{4z}{z^2 + 2}, \frac{2}{z}. \tag{2.57}$$

Hence from (2.51) and (2.57) a solution to (2.49), with $u_t \neq 0$, is

$$u = \frac{4x}{x^2 + 2t}.$$

The system (2.55) corresponds to a split of the PDE (2.49) which among other possibilities includes

$$u_{xx} - u_t = \frac{2t}{x}uu_t + \frac{u^2}{x},$$

$$-u^2\frac{u_{xt}}{u_t} - \frac{u^3}{2} = \frac{2t}{x}uu_t + \frac{u^2}{x}.$$

4. CONCLUSION

We have shown that solutions to practical PDEs can be found by classical symmetry reductions of a larger system of equations by the addition of symmetry-enhancing or symmetry-preserving constraints. These constraints are chosen so that the system has a larger symmetry group or an identical symmetry group compared to that of the original single equation. Obviously, this method could be applied to solve many other nonlinear partial differential equations.

Of course *not all* similarity solutions can be obtained by standard symmetry techniques. The possibility of new reductions by differential constraints remains open [17]. However, by the simple device of adding symmetry-enhancing constraints, we can obtain new solutions that were previously thought to be unrelated to classical symmetries.

We have also demonstrated how the addition of symmetry-preserving constraints can enable us to find new solutions to PDEs by leading to simpler quotient equations.

Naturally, the techniques of symmetry-enhancing and symmetry-preserving constraints may be broadened by expanding the class of allowable symmetries to contact symmetries and nonlocal symmetries.

REFERENCES

1. W. F. Ames, ‘‘Nonlinear Partial Differential Equations in Engineering,’’ Vol. 2, Academic Press, San Diego, 1972.
2. D. J. Arrigo, P. Broadbridge, and J. M. Hill, Nonclassical symmetry solutions and the methods of Bluman–Cole and Clarkson–Kruskal, *J. Math. Phys.* **34**, No. 10 (1993), 4692–4703.
3. G. W. Bluman, and J. D. Cole, The general similarity solution of the heat equation, *J. Math. Mech.* **18** (1969), 1025–1042.
4. D. Bohm, A suggested interpretation of the quantum theory in terms of ‘‘hidden variables I,’’ *Phys, Rev.* **85**, No. 2 (1952), 166–180.
5. R. E. Boisvert, W. F. Ames, and U. N. Srivastava, Group properties and new solutions of the Navier–Stokes equations, *J. Eng. Math.* **17** (1983), 203–221.
6. G. I. Burde, The construction of special explicit solutions of the boundary-layer equations. Steady flows, *Quart. J. Mech. Appl. Math.* **47** (1994), 247–260.
7. G. I. Burde, New similarity reductions of the steady-state boundary layer equations, *J. Phys. A.* **29** (1996), 1665–1683.
8. P. A. Clarkson and M. D. Kruskal, New similarity reductions of the Boussinesq equation, *J. Math. Phys.* **30** (1989), 2201–2213.
9. J. M. Goard and P. Broadbridge, Nonlinear superposition principles obtained by Lie symmetry methods, *J. Math. Anal. Appl.* **214** (1997), 633–657.
10. W. Hereman, Review of symbolic software of Lie symmetry analysis, *Math. Comput. Modelling* **25**, No. 8/9 (1997), 115–132.
11. S. Hood, New Exact Solutions of Burger’s equation—An extension to the direct method of Clarkson and Kruskal, *J. Math. Phys.* **36**, No. 4 (1995), 1971–1990.
12. N. H. Ibragimov, ‘‘CRC Handbook of Lie Group Analysis of Differential Equations,’’ Vols. 1–3, CRC Press, Boca Raton, FL, 1994–1996.
13. E. L. Mansfield, ‘‘DIFFGROB2: A Symbolic Algebra Package for Analysing Systems of PDE using MAPLE, User’s Manual for Release 2,’’ University of Exeter, Department of Mathematics, 1993.
14. W. Miller Jr., ‘‘Symmetry and Separation of Variables,’’ Addison-Wesley, Reading, MA, 1977.
15. M. C. Nucci and P. A. Clarkson, The nonclassical method is more general than the direct method for symmetry reductions: An example of the Fitzhugh–Nagumo equation, *Phys. Lett.* **164A** (1992), 49–56.
16. P. J. Olver and P. Rosenau, The construction of special solutions to partial differential equations, *Phys. Lett.* **114A** (1986), 107–112.
17. P. J. Olver, Direct reduction and differential constraints, *Proc. Roy. Soc. London A Ser.* **444** (1994), 509–523.
18. E. Pucci, Similarity reductions of partial differential equation, *J. Phys. A: Math. Gen.* **25** (1992), 2631–2640.
19. D. E. Rutherford, ‘‘Fluid Mechanics,’’ Oliver & Boyd, Edinburgh, 1959, Chaps. 2, 3.