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Some Theorems about Projective Modules over Polynomial Rings

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1. INTRODUCTION

Let R be a commutative noetherian ring of dimension d, let A denote the polynomial ring $R[X_1,..., X_n]$ and let P be a finitely generated projective A-module. Bass [3, Question $(\mathrm{XIV})_n$] has asked whether the following statements are true:

(1) If $\operatorname{rank} P \geqslant d+1$, then P has a unimodular element.

(2) If $\operatorname{rank} P \geqslant d+1$ and if Q is a finitely generated projective A-module such that $P \oplus A \approx Q \oplus A$, then $P \approx Q$.

Question (1) was originally raised in Section 9 of [4]. For $n=1$, the statements (1) and (2) were also conjectured by Eisenbud and Evans and settled affirmatively by Plumstead (in his thesis [8]), who in turn conjectured the truth of (1) and (2) for arbitrary n.

When P is stably extended from R, (1) and (2) both follow from Theorem 1.1 of [12]. When $d=1$, the statements (1) and (2) are equivalent, and were proved to be true by Kang [6] under the assumptions that R_{red} have finite normalization and $1/2 \in R$.

In this paper we prove the validity of conjecture (1). As a consequence one gets: if P is a projective $R[X_1,..., X_n]$-module of rank $\leqslant r$, then P can be generated by $r+d$ elements. To see this one can use Theorem 1.1 of [12] and the argument in the proof of [1, Corollary 3.8, p. 185].

As for conjecture (2), we prove its validity when R is two-dimensional normal or an affine algebra with isolated singularities.

In Section 2 we record some definitions and results. In Section 3 we prove the validity of the first conjecture (Theorem 3.1). In Section 4 we prove some results about lifting automorphisms of projective modules (Proposition 4.1 and Theorem 4.6) and use these results to prove the validity of the second

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conjecture when R is two-dimensional normal (Corollary 4.9). Plumstead's patching technique is an important tool in our proofs.

2. Preliminaries

Throughout this paper all rings will be commutative and all modules will be finitely generated.

In this section we collect some definitions and results for later use; R will denote a commutative ring.

(2.1) Given a projective R-module P and an element $p \in P$ we define $O_P(p) = \{\varphi(p) \mid \varphi \in \mathrm{Hom}_R(P, R)\}$. We say that p is *unimodular* if $O_P(p) = R$. The set of all unimodular elements of P will be denoted by $\mathrm{Um}(P)$.

(2.2) Let P be a projective R-module of *constant* rank r. Let σ be an endomorphism of P. Then $\Lambda^r\sigma$ is a homothety in $\Lambda^r P$ by some $a \in R$. We define a to be the *determinant* of σ. The group of automorphisms of P of determinant 1 will be denoted by $SL(P)$.

(2.3) Lemma. *Let P be a projective $R[X]$-module and let p be an element of P such that $O_P(p)$ contains a monic $f(X)$ and the image of p in P/JP is unimodular, where J denotes the Jacobson radical of R. Then p is unimodular.*

Proof. Since the image of p in P/JP is unimodular and P is projective, $O_P(p)$ contains an element of the type $1 + g(X)$ with $g(X) \in JR[X]$. It suffices to show that $f(X)$ and $1 + g(X)$ are comaximal in $R[X]$. This follows from the fact that, $R' = R[X]/(f)$ being a finite extension of R, JR' is contained in the Jacobson radical of R'.

(2.4) Let I be an ideal of $R[X]$. Define $l(I)$ to be the ideal of R consisting of the leading coefficients of elements of I.

If R is noetherian, then $\mathrm{ht}\, l(I) \geqslant \mathrm{ht}\, I$ [2, Lemma 2 of Section 4].

(2.5) Lemma. *Let R_1 and R_2 be noetherian rings of dimension d_1 and d_2, respectively. Let I_i be an ideal of $R_i[X_1,..., X_n]$ with $\mathrm{ht}\, I_i \geqslant d_i + 1$ for $i = 1, 2$. Then there exist positive integers $r_1,..., r_{n-1}$ such that, denoting $X_j + X_n^{r_j}$ by X_j' $(1 \leqslant j \leqslant n-1)$, I_i contains a monic in X_n with coefficients in $R_i[X_1',..., X_{n-1}']$ for $i = 1, 2$.*

This lemma is a variant of [2, Lemma 3 of Section 4].

Proof. We shall prove the lemma by induction on n. The case $n=1$ follows immediately from (2.4). Assume $n \geqslant 2$. Write B_i for $R_i[X_1,..., X_{n-2}, X_n]$. Applying (2.4), taking $R = B_i$, we see that $l(I_i)$ is an ideal of B_i of height $\geqslant d_i + 1$. By the induction hypothesis, there exist positive integers $r_1,..., r_{n-2}$ such that $l(I_i)$ contains a monic $f_i(X_n)$ (for $i = 1, 2$) with coefficients in $R_i[X'_1,..., X'_{n-2}]$, where $X'_j = X_j + X_n^{r_j}$ $(1 \leqslant j \leqslant n-2)$. Let $g_i(X_{n-1})$ be an element of I_i with $f_i(X_n)$ as its leading coefficient. It is clear that if r_{n-1} is a sufficiently large integer, then g_i is a monic in X_n with coefficients in $R_i[X'_1,..., X'_{n-1}]$, where $X'_{n-1} = X_{n-1} + X_n^{r_{n-1}}$.

Next we state a result, a proof of which follows from the Remark following Theorem A of [5].

(2.6) *Let R be noetherian and let P be a projective R-module of rank $\geqslant r$ and let $p = (p_1, a)$ be an element of $P \oplus R$. Then there exists a $p_2 \in P$ such that* ht $O_P(p_1 + ap_2) \geqslant \min(r, \text{ht } O_{P \oplus R}(p))$.

We conclude this section with a result which can be proved using ideas in the proof of Theorem 1 of [9].

(2.7) *Let P be a projective module over $R[X_1,..., X_n]$. Let $J(R, P)$ be the set of those elements a of R such that P_a is extended from R_a. Then $J(R, P)$ is an ideal of R and $J(R, P) = \sqrt{J(R, P)}$.*

3. Existence of Unimodular Elements

The aim of this section is to prove the following theorem.

Theorem 3.1. *Let R be a noetherian ring of dimension d and let P be a projective module of rank $\geqslant d+1$ over $R[X_1,..., X_n]$. Then P contains a unimodular element.*

For the proof of this theorem we shall need the following result of Plumstead [8, Proposition 1 of Section II] which we state only for projective modules.

Proposition 3.2. *Let B be any ring, $A = B[Y]$ and P a projective A-module. Let p' be a unimodular element of $\bar{P}$ ("bar" meaning "modulo Y"). Let s be an element of B and $T = 1 + sB$. Let p_1 (resp. p_2) be a unimodular element of P_s (resp. P_T) such that $\bar{p}_1 = p'_s$ (resp. $\bar{p}_2 = p'_T$). Set $N_1 = P_s/A_s p_1$ and $N_2 = P_T/A_T p_2$. Assume further that $(N_1)_T$ and $(N_2)_s$ are extended from B_{Ts}. Then P has a unimodular element p such that $\bar{p} = p'$.*

Proof of Theorem 3.1. The case $d=0$ is covered by Quillen–Suslin Theorem (Theorem 4 of [9] and Theorem 3 of [10]). So assume $d \geqslant 1$.

We prove the theorem by induction on n, the case $n=0$ being a well-known theorem of Serre [1, Corollary 2.7, p. 173].

Without loss of generality we may assume that R is reduced and that P is of constant rank.

We shall write A for $R[X_1,..., X_n]$. Let S be the set of non-zero-divisors of R. Since R_S is a finite product of fields, it follows from the Quillen–Suslin theorem that P_S is A_S-free. Therefore there exists an $s \in S$ such that P_s is free over A_s.

Since $A/(sX_n) = R[X_n]/(sX_n)[X_1,..., X_{n-1}]$ and $\dim R[X_n]/(sX_n) = \dim R$, by the induction hypothesis P/sX_nP contains a unimodular element, say, $\tilde{p}$. Let $p' \in P$ be a lift of $\tilde{p}$. Applying (2.6) to the unimodular element (p', sX_n) of $P \oplus A$ we can find a $p'' \in P$ such that $\operatorname{ht} O_P(p' + sX_n p'') \geqslant d+1$. Let p denote $p' + sX_n p''$. Note that p is a lift of $\tilde{p}$.

In the rest of the proof "bar" will denote "modulo X_n." We shall show that the unimodular element $\bar{p}$ in $\bar{P}$ can be lifted to a unimodular element in P.

Since P_s is free over A_s $(=R_s[X_1,..., X_n])$ and there is a retraction $\bar{A}_s \to A_s$, $\bar{p}$ can be lifted to a unimodular element p_1 of P_s. Hence we have a decomposition $P_s = Q \oplus A_s p_1$. Let $p = (q, ap_1)$. Since $\operatorname{ht} O_{P_s}(p) \geqslant \operatorname{ht} O_P(p) \geqslant d+1$ and $\operatorname{rank} Q \geqslant d$, by (2.6) there exists a $q_1 \in Q$ such that $\operatorname{ht} O_Q(q + aq_1) \geqslant d$.

Let $T' = 1 + sR$. Since $\dim R_{T's} \leqslant d-1$ and $Q_{T'}$ is stably free of rank $\geqslant d$ over $A_{T's}$, by [12, Theorem 1.1] $Q_{T'}$ is free.

Further $\operatorname{ht} O_{Q_{T'}}(q + aq_1) \geqslant d$. Therefore applying Lemma 2.5 to the ideals $O_P(p)$ and $O_{Q_{T'}}(q + aq_1)$ of A and $A_{T's}$, respectively, we can find a change of variables $X_i \to X_i' = X_i + X_n^{r_i}$ (for $1 \leqslant i \leqslant n-1$) and $X_n \to X_n$ such that $O_P(p)$ contains a monic $f(X_n)$ with coefficients in B and $O_{Q_{T'}}(q + aq_1)$ contains a monic $g(X_n)$ with coefficients in $B_{T's}$, where B denotes $R[X_1',..., X_{n-1}']$.

Let N denote P/Ap. Since p can be mapped into $(q + aq_1, ap_1)$ by an (elementary) automorphism of P_s $(=Q \oplus A_s p_1)$, we have $N_s \simeq P_s/A_s(q + aq_1, ap_1)$. Since the element $q + aq_1$ of the $B_{T's}[X_n]$-module $Q_{T'}$ becomes unimodular after inverting g, we see that $P_{T's}/A_{T's}(q + aq_1, ap_1)$ becomes isomorphic to $Q_{T'}$ after inverting g. Therefore $N_{T'sg}$ is isomorphic to $Q_{T'g}$ and hence free over $B_{T's}[X_n]_g$.

Let T denote $1 + sB$. We claim that p is a unimodular element of P_T. To see this we first recall that $O_{P_T}(p)$ contains a monic polynomial from $B_T[X_n]$, namely, $f(X_n)$. Further, the image of p in p_T/sP_T $(=P/sP)$ is unimodular. Therefore, since s is in the Jacobson radical of B_T, it follows from Lemma 2.3 that p is unimodular.

It follows in particular that N_T is projective. Since $T' \subset T$, N_{Tsg} is free over $B_{Ts}[X_n]_g$. Therefore, by (Theorem 3 of [9] and Theorem 1 of [10]), N_{Ts} is free over $B_{Ts}[X_n]$. Applying Proposition 3.2 we find a unimodular element in P which lifts $\bar{p}$.

4. Lifting Automorphisms and a Theorem on Cancellation

Let A be a ring and P a projective A-module. Given a $\varphi \in P^*$ ($= \operatorname{Hom}_A(P, A)$) and a $p \in P$, we define an endomorphism φ_p of P as the composite $P \to^{\varphi} A \to^{p} P$. If $\varphi(p) = 0$, then $\varphi_p^2 = 0$ and $1 + \varphi_p$ is an automorphism of P. We shall say that an automorphism of P is a *transvection* if it is of the form $1 + \varphi_p$ with $\varphi(p) = 0$ and φ is unimodular in P^* or p is unimodular in P.

Proposition 4.1. *Let A be a ring, L an ideal of A and P a projective A-module. Then any transvection of P/LP can be lifted to a (unipotent) automorphism of P.*

Proof. Let $\varphi' \in (P/LP)^*$ and $p' \in P/LP$ be such that $\varphi'(p') = 0$. Assume that p' is unimodular. Let $p \in P$ (resp. $\theta \in P^*$) be a lift of p' (resp. φ'). Then we have $\theta(p) = a$ for some $a \in L$. Since p' is unimodular, there exists a $\psi \in P^*$ such that $\psi(p) = 1 + b$ for some $b \in L$. Set $\varphi = (1 + b)\theta - a\psi$. Then φ is a lift of φ' and $\varphi(p) = 0$. Consequently, $1 + \varphi_p$ is an automorphism of P lifting $1 + \varphi'_{p'}$.

The case in which φ' is unimodular is settled similarly.

Corollary 4.2. *With the notation as in Proposition 4.1, assume that P/LP is free of rank r and let σ' be an automorphism of P/LP which belongs to $E_r(A/L)$, when considered as a matrix with respect to some basis of P/LP. Then σ' can be lifted to an automorphism of P.*

Remark 4.3. In the situation of Proposition 4.1, assume that P has a unimodular element and that the natural map $\operatorname{Um}(P) \to \operatorname{Um}(P/LP)$ is surjective (consequently, P^* has a unimodular element and $\operatorname{Um}(P^*) \to \operatorname{Um}(P^*/LP^*)$ is surjective). Then any transvection of P/LP can be lifted to a *transvection* of P. The surjectivity of $\operatorname{Um}(P) \to \operatorname{Um}(P/LP)$ is realized if, for instance, P contains a unimodular element and the group generated by the transvections of P/LP acts transitively on $\operatorname{Um}(P/LP)$.

As an application of Proposition 4.1 we shall prove the following theorem.

Theorem 4.4. *Let B be a ring, P a projective module of constant rank r over $B[Y_1, \ldots, Y_m]$ and I an ideal of B such that P_b is extended from B_b for all $b \in I$. Assume further that $\overline{P}/I\overline{P}$ is B/I-free (where "bar" means* "mod$(Y_1,..., Y_m)$"). *Let* $\sigma' \in SL(\bar{P})$ *be such that, when considered as a matrix with respect to some basis of* $\bar{P}/I\bar{P}$, $\sigma'(\text{mod } I)$ *belongs to* $E_r(B/I)$. *Then* σ' *can be lifted to an automorphism of* P.

Moreover, if we assume P *to have a unimodular element and* $SL_r(B/I) = E_r(B/I)$, *then* $\text{Aut}(P) \to \text{Aut}(\bar{P})$ *is surjective.*

For the proof of this theorem, we shall need the following result of Plumstead [8, Lemma 2 of Section II]:

PROPOSITION 4.5. *Let* B *be any ring,* $A = B[Y_1,..., Y_m]$ *and let* P, Q *be projective* A*-modules. Let* $\sigma': \bar{Q} \to \bar{P}$ *be a* B*-isomorphism ("bar" meaning "modulo* $(Y_1,..., Y_m)$"). *Let* s_1 *and* s_2 *be elements of* B *satisfying* $Bs_1 + Bs_2 = B$. *Let* $\sigma_i: Q_{s_i} \to P_{s_i}$ *(for* $i = 1, 2$) *be an* A_{s_i}*-isomorphism such that* $\bar{\sigma}_i = \sigma'_{s_i}$. *Assume further that* $P_{s_1 s_2}$ *is extended from* $B_{s_1 s_2}$. *Then there exists an* A*-isomorphism* $\sigma: Q \to P$ *such that* $\bar{\sigma} = \sigma'$.

Plumstead proved this result for one variable, but his proof goes through for several variables.

Proof of Theorem 4.4. Let $\{\tilde{e}_1,..., \tilde{e}_r\}$ be a basis of $\bar{P}/I\bar{P}$ with respect to which $\sigma'(\text{mod } I)$ belongs to $E_r(B/I)$. Let S denote $1 + I$. Then $B_S/IB_S = B/I$. Let $e_i \in \bar{P}_S$ be a lift of $\tilde{e}_i$ for $1 \leqslant i \leqslant r$. Since IB_S is in the Jacobson radical of B_S, $\bar{P}_S$ is B_S-free with $\{e_1,..., e_r\}$ as a basis and σ'_S, considered as a matrix with respect to $\{e_1,..., e_r\}$, belongs to $E_r(B_S)$. Applying Corollary 4.2 we find an automorphism σ^* of P_S which lifts σ'_S. Therefore we can find $1 + b \in S$ and an automorphism σ_1 of P_{1+b} such that $\bar{\sigma}_1 = \sigma'_{1+b}$.

Since P_b is extended from B_b, we have an automorphism σ_2 of P_b such that $\bar{\sigma}_2 = \sigma'_b$. Applying Proposition 4.5 we get an automorphism σ of P lifting σ'.

Next assume that $SL_r(B/I) = E_r(B/I)$ and that P has a unimodular element, say, p. Write $P = B[Y_1,..., Y_m] p \oplus N$. Let $\sigma' \in \text{Aut}(\bar{P})$ and let $u = \det \sigma'$. Let τ denote the automorphism of P which is identity on N and sends p to $u^{-1}p$. Then $\det(\sigma'\bar{\tau}) = 1$ and therefore $\sigma'\bar{\tau}(\text{mod } I)$ belongs to $E_r(B/I)$. Hence, by the first part of the theorem, $\sigma'\bar{\tau}$ can be lifted to an automorphism σ of P showing that σ' can be lifted to an automorphism of P, namely, $\sigma\tau^{-1}$.

THEOREM 4.6. *Let* R *be a noetherian ring. Let* P *be a projective module of constant rank* $r \geqslant 3$ *over* $R[X_1,..., X_n][Y_1,..., Y_m]$. *Assume that* $P_{\mathfrak{p}}$ *is extended from* $R_{\mathfrak{p}}$ *for all prime ideals of* R *which are not maximal. Then any element of* $SL(P/(Y_1,..., Y_m)P)$ *can be lifted to an element of* $\text{Aut}(P)$.

Moreover, if P *has a unimodular element, then the canonical map* $\text{Aut}(P) \to \text{Aut}(P/(Y_1,..., Y_m)P)$ *is surjective.*

Proof. Consider the ideal $J(R, P)$ of R as defined in (2.7). By the hypothesis of the theorem, $R/J(R,P)$ is a finite product of fields, say, $k_1 \times \cdots \times k_t$. Let B denote $R[X_1,..., X_n]$ and let I denote $J(R,P)B$. Clearly $I \subset J(B,P)$. Moreover, $B/I = (k_1 \times \cdots \times k_t)[X_1,..., X_n]$ so that projective modules of constant rank over B/I are free (Theorem 4 of [9] and Theorem 3 of [10]) and $SL_r(B/I) = E_r(B/I)$ [11, Corollary 6.7]. Therefore, in view of Theorem 4.4, we are through.

COROLLARY 4.7. *Let R be a noetherian ring of dimension d. Let P be a projective module of constant rank $\geqslant \max(d+1, 3)$ over $R[X_1,..., X_n][Y_1,..., Y_m]$. Then the canonical map $\operatorname{Aut}(P) \to \operatorname{Aut}(P/(Y_1,..., Y_m)P)$ is surjective in each of the following cases*:

(1) $d \leqslant 1$,

(2) $d = 2$ *and R is normal*,

(3) *R is an affine algebra over a field with R_{red} having isolated singularities.*

Proof. First we note that for any non-maximal prime ideal $\mathfrak{p}$ of R, $P_{\mathfrak{p}}$ is extended from $R_{\mathfrak{p}}$. This follows from the Quillen–Suslin theorem in cases (1) and (2), and from a result of Lindel [7, Theorem] in case (3). Next, since $\operatorname{rank} P \geqslant d+1$, P has a unimodular element by Theorem 3.1. So, applying Theorem 4.6, we are through.

THEOREM 4.8. *Let R be a noetherian ring of dimension d. Let P and Q be projective modules of rank $\geqslant d+1$ over $R[Y_1,..., Y_n]$ such that $P \oplus R[Y_1,..., Y_n] \approx Q \oplus R[Y_1,..., Y_n]$. Assume that $P_{\mathfrak{p}}$ is extended from $R_{\mathfrak{p}}$ for all non-maximal prime ideals $\mathfrak{p}$ of R. Then $P \approx Q$.*

Proof. We can assume that P and Q are of constant rank, say, r. If $d \leqslant 1$, then the theorem follows from Theorem 3.1. Therefore we assume $d \geqslant 2$. We proceed by induction on n, the case $n = 0$ being a theorem of Bass [1, Corollary 3.5, p. 184].

Let A denote $R[Y_1,..., Y_n]$. In what follows, "bar" will mean "modulo Y_n." By the induction hypothesis, there exists an isomorphism $\sigma' : \bar{Q} \to \bar{P}$. We shall show that there is an isomorphism $\sigma : Q \to P$ with $\bar{\sigma} = \sigma'$.

Let $\theta : Q \oplus A \to P \oplus A$ be an isomorphism and let $\theta(0, 1) = (p, a)$. Let us write J for $J(R,P)$. By the hypothesis of the theorem, R/J is a finite product of fields. Therefore P/JP is a free A/JA-module of rank $\geqslant d+1 \geqslant 3$. Further, since $SL_r(A/JA) = E_r(A/JA)$, the group generated by the transvections of $P/JP \oplus A/JA$ acts transitively on $\operatorname{Um}(P/JP \oplus A/JA)$. Since transvections of $P/JP \oplus A/JA$ can be lifted to automorphisms of $P \oplus A$ (Proposition 4.1), we may modify θ by an automorphism of $P \oplus A$ and assume that $p(\operatorname{mod} J)$ is a unimodular element of P/JP and $a \in JA$. Further, replacing p by $p + aq$ for a suitable $q \in P$ (using (2.6)), we can assume that $\operatorname{ht} O(p) \geqslant d+1$. Using

Lemma 2.5 choose a change of variables $Y_i \to X_i$ (for $1 \leqslant i \leqslant n-1$) and $Y_n \to Y_n$ such that $O(p)$ contains a monic $f(Y_n) \in B[Y_n]$, where B denotes $R[X_1,..., X_{n-1}]$.

Let $S = 1 + JB$. Then, by Lemma 2.3, p is a unimodular element of P_S. Therefore (p, a) can be mapped into $(0, 1)$ by an automorphism of $P_S \oplus A_S$ showing that there exists an isomorphism $\tau: Q_S \to P_S$. By Theorem 4.4, the automorphism $\sigma'_S \bar{\tau}^{-1}$ of $\bar{P}_S$ can be lifted to an automorphism τ_1 of P_S so that $\tau_1 \tau: Q_S \to P_S$ is an isomorphism lifting σ'_S. Therefore, there exists $1 + b \in S$ and an isomorphism $\sigma_1: Q_{1+b} \to P_{1+b}$ lifting σ'_{1+b}.

Next we note that $J(R, P) = J(R, Q)$ because P and Q are stably isomorphic and have rank $\geqslant d+1$ [12, Corollary 1.2]. Therefore P_b and Q_b are both extended from B_b. Hence there exists an isomorphism $\sigma_2: Q_b \to P_b$ lifting σ'_b. Now using Proposition 4.5 we find an isomorphism $\sigma: Q \to P$ which lifts σ'.

COROLLARY 4.9. *Let R be a noetherian ring of dimension d. Let P and Q be stably isomorphic projective modules of rank $\geqslant d+1$ over $R[Y_1,..., Y_n]$. Then $P \approx Q$ in each of the following cases:*

(1) $d \leqslant 1$,

(2) *$d = 2$ and R is normal,*

(3) *R is an affine algebra over a field with R_{red} having isolated singularities.*

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