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LINEAR ALGEBRA AND ITS APPLICATIONS

Maximal abelian subalgebras of $e(p,q)$ algebras

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Abstract

Maximal abelian subalgebras (MASAs) of one of the classical real inhomogeneous Lie algebras are constructed, namely those of the pseudoeuclidean Lie algebra $e(p,q)$. Use is made of the semidirect sum structure of $e(p,q)$ with the translations $T(p+q)$ as an abelian ideal. We first construct splitting MASAs that are themselves direct sums of abelian subalgebras of $o(p,q)$ and of subalgebras of $T(p+q)$. The splitting subalgebras are used to construct the complementary nonsplitting ones. Here the results are less complete than in the splitting case. We present general decomposition theorems and construct indecomposable MASAs for all algebras $e(p,q)$, $p \geqslant q \geqslant 0$. The case of $q=0$ and 1 were treated earlier in a physical context. The case $q=2$ is analyzed here in detail as an illustration of the general results. © 1999 Elsevier Science Inc. All rights reserved.

Résumé

Les sous-algèbres maximales abéliennes (SAMAs) d'une algèbre réelle classique non-homogène sont construites, en particulier, celles d'algèbre de Lie pseudo-euclidienne $e(p,q)$. On utilise la structure de la somme semi-directe de $e(p,q)$ avec les translations $T(p+q)$ qui représente un idéal abélien. Nous avons construit, en premier, les SAMAs "splitting", qui sont des sommes directes des sous-algèbres abéliennes de $o(p,q)$ et de sous-algèbres de $T(p+q)$. Les sous-algèbres "splitting" sont utilisées pour construire les

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sous-algèbres complémentaire – "nonsplitting". Les résultats ne sont pas explicites comme dans le cas des SAMAs "splitting". Nous présentons les théorèmes généraux de décomposition et nous construisons les SAMAs indécomposables pour toutes les algèbres $e(p,q)$, $p \geqslant q \geqslant 0$. Les cas de $q = 0$ et 1 sont déjà traités dans un context physique. Le cas $q = 2$ est analysé ici en détail comme une illustration des résultats généraux.

1. Introduction

The purpose of this paper is to present a classification of the maximal abelian subalgebras (MASAs) of the pseudoeuclidean Lie algebra $e(p,q)$. Since this Lie algebra can be represented by a specific type of real matrices of dimension $(p+q+1) \times (p+q+1)$, the subject of this paper is placed squarely within a classical problem of linear algebra, the construction of sets of commuting matrices.

Most of the early papers in this direction [1–3] as well as more recent ones [4–8] were devoted to commuting matrices within the set of all matrices of a given dimension. In other words, they studied abelian subalgebras of the Lie algebras $\mathrm{gl}(n,\mathbb{C})$ and $\mathrm{gl}(n,\mathbb{R})$. For a historical review with many references see the book by Suprunenko and Tyshkevich [9].

Maltsev constructed all maximal abelian subalgebras of maximal dimension for all complex finite-dimensional simple Lie algebras [10]. An important subclass of MASAs are Cartan subalgebras, i.e. self-normalizing MASAs [11]. The simple complex Lie algebras, as well as the compact ones, have just one conjugacy class of Cartan subalgebras. The real noncompact forms of the simple Lie algebras can have several conjugacy classes of them. They have been classified by Kostant [12] and Sugiura [13].

This paper is part of a series, the aim of which is to construct all MASAs of the classical Lie algebras. Earlier papers were devoted to the classical simple Lie algebras such as $\mathrm{sp}(2n,\mathbb{R})$ and $\mathrm{sp}(2n,\mathbb{C})$ [14], $\mathrm{su}(p,q)$ [15], $o(n,\mathbb{C})$ [16] and $o(p,q)$ [17]. General results for MASAs of classical simple Lie algebras are presented in Ref. [18]. More recently MASAs of some inhomogeneous classical Lie algebras were studied, namely those of $e(n,\mathbb{C})$ [19], $e(p,0)$ and $e(p,1)$ [20]. Here we consider $e(p,q)$ for all $p \geqslant q \geqslant 0$. The two special cases, $q = 0$ and $q = 1$, treated earlier are of particular importance in physics and are also much simpler than the general case.

The motivation for a study of MASAs was discussed in previous papers [14–20]. As a mathematical problem the classification of MASAs is an extension of the classification of individual elements of Lie algebras into conjugacy classes [21–23]. A classification of MASAs of classical Lie algebras is an important ingredient in the classification of all subalgebras of these algebras.

In applications in the theory of partial differential equations, MASAs provide coordinate systems in which invariant equations allow the separation of variables. More specifically, they provide "ignorable variables" not figuring in the corresponding metric tensors, when considering Laplace–Beltrami or Hamilton–Jacobi equations. In quantum physics they provide complete sets of commuting operators. In classical physics they provide integrals of motion in involution.

The classification problem is formulated in Section 2, where we also present some necessary definitions and explain the classification strategy. Section 3 contains a brief summary of the known results on MASAs of $o(p,q)$ [17]. They are needed in the rest of this paper and we reproduce them in a condensed form to make the paper self-contained. Section 4 is devoted to splitting subalgebras of $e(p,q)$, i.e. subalgebras that are direct sums of subalgebras of the algebra $o(p,q)$ and those of the translation algebra $T(p+q)$. The complementary case of nonsplitting MASAs of $e(p,q)$ is the subject of Section 5. The results on MASAs of $e(p,q)$ obtained in Sections 4 and 5 are reformulated in terms of a decomposition of the underlying linear space $S(p,q)$ in Section 6. Indecomposable MASAs of $e(p,q)$ are described in the same section. Section 7 is devoted to a special case in which all results are entirely explicit, namely MASAs of $e(p,2)$.

2. General formulation

2.1. Some definitions

The pseudoeuclidean Lie algebra $e(p,q)$ is the semidirect sum of the pseudoorthogonal Lie algebra $o(p,q)$ and an abelian algebra $T(n)$ of translations

$$e(p,q) = o(p,q) \niplus T(n), \quad n = p+q. \tag{2.1}$$

We will make use of the following matrix representation of the Lie algebra $e(p,q)$ and the corresponding Lie group $E(p,q)$. We introduce an "extended metric"

$$K_e = \begin{pmatrix} K & 0 \\ 0 & 0_1 \end{pmatrix}, \tag{2.2}$$

where K satisfies

$$K = K^{\mathrm{T}} \in \mathbb{R}^{n\times n}, \quad n = p+q, \quad \det K \neq 0, \tag{2.3}$$

$$\operatorname{sgn} K = (p,q), \quad p \geqslant q \geqslant 0. \tag{2.4}$$

Here sgn K denotes the signature of K, where p and q are the numbers of positive and negative eigenvalues, respectively. Then $X_e \in e(p,q)$ and $H \in E(p,q)$ are represented as

$$X_e(X,\alpha) \equiv X_e = \begin{pmatrix} X & \alpha^T \\ 0 & 0 \end{pmatrix}, \quad X \in \mathbb{R}^{n\times n}, \quad \alpha \in \mathbb{R}^{1\times n}, \tag{2.5}$$

$$H = \begin{pmatrix} G & a^T \\ 0 & 1 \end{pmatrix}, \quad G \in \mathbb{R}^{n\times n}, \quad a \in \mathbb{R}^{1\times n}, \tag{2.6}$$

$$XK + KX^T = 0, \quad GKG^T = K, \quad X_eK_e + K_eX_e^T = 0. \tag{2.7}$$

The vector $\alpha \in \mathbb{R}^{1\times n}$ represents the translations. We say that the translations are positive, negative or zero (isotropic) length if

$$\alpha K\alpha^T > 0, \quad \alpha K\alpha^T < 0, \quad \alpha K\alpha^T = 0, \tag{2.8}$$

respectively.

We will be classifying maximal abelian subalgebras of the pseudoeuclidean Lie algebra $e(p,q)$ into conjugacy classes under the action of the pseudoeuclidean Lie group $E(p,q)$. Let us define some basic concepts.

Definition 2.1. The centralizer $\mathrm{cent}(L_0,L)$ of a Lie algebra $L_0 \subset L$ is the subalgebra of L consisting of all elements in L, commuting elementwise with L_0

$$\mathrm{cent}(L_0,L) = \{e \in L \mid [e,L_0] = 0\}. \tag{2.9}$$

Definition 2.2. A maximal abelian subalgebra L_0 (MASA) of L is an abelian subalgebra, equal to its centralizer

$$[L_0,L_0] = 0, \quad \mathrm{cent}(L_0,L) = L_0. \tag{2.10}$$

Definition 2.3. A normalizer group $\mathrm{Nor}(L_0,G)$ in the group G of the subalgebra $L_0 \subseteq L$ is

$$\mathrm{Nor}(L_0,G) = \{g \in G \mid gL_0g^{-1} \subseteq L_0\}. \tag{2.11}$$

Definition 2.4. A splitting subalgebra L_0 of the semidirect sum

$$L = F \niplus N, \quad [F,F] \subseteq F, \quad [F,N] \subseteq N, \quad [N,N] \subseteq N \tag{2.12}$$

is itself a semidirect sum of a subalgebra of F and a subalgebra of N,

$$L_0 = F_0 \niplus N_0, \quad F_0 \subseteq F, \quad N_0 \subseteq N. \tag{2.13}$$

All other subalgebras of $L = F \niplus N$ are called *nonsplitting subalgebras*.

An *abelian splitting subalgebra* of $L = F \ni N$ is a direct sum

$$L_0 = F_0 \oplus N_0, \quad F_0 \subseteq F, \quad N_0 \subseteq N. \tag{2.14}$$

Definition 2.5. A maximal abelian nilpotent subalgebra (MANS) M of a Lie algebra L is a MASA, consisting entirely of nilpotent elements, i.e. it satisfies

$$[M, M] = 0, \quad [[[L, M]M] \dots]_m = 0 \tag{2.15}$$

for some finite number m (we commute M with L m-times). A MANS is represented by nilpotent matrices in any finite dimensional representation.

2.2. Classification strategy

The classification of MASAs of $e(p,q)$ is based on the fact that $e(p,q)$ is the semidirect sum of the Lie algebra $o(p,q)$ and an abelian ideal $T(n)$ (the translations). We use here a procedure related to one used earlier [19] for $e(n, C)$ and [20] for $e(p, 1)$. It proceeds in five steps.

1. Classify subalgebras $T(k_+, k_-, k_0)$ of $T(n)$. They are characterized by a triplet (k_+, k_-, k_0), where k_+, k_- and k_0 are the number of positive length, negative length and isotropic vectors, respectively.
2. Find the centralizer $C(k_+, k_-, k_0)$ of $T(k_+, k_-, k_0)$ in $o(p,q)$
$$C(k_+, k_-, k_0) = \{X \in o(p,q) \mid [X, T(k_+, k_-, k_0)] = 0\}. \tag{2.16}$$
3. Construct all MASAs $M(k_+, k_-, k_0)$ of $C(k_+, k_-, k_0)$ and classify them under the action of the normalizer $\mathrm{Nor}[T(k_+, k_-, k_0), G]$ of $T(k_+, k_-, k_0)$ in the group $G \sim E(p,q)$.
4. Obtain a representative list of all splitting MASAs of $e(p,q)$ as direct sums
$$M(k_+, k_-, k_0) \oplus T(k_+, k_-, k_0) \tag{2.17}$$
and keep only those amongst them that are indeed maximal (and mutually inequivalent).
5. Construct all nonsplitting MASAs from splitting ones as described below in Section 5.1.

3. Results on MASAs of $o(p,q)$

3.1. General results

Let us briefly sum up some known [17] results on MASAs of $o(p,q)$ that we shall need below. We shall represent these MASAs by matrix sets $\{X, K\}$ with notations as in Eqs. (2.3)–(2.7).

Definition 3.1. A MASA of $o(p,q)$ is called orthogonally decomposable (OD) if all matrices in the set $\{X, K\}$ can be simultaneously represented by block diagonal matrices with the same decomposition pattern. It is called orthogonally indecomposable (OID) otherwise.

Proposition 3.1. *Every OD MASA of $o(p,q)$ can be represented by a matrix set*

$$
\begin{aligned}
&X = \mathrm{diag}(X_1, X_2, \ldots, X_k), \qquad K = \mathrm{diag}(K_{p_1,q_1}, K_{p_2,q_2}, \ldots, K_{p_k,q_k}),\\
&X_j K_{p_j,q_j} + K_{p_j,q_j} X_j^{\mathrm{T}} = 0, \qquad X_j, K_{p_j,q_j} \in \mathbb{R}^{(p_j+q_j)\times(p_j+q_j)},\\
&K_{p_j,q_j} = K_{p_j,q_j}^{\mathrm{T}}, \qquad \operatorname{sgn} K_{p_j,q_j} = (p_j\ q_j), \qquad\qquad (3.1)\\
&\det K_{p_j,q_j} \neq 0, \quad 1 \leqslant j \leqslant k, \quad 2 \leqslant k \leqslant [(p+q+1)/2],\\
&\sum_{j=1}^{k} p_j = p, \quad \sum_{j=1}^{k} q_j = q, \quad p_1+q_1 \geqslant p_2+q_2 \geqslant \cdots \geqslant p_k+q_k \geqslant 1,
\end{aligned}
$$

where:

(i) *For each j, the matrix set $\{X_j, K_{p_j,q_j}\}$ represents an OID MASA of $o(p_j,q_j)$; let us call it M_{p_j,q_j}.*

(ii) *At most one of the MASAs M_{p_j,q_j} is a maximal abelian nilpotent subalgebra (MANS) of $o(p_j,q_j)$. In particular only one pair (p_j,q_j) can satisfy $p_j+q_j=1$. The corresponding pair $\{X,K\}$ is $(0,1)$ and represents a MANS of $o(1,0)$ or $o(0,1)$.*

To obtain representatives of all $O(p,q)$ classes of OD MASAs of $o(p,q)$ we let M_{p_j,q_j}, for all j, run independently through all representatives of $O(p_j,q_j)$ conjugacy classes of OID MASAs of $o(p_j,q_j)$, subject to the restriction (ii). *Conversely, each such matrix set represents a conjugacy class of OD MASAs of $o(p,q)$.*

The problem of classifying MASAs of $o(p,q)$ is thus reduced to the classification of OID MASAs. Under the field extension from $\mathbb{R}$ to $\mathbb{C}$ an OID MASA can remain OID, or become orthogonally decomposable. In the first case we call it *absolutely orthogonally decomposable* (AOID) in the second *nonabsolutely orthogonally indecomposable* (NAOID). The following types of orthogonally indecomposable MASAs of $o(p,q)$ exist:

1. Maximal abelian nilpotent subalgebras (MANSs). They exist for all values of (p,q), $\min(p,q) \geqslant 1$. They are discussed below in Section 3.2. They are AOID MASAs.
2. MASAs that are decomposable but not orthogonally decomposable (AOID but D). They stay OID when considered over $\mathbb{C}$. They exist for all values of $p=q \geqslant 1$. Their canonical form is

$$M = \left\{ X_{p,p} = \begin{pmatrix} A & \\ & -A^{\mathrm{T}} \end{pmatrix}, \ K = \begin{pmatrix} & I_p \\ I_p & \end{pmatrix} \right\}, \tag{3.2}$$

where $A = \mathbb{R}I_p \oplus$ MANS of $sl(p, \mathbb{R})$.

3. MASAs that are indecomposable over $\mathbb{R}$ but become orthogonally decomposable after field extension to $\mathbb{C}$ (NAOID, ID but NAID). They exist for $p = 2k$, $q = 2l$, $\min(k, l) \geqslant 1$. Their canonical form is

$$M = \mathbb{R}Q \oplus \text{MANSs of } \mathrm{su}(k, l), \qquad K = \begin{pmatrix} I_{2k} & \\ & -I_{2l} \end{pmatrix},$$

$$Q = \mathrm{diag}(F_2, \ldots, F_2) \in \mathbb{R}^{2(k+l) \times 2(k+l)}, \quad F_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.3}$$

4. MASAs that are indecomposable over $\mathbb{R}$ and decomposable over $\mathbb{C}$ (but not orthogonally decomposable even over $\mathbb{C}$) (OID, AOID but NAID). They exist for $p = q = 2k$, $k \geqslant 1$. Their canonical form is

$$M = \mathbb{R}Q \oplus \text{OID but D MASAs of } \mathrm{su}(k, k)$$

with Q as in Eq. (3.3). An exception is the case of $o(2)$, itself abelian. Thus, for $p = 2$, $q = 0$ or $p = 0$, $q = 2$, $o(2)$ is AOID but NAID.

5. Decomposable MASAs that become orthogonally decomposable over $\mathbb{C}$ (NAOID and D). They occur only for $p = q = 2k$, $k \geqslant 1$. Their canonical form is

$$M = \left\{ X = \begin{pmatrix} A & \\ & -A^{\mathrm{T}} \end{pmatrix}, \ K = \begin{pmatrix} & I_{2k} \\ I_{2k} & \end{pmatrix} \right\}, \tag{3.4}$$

where

$$A = \mathbb{R}Q_{2k} \oplus \text{MANSs of } \mathrm{sl}(2k, \mathbb{C}).$$

3.2. *MANSs of $o(p, q)$*

A MANS M of a classical Lie algebra is characterized by its Kravchuk signature, which we will denote KS [3,9,17,18]. It is a triplet of integers

$$(\lambda\ \mu\ \lambda), \qquad 2\lambda + \mu = n, \qquad \mu \geqslant 0, \qquad 1 \leqslant \lambda \leqslant q \leqslant p, \tag{3.5}$$

where λ is the dimension of the kernel of M, equal to the codimension of the image of M. A MANS can be transformed into the Kravchuk normal form

$$N = \begin{pmatrix} 0 & A & Y \\ 0 & S & -\tilde{K}A^{\mathrm{T}} \\ 0 & 0 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} & & I_\lambda \\ & \tilde{K} & \\ I_\lambda & & \end{pmatrix},$$

$$A \in \mathbb{R}^{\lambda\times\mu}, \quad Y = -Y^{\mathrm{T}} \in \mathbb{R}^{\lambda\times\lambda}, \quad S\tilde{K} + \tilde{K}S^{\mathrm{T}} = 0, \tag{3.6}$$

$$S \in \mathbb{R}^{\mu\times\mu}, \quad \tilde{K} = \tilde{K}^{\mathrm{T}} \in \mathbb{R}^{\mu\times\mu}, \quad \operatorname{sgn}\tilde{K} = (p-\lambda, q-\lambda)$$

and S nilpotent.

There are two types of MANS of $o(p,q)$:

(i) Free-rowed MANS. The first row of A has μ free real entries. All other entries in A and S depend linearly on those μ free entries.

(ii) Non-free-rowed MANS. Any combination of rows of A contains less than μ free real entries.

The results on free-rowed MANS of $o(p,q)$ [17] are stated in the following proposition.

Proposition 3.2. *A representative list of $O(p,q)$ conjugacy classes of free-rowed MANSs of $o(p,q)$ with Kravchuk signature $(\lambda\ \mu\ \lambda)$ is given by the matrix sets*

$$N = \begin{pmatrix} 0 & A & Y \\ 0 & 0 & -\tilde{K}A^{\mathrm{T}} \\ 0 & 0 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} & & I_\lambda \\ & \tilde{K} & \\ I_\lambda & & \end{pmatrix}, \tag{3.7}$$

$$A = \begin{pmatrix} \alpha Q_1 \\ \alpha Q_2 \\ \vdots \\ \alpha Q_\lambda \end{pmatrix}, \quad \alpha \in \mathbb{R}^{1\times\mu}, \qquad Y = -Y^{\mathrm{T}} \in \mathbb{R}^{\lambda\times\lambda}, \tag{3.8}$$

$$Q_i \in \mathbb{R}^{\mu\times\mu}, \quad Q_i\tilde{K} = \tilde{K}Q_i^{\mathrm{T}}, \quad [Q_i, Q_j] = 0, \tag{3.9}$$
$$Q_1 = I, \quad \operatorname{Tr} Q_i = 0, \qquad 2 \leqslant i \leqslant \lambda.$$

The entries in α and Y are free. The matrices Q_i are fixed and form an abelian subalgebra of the Jordan algebra jo$(p-\lambda, q-\lambda)$. *In the case $\lambda = 2$ we must have $Q_2 \neq 0$. There exists a $\lambda_1 \in \mathbb{Z}, 1 \leqslant \lambda_1 \leqslant \lambda$ such that $Q_1, \ldots, Q_{\lambda_1}$ are linearly independent and $Q_\nu = 0$, $\lambda_1 + 1 \leqslant \nu \leqslant \lambda$.*

Proofs of Propositions 3.1 and 3.2 and details about MASAs of $o(p,q)$ are given in Ref. [17]. The results on non-free-rowed MANS of $o(p,q)$ are less complete and we shall not reproduce them here [17].

4. Splitting MASAs of $e(p, q)$

4.1. General comments on MASAs of $e(p, q)$

A MASA of $e(p, q)$ will be represented by a matrix set $\{X_e, K_e\}$

$$X_e = \begin{pmatrix} N & & & & & & \xi^{\mathrm{T}} \\ & X_{p_1,q_1} & & & & & \delta_1^{\mathrm{T}} \\ & & \ddots & & & & \vdots \\ & & & X_{p_j,q_j} & & & \delta_j^{\mathrm{T}} \\ & & & & 0_{k_+} & & x^{\mathrm{T}} \\ & & & & & 0_{k_-} & y^{\mathrm{T}} \\ & & & & & & 0_1 \end{pmatrix}, \tag{4.1}$$

$$K_e = \begin{pmatrix} K_0 & & & & & & \\ & K_{p_1,q_1} & & & & & \\ & & \ddots & & & & \\ & & & K_{p_j,q_j} & & & \\ & & & & I_{k_+} & & \\ & & & & & -I_{k_-} & \\ & & & & & & 0_1 \end{pmatrix}, \tag{4.2}$$

$$p = p_0 + k_0 + \sum_{i=1}^{j} p_i + k_+, \qquad q = q_0 + k_0 + \sum_{i=1}^{j} q_i + k_-, \tag{4.3}$$

where $M_{p_i,q_i} = \{X_{p_i,q_i}, K_{p_i,q_i}\}$, $i = 1, \ldots, j$, is an OID MASA of $o(p_i, q_i)$, that is not a MANS. The vector ξ has the following form:

$$\xi = \begin{pmatrix} z^{\mathrm{T}} \\ \beta^{\mathrm{T}} \\ \gamma^{\mathrm{T}} \end{pmatrix}, \quad z, \gamma \in \mathbb{R}^{1\times k_0}, \ \beta \in \mathbb{R}^{1\times(p_0+q_0)} \tag{4.4}$$

and N is a MANS of $o(p_0 + k_0, q_0 + k_0)$ with Kravchuk signature $(k_0\ p_0 + q_0\ k_0)$ and is given by

$$N = \begin{pmatrix} 0_{k_0} & A & Y \\ 0 & S & -K_{p_0,q_0}A^{\mathrm{T}} \\ 0 & 0 & 0_{k_0} \end{pmatrix}, \quad K_0 = \begin{pmatrix} 0 & 0 & I_{k_0} \\ 0 & K_{p_0,q_0} & 0 \\ I_{k_0} & 0 & 0 \end{pmatrix}, \tag{4.5}$$

$$Y = -Y^{\mathrm{T}}, \qquad SK_{p_0,q_0} + K_{p_0,q_0}S^{\mathrm{T}} = 0$$
$$A \in \mathbb{R}^{k_0\times(p_0+q_0)}, \qquad S \in \mathbb{R}^{(p_0+q_0)\times(p_0+q_0)}, \qquad Y \in \mathbb{R}^{k_0\times k_0}, \tag{4.6}$$
$$K_{p_0,q_0} = K^{\mathrm{T}}_{p_0,q_0}, \qquad \operatorname{sgn} K_{p_0,q_0} = (p_0, q_0).$$

The entries in z, x and y are free and represent the positive, negative and zero length translations contained in $T(k_+, k_-, k_0)$. The entries in β, γ and δ_i are linearly dependent on the free entries in A, Y and X_{p_i,q_i}. If they are nonzero (and cannot be annulled by an $E(p,q)$ transformation), we have a nonsplitting MASA. This case will be discussed in Section 5.

4.2. Basic results on splitting MASAs

In this section we shall construct all splitting MASAs of $e(p,q)$.

Theorem 4.1. *Every splitting MASA of $e(p,q)$ is characterized by a partition*

$$p = p_0 + k_+ + k_0 + \sum_{i=1}^{j} p_i, \qquad q = q_0 + k_- + k_0 + \sum_{i=1}^{j} q_i, \tag{4.7}$$
$$k_0 + k_+ + k_- \neq p + q - 1, \quad 0 \leqslant k_0 \leqslant q.$$

A representative list of $E(p,q)$ conjugacy classes of MASAs of $e(p,q)$ is given by the matrix sets $\{X_e, K_e\}$ of Eqs. (4.1) *and* (4.2) *with*

$$\delta_i = 0, \quad i = 1, \ldots, j, \qquad \xi = \begin{pmatrix} z^{\mathrm{T}} \\ 0 \\ 0 \end{pmatrix}. \tag{4.8}$$

If $k_0 = 0$ then the MANS N is absent. M_{p_i,q_i} is an orthogonally indecomposable MASA of $o(p_i, q_i)$ which is not a MANS. Running through all possible partitions, all MANSs $\{N, K_0\}$ and all MASAs M_{p_i,q_i} we obtain a representative list of all splitting MASAs of $e(p,q)$.

Proof. We start by choosing a subalgebra $T(k_+, k_-, k_0)$. Calculating the centralizer of $T(k_+, k_-, k_0)$ in $o(p,q)$ gives us

$$C(k_+, k_-, k_0) = \begin{pmatrix} \tilde{M} & & \\ & 0_{k_+} & \\ & & 0_{k_-} \end{pmatrix}, \qquad K = \begin{pmatrix} \tilde{K} & & \\ & I_{k_+} & \\ & & -I_{k_-} \end{pmatrix}, \tag{4.9}$$
$$\operatorname{sgn} \tilde{K} = (p - k_+, q - k_-).$$

$\tilde{M}$ is a subalgebra of $o(p - k_+, q - k_-)$ which commutes with the translations corresponding to $\xi = (z, 0)$, $\xi \in \mathbb{R}^{1\times(p+q-k_+-k_-)}$, $z \in \mathbb{R}^{1\times k_0}$, and with no other translations. To obtain a MASA of $e(p,q)$ we must complement $T(k_+,k_-,k_0)$ by a MASA $F(k_+,k_-,k_0)$ of the centralizer $C(k_+,k_-,k_0)$. $F(k_+,k_-,k_0)$ must not commute with any further translations, hence $F(k_+,k_-,k_0)$ is either a MANS of $o(p-k_+,q-k_-)$ with KS $(k_0, p-k_+-k_0+q-k_--k_0, k_0)$ or an orthogonally decomposable MASA containing a MANS N with KS $(k_0\ \mu\ k_0)$. For $k_0=0$ the MANS N is absent. This leads to Eq. (4.8) and each $M_{p_i,q_i} = \{X_{p_i,q_i}, K_{p_i,q_i}\}$ is an OID MASA of $o(p_i,q_i)$ of the type 2, 3, 4, or 5, listed in Section 3.1. □

5. Nonsplitting MASAs of $e(p,q)$

5.1. General comments

First we describe the general procedure for finding nonsplitting MASAs of $e(p,q)$.

Every nonsplitting MASA $M(k_+,k_-,k_0)$ of $e(p,q)$ is obtained from a splitting one by the following procedure:

1. Choose a basis for $F(k_+,k_-,k_0)$ and $T(k_+,k_-,k_0)$ e.g. $F(k_+,k_-,k_0) \sim \{B_1,\dots,B_J\}$, $T(k_+,k_-,k_0) \sim \{X_1,\dots,X_L\}$.
2. Complement the basis of $T(k_+,k_-,k_0)$ to a basis of $T(n)$.

$$T(n)/T(k_+,k_-,k_0) = \{Y_1,\dots,Y_N\}, \qquad L+N=n.$$

3. Form the elements

$$\tilde{B}_a = B_a + \sum_{j=1}^{N} \tilde{\alpha}_{aj} Y_j, \qquad a=1,\dots,J, \tag{5.1}$$

   where the constants $\tilde{\alpha}_{aj}$ are such that $\tilde{B}_a$ form an abelian Lie algebra $[\tilde{B}_a,\tilde{B}_b]=0$. This provides a set of linear equations for the coefficients $\tilde{\alpha}_{aj}$. Solutions $\tilde{\alpha}_{aj}$ are called 1-cocycles and they provide abelian subalgebras $\tilde{M}(k_+,k_-,k_0) \sim \{\tilde{B}_a, X_b\} \subset e(p,q)$.
4. Classify the subalgebras $\tilde{M}(k_+,k_-,k_0)$ into conjugacy classes under the action of the group $E(p,q)$. This can be done in two steps.

(i) Generate trivial cocycles t_{aj}, called coboundaries, using the translation group $T(n)$

$$e^{\theta_j P_j} \tilde{B}_a e^{-\theta_j P_j} = \tilde{B}_a + \theta_j [P_j, \tilde{B}_a] = \tilde{B}_a + \sum_j t_{aj} P_j. \tag{5.2}$$

coboundaries should be removed from the set of the cocycles. If we have $\tilde{\alpha}_{aj} = t_{aj}$ for all (a,j) the algebra is splitting (i.e. equivalent to a splitting one).

(ii)Use the normalizer of the original splitting subalgebra in the group $O(p,q)$ to further simplify and classify the nontrivial cocycles.

The general form of a nonsplitting MASA of $e(p,q)$ is $M_e = \{X_e, K_e\}$ given by Eqs. (4.1) and (4.2). Requiring commutativity $[X_e, X'_e] = 0$ leads to

$$X_{p_i q_i}\delta_i'^{\mathrm{T}} = X'_{p_i q_i}\delta_i^{\mathrm{T}}, \qquad N\xi'^{\mathrm{T}} = N'\xi^{\mathrm{T}}. \tag{5.3}$$

From Eqs. (5.3) we see that the entries in δ_i depend linearly only on $X_{p_i q_i}$, i.e. only on the MASA $M_{p_i q_i}$ of $o(p_i, q_i)$.

Each $M_{p_i q_i}$ belongs to one of the four types of OID MASAs of $o(p_i, q_i)$ which were listed in Section 3.1 – AOID but D MASAs, AOID but NAID MASAs, NAOID ID but NAID MASAs or NAOID but D MASAs.

We will make use of the following result.

Lemma 5.1. *If M is a MASA of $o(p,q)$ when considered over $\mathbb{R}$, then it will also be a MASA of $o(n,\mathbb{C})$, $n = p + q$, when considered over $\mathbb{C}$.*

If any of the vectors δ_i were nonzero then after field extension we would obtain a nonsplitting MASA of $e(n,\mathbb{C})$ of a type that does not exist [19]. This implies that all of the δ_i's are zero.

Any further study of nonsplitting MASAs of $e(p,q)$ is reduced to studying the matrices

$$X_e = \begin{pmatrix} N & & & & & & \xi^{\mathrm{T}} \\ & M_{p_1 q_1} & & & & & 0 \\ & & \ddots & & & & \vdots \\ & & & M_{p_j q_j} & & & 0 \\ & & & & 0_{k_0} & & 0 \\ & & & & & 0_k & 0 \\ & & & & & & 0_1 \end{pmatrix} \tag{5.4}$$

with ξ and N as in Eqs. (4.4) and (4.5), respectively. Further, we can see from Eqs. (5.3) and (5.4) that the study of nonsplitting MASAs is in fact reduced to the study of nonsplitting MASAs of $e(p_0 + k_0, q_0 + k_0)$ for which the projection onto the subalgebra $o(p_0 + k_0, q_0 + k_0)$ is a MANS with Kravchuk signature $(k_0\ \mu\ k_0)$, $\mu = p_0 + q_0$. Further classification is performed under the group $E(p_0 + k_0, q_0 + k_0)$.

The MASAs of $e(p_0 + k_0, q_0 + k_0)$ to be considered will thus be represented by the matrix sets $\{X_e, K_e\}$

$$X_e = \begin{pmatrix} 0_{k_0} & A & Y & z^{\mathrm{T}} \\ 0 & S & -K_{p_0,q_0}A^{\mathrm{T}} & \beta^{\mathrm{T}} \\ 0 & 0 & 0_{k_0} & \gamma^{\mathrm{T}} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K_e = \begin{pmatrix} & & I_{k_0} & \\ & K_{p_0,q_0} & & \\ I_{k_0} & & & \\ & & & 0_1 \end{pmatrix}, \tag{5.5}$$

where $Y = -Y^{\mathrm{T}}$, and $\beta \in \mathbb{R}^{1\times\mu}$, $\gamma \in \mathbb{R}^{1\times k_0}$ depend linearly on the free entries in A and Y. Using the commutativity $[X_e, X'_e] = 0$ we obtain

$$\begin{aligned} &A\beta'^{\mathrm{T}} + Y\gamma'^{\mathrm{T}} = A'\beta^{\mathrm{T}} + Y'\gamma^{\mathrm{T}}, \\ &S\beta'^{\mathrm{T}} - K_{p_0,q_0}A^{\mathrm{T}}\gamma'^{\mathrm{T}} = S'\beta^{\mathrm{T}} - K_{p_0,q_0}A'^{\mathrm{T}}\gamma^{\mathrm{T}}. \end{aligned} \tag{5.6}$$

The translations

$$\Pi = \begin{pmatrix} 0_{k_0} & 0 & 0 & 0 \\ 0 & 0_{p_0,q_0} & 0 & \tau^{\mathrm{T}} \\ 0 & 0 & 0_{k_0} & \zeta^{\mathrm{T}} \\ 0 & 0 & 0 & 0_1 \end{pmatrix}, \qquad \tau \in \mathbb{R}^{1\times\mu}, \quad \zeta \in \mathbb{R}^{1\times k_0} \tag{5.7}$$

will be used to remove coboundaries from β and γ and the remaining cocycles will be classified under the action of the normalizer of the MANS N in the group $O(p_0 + k_0, q_0 + k_0)$.

The situation will be very different for free-rowed and non-free-rowed MANS of $o(p_0 + k_0, q_0 + k_0)$. The two cases will be treated separately.

5.2. Nonsplitting MASAs of $e(p_0 + k_0, q_0 + k_0)$ related to free-rowed MANSs

Let N be a free-rowed MANS of $o(p_0 + k_0, q_0 + k_0)$. The corresponding nonsplitting MASAs of $e(p_0 + k_0, q_0 + k_0)$ can be represented as follows.

Theorem 5.1. *A nonsplitting MASA of $e(p,q)$ must contain a MANS of $o(p_0 + k_0, q_0 + k_0)$ with $1 \leqslant k_0 \leqslant q$, $\min(p_0 + k_0, q_0 + k_0) \geqslant 1$. All nonsplitting MASAs of $e(p_0 + k_0, q_0 + k_0)$ for which the projection onto $o(p_0 + k_0, q_0 + k_0)$ is a free rowed MANS N with Kravchuk signature $(k_0\ \mu\ k_0)$, $\mu = p_0 + q_0$ can be represented by the matrix sets $\{X_e, K_e\}$ of Eq.* (5.5) *with $S = 0$ and A and Y as in Eq.* (3.8).

1. *For $k_0 \geqslant 3$ we have*

$$\beta = a\Lambda, \qquad \gamma = 0 \tag{5.8}$$

$\Lambda \in \mathbb{R}^{\mu\times\mu}$ *satisfies the following conditions:*

$$\Lambda = \Lambda^{\mathrm{T}}, \qquad Q_j\Lambda K^{-1}_{p_0,q_0} = \Lambda K^{-1}_{p_0,q_0}Q_j. \tag{5.9}$$

2. $k_0 = 2$, $\mu \geqslant 2$. *Λ satisfies Eq. (5.9) for $j = 2$ and*

$$\beta = \alpha\Lambda + y\rho, \qquad \gamma = \begin{pmatrix} 0 \\ \alpha\rho^{\mathrm{T}} \end{pmatrix}, \qquad \rho = (1,0,\ldots,0) \tag{5.10}$$

for the following Q:

$$Q = \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \qquad K_{p_0,q_0} = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & K_{p_0-1,q_0-1} \end{pmatrix}. \tag{5.11}$$

For all the other Q

$$\beta = \alpha\Lambda, \qquad \gamma = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.12}$$

3. $k_0 = 2$, $\mu = 1$

$$\beta = y\rho, \qquad \gamma^{\mathrm{T}} = \begin{pmatrix} 0 \\ a\rho + p_2 y \end{pmatrix}, \tag{5.13}$$

where (ρ, p_2) is $(1,0)$, $(0,1)$, or $(1,1)$.

4. $k_0 = 2$, $\mu = 0$, *there is no β and we have*

$$\gamma^{\mathrm{T}} = \begin{pmatrix} y \\ 0 \end{pmatrix}. \tag{5.14}$$

5. $k_0 = 1$, $\mu \geqslant 2$

$$\beta = \alpha\Lambda, \qquad \Lambda = \Lambda^{\mathrm{T}}, \qquad \gamma = 0. \tag{5.15}$$

6. $k_0 = 1$, $\mu = 1$

$$\beta = 0, \qquad \gamma = a. \tag{5.16}$$

The case $k_0 = 1$, $\mu = 0$ is not allowed. Two free-rowed nonsplitting MASAs of $e(p_0 + k_0, q_0 + k_0)$, $M(p_0, q_0, k_0, \Lambda)$ and $M'(p_0, q_0, k_0, \Lambda')$, are $E(p_0 + k_0, q_0 + k_0)$ conjugated (for cases 1 and 5) if the matrices Λ, Λ' characterizing them satisfy

$$A' = \frac{1}{g_1} G_2 \left(A - \sum_{k=1}^{k_0} \theta_k Q_k K_{p_0,q_0} \right) G_2^{-1} \tag{5.17}$$

for some $g_1, g_j \in \mathbb{R}$, $\theta_k \in \mathbb{R}$, $G_2 \in o(p_0, q_0)$ *such that*

$$Q_j = \frac{1}{g_1} g_j G_2 Q_j G_2^{-1}. \tag{5.18}$$

Proof. 1. $k_0 \geqslant 3$. We start from a free-rowed MANS in Eq. (5.5). Requiring commutativity $[X_e, X'_e] = 0$ leads to the following equations:

$$(\alpha Q_j)\beta'^T + y_{ja}\gamma'_a = (\alpha' Q_j)\beta^T + y'_{ja}\gamma_a, \qquad (Q_j\alpha^T)\gamma'_j = (Q_j\alpha'^T)\gamma_j. \tag{5.19}$$

The entries in β, γ are linearly dependent on those in Y and α, i.e.

$$\begin{aligned} \beta &= \alpha A + \sum_{1 \leqslant i<k \leqslant k_0} y_{ik}\rho_{ik}, \quad A \in \mathbb{R}^{p \times p}, \ \rho_{ik} \in \mathbb{R}^{1 \times p}, \\ \gamma &= \alpha W + \sum_{1 \leqslant i<k \leqslant k_0} y_{ik}P_{ik}, \quad W \in \mathbb{R}^{p \times k_0}, \ P_{ik} \in \mathbb{R}^{1 \times k_0}. \end{aligned} \tag{5.20}$$

We substitute β and γ into Eq. (5.19) and compare coefficients of $\alpha_i\alpha'_j$, for i and j fixed. First consider the case $j = 1$. We obtain that

$$\begin{aligned} &A = A^T; \qquad P_{ik,a} = 0, \quad 2 \leqslant i < k, \quad 1 < a: \qquad P_{1k,a} = P_{1a,k}, \\ &\rho_{ik} = 0, \quad 2 \leqslant i < k; \qquad W_a = \rho_{1a}, \quad a \geqslant 2, \\ &Q_j A K^{-1}_{p_0,q_0} = A K^{-1}_{p_0,q_0} Q_j. \end{aligned} \tag{5.21}$$

For $j = 2$ we obtain

$$\begin{aligned} &P_{ik,1} = 0, \quad 3 \leqslant i < k, \qquad P_{12,a} = -P_{2a,1}, \\ &\rho_{1k} = 0, \quad k \geqslant 3, \qquad W_1 = -Q_2\rho_{12}^T. \end{aligned} \tag{5.22}$$

And for $j = 3$ we get

$$W = 0, \qquad \rho_{ik} = 0, \qquad P_{ik} = 0 \qquad \text{for} \quad k_0 \geqslant 3. \tag{5.23}$$

Using the translations we obtain the coboundaries θ_i

$$e^{\theta_i P_i} Z e^{-\theta_i P_i} = Z - \theta_i [Z, P_i]. \tag{5.24}$$

This leads to replacing A by

$$A' = A - \sum_{k=1}^{k_0} \theta_k Q_k K_{p_0,q_0}. \tag{5.25}$$

All θ_i are free and can be used to remove all coboundaries. In particular, if K_{p_0,q_0} is chosen to satisfy $\mathrm{Tr}\, K_{p_0,q_0} \neq 0$ we can use θ_1 to make A traceless. Eq. (5.17) corresponds to transformations of A using the normalizer of N in $E(p,q)$.

2. $k_0 = 2$, $\mu \geqslant 2$. Here there is only one matrix $Q = Q_2$, the vector γ is $\gamma = (\gamma_1, \gamma_2)$ and

$$Y = \begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix}.$$

We have

$$\beta = \alpha A + y\rho, \qquad \rho \in \mathbb{R}^{1\times\mu}, \tag{5.26}$$

$$\gamma_1 = \alpha w_1^{\mathrm{T}} + p_1 y, \quad \gamma_2 = \alpha w_2^{\mathrm{T}} + p_2 y, \quad w_1, w_2 \in \mathbb{R}^{1\times\mu}, \quad p_1, p_2 \in \mathbb{R}. \tag{5.27}$$

From $[X_e, X_e'] = 0$ we obtain that

$$A = A^{\mathrm{T}}, \qquad QAK_{p_0,q_0}^{-1} = AK_{p_0,q_0}^{-1}Q, \tag{5.28}$$

$$\beta = \alpha A + y\rho, \qquad \gamma = \begin{pmatrix} -\alpha Q\rho^{\mathrm{T}} \\ \alpha\rho^{\mathrm{T}} \end{pmatrix}. \tag{5.29}$$

Eq. (5.19) for $j = 2$ leads to

$$[Q^{\mathrm{T}}(\alpha^{\mathrm{T}}\alpha' - \alpha'^{\mathrm{T}}\alpha) + (\alpha'^{\mathrm{T}}\alpha - \alpha^{\mathrm{T}}\alpha')Q]\rho^{\mathrm{T}} = 0. \tag{5.30}$$

Writing Eq. (5.30) [illegible]mponents and choosing α and α' such that $\alpha_a = 1, \alpha'_b = 1$ and all other [illegible] vanish, we obtain

$$(Q^{\mathrm{T}})_{ia}\rho_b - (Q^{\mathrm{T}})_{ib} [illegible] \sum_{k=1}^{\mu}(\delta_{ib}Q_{ak} - \delta_{ia}Q_{bk})\rho_k = 0 \qquad \forall i, a, b. \tag{5.31}$$

This provides us with [illegible]pes of relations

$$Q_{ai}\rho_b - Q_{bi}\rho_a = [illegible] \neq i, \quad b \neq i, \tag{5.32}$$

$$-Q_{ii}\rho_a + Q_{ai}\rho_i + \sum_{k=1} Q_{ak}\rho_k = 0, \qquad a \neq i. \tag{5.33}$$

The matrix Q is block diagonal,

$$Q = \mathrm{diag}(J_1, J_2, \ldots, J_r), \qquad \sum_{i=1}^{r} \dim J_i = \mu,$$
$$\dim J_1 \geqslant \dim J_2 \geqslant \cdots \geqslant \dim J_r \geqslant 1, \tag{5.34}$$

where each J_i is an indecomposable element of a Jordan algebra $\mathrm{jo}(p_i, q_i)$, $p_i + q_i = \dim J_i$ (see e.g Ref. [23]). The matrix K_{p_0,q_0} has the same block structure. Possible forms of elementary blocks in Q are

$$J_i(q_i) = \begin{pmatrix} q_i & 1 & & \\ & q_i & 1 & \\ & & \ddots & 1 \\ & & & q_i \end{pmatrix},$$

$$J_i(r_i + s_i) = \begin{pmatrix} r_i & s_i & 1 & 0 & & & & \\ -s_i & r_i & 0 & 1 & & & & \\ & & \ddots & & \ddots & & & \\ & & & & & & 1 & 0 \\ & & & & & & 0 & 1 \\ & & & & & & r_i & s_i \\ & & & & & & -s_i & r_i \end{pmatrix}. \tag{5.35}$$

After complexification the second type of block reduces to the first one, so it actually suffices to consider the first type of block only (see Lemma 5.1).

Let us first assume $\dim J_1 \geqslant 3$. Writing relation (5.33) for $i = 1$ and $2 \leqslant a \leqslant r$ we obtain $\rho_3 = \rho_4 = \cdots = \rho_\mu = 0$. Taking $a = 1, i = 2$ in Eq. (5.32) we then obtain $\rho_2 = 0$. Taking $a = 1, b = 2, i = 3$ in Eq. (5.33) we obtain $\rho_1 = 0$. Thus, if the largest block $J_1(q)$ satisfies $\dim J_1(q) \geqslant 3$, we have $\rho = 0$.

Now let us assume $\dim J_1(q) = 2$ so that all other blocks have dimension 2 or 1. By the same argument we have $\rho_3 = \rho_4 = \cdots = \rho_\mu = 0$ and also $\rho_2 = 0$. If Q has the form (5.11), then all relations (5.32) and (5.33) are satisfied and ρ_1 remains free. If any of the other diagonal elements, say Q_{33} is not zero, then relation (5.33) for $i = 3, a = 1$ implies $\rho_1 = 0$. If we have $q \neq 0$ in $J_1(q)$, then at least one other diagonal element of Q must satisfy $Q_{aa} \neq 0, a \geqslant 3$, since we have $\operatorname{Tr} Q = 0$.

Finally, let Q be diagonal. We have $Q \neq 0, \operatorname{Tr} Q = 0$, hence at least two diagonal elements are nonzero. Relations (5.32) and (5.33) then imply $\rho_i = 0, i = 1, \ldots, \mu$.

Using the normalizer $G = \operatorname{diag}(g_1, g_2, G_2, g_1^{-1}, g_2^{-1})$ we normalize ρ_1 to $\rho_1 = 1$ for $\rho_1 \neq 0$.

3. $k_0 = 2$, $\mu = 1$. There is no matrix Q and we have

$$\beta = \Lambda a + \rho y, \qquad \lambda \in \mathbb{R} \tag{5.36}$$

$$\gamma_1 = a w_1 + p_1 y, \qquad \gamma_2 = a w_2 + p_2 y, \qquad w_1, w_2, p_1, p_2 \in \mathbb{R}.$$

Condition $[X_e, X_e'] = 0$ implies $w_1 = 0$, $p_1 = 0$ and after removing the coboundaries we obtain

$$\beta = \rho y, \qquad \gamma_1 = 0, \qquad \gamma_2 = a\rho + p_2 y. \tag{5.37}$$

Using the normalizer $G = \mathrm{diag}(g_1, g_2, g_3, g_4, g_5, 1)$, satisfying $G\tilde{K}_0 G^{\mathrm{T}} = \tilde{K}_0$, we can normalize (ρ, p_2) to one of the following: $(1,0), (1,1), (0,1)$.

4. $k_0 = 2$, $\mu = 0$. Using the normalizer $G = \mathrm{diag}(g_1, G_2, 1/g_1, 1)$ we obtain Eq. (5.14).

5. $k_0 = 1$, $\mu \geqslant 2$. In this case $Y = 0$ and $A = \alpha \in \mathbb{R}^{1 \times \mu}$ in Eq. (5.5). Then we have

$$\beta = \alpha A, \qquad \gamma = \alpha w^{\mathrm{T}}, \quad \beta \in \mathbb{R}^{1 \times \mu}, \quad \gamma \in \mathbb{R}. \tag{5.38}$$

From $[X_0, X_0'] = 0$ we obtain that

$$A = A^{\mathrm{T}}, \qquad w = 0. \tag{5.39}$$

Removing the coboundaries leads to replacing A by

$$A' = A - \theta K_{p_0, q_0}, \tag{5.40}$$

where θ can be chosen to annul trace of A (if $\mathrm{Tr}\, K_{p_0, q_0} \neq 0$).

6. $k_0 = 1$, $\mu = 1$. The proof is trivial and can be found in Ref. [20].

Using the normalizer of the splitting MASA (4.8) in the group $E(p_0 + k_0, q_0 + k_0)$ we can simplify A further. The normalizer is represented by block diagonal matrices

$$G = \mathrm{diag}(G_1, G_2, G_1^{-1}, 1). \tag{5.41}$$

Choosing $G_1 = \mathrm{diag}(g_1, \ldots, g_{k_0})$, G_2 satisfying $G_2 K_{p_0, q_0} G_2^{\mathrm{T}} = K_{p_0, q_0}$ leads to Eqs. (5.17) and (5.18).

This completes the proof of the Theorem 5.1. □

5.3. Nonsplitting MASAs of $e(p_0 + k_0, q_0 + k_0)$ related to non-free-rowed MANSs

The general study of non-free rowed MASAs of $o(p,q)$ is less well developed. Many different series of MASAs of $o(p,q)$ exist. We will consider only two of them, which we denote $A(2k+1, 0)$ and $A(2k+1, 1)$, by analogy with series of non-free-rowed MANSs of $o(n, \mathbb{C})$ [16].

1. The series $A(2k+1,0)$ of $o(p,q)$ is represented by the matrix set

$$X = \begin{pmatrix} 0 & a_1 & 0 & a_2 & \dots & a_k & 0 \\ & \ddots & \ddots & \ddots & \ddots & & a_k \\ & & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & \ddots & a_2 \\ & & & & \ddots & \ddots & 0 \\ & & & & & \ddots & a_1 \\ & & & & & & 0 \end{pmatrix}, \tag{5.42}$$

$$K = F_{2k+1} = \begin{pmatrix} & & & & \epsilon \\ & & & -\epsilon & \\ & & \epsilon & & \\ & \iddots & & & \\ -\epsilon & & & & \\ \epsilon & & & & \end{pmatrix}, \tag{5.43}$$

where all a_i's are free. Thus for $\epsilon = 1$ we have

$$M \subset \begin{cases} o(k+1,k) & \text{for } k \text{ even}, \\ o(k,k+1) & \text{for } k \text{ odd}, \end{cases}$$

and for $\epsilon = -1$ we have

$$M \subset \begin{cases} o(k+1,k) & \text{for } k \text{ odd}, \\ o(k,k+1) & \text{for } k \text{ even}. \end{cases}$$

The splitting MASA of $e(p,q)$ for this series (in accordance with Theorem 4.1) is written as follows:

$$X_e = \begin{pmatrix} 0 & a_1 & 0 & a_2 & \dots & a_k & 0 & \alpha \\ & \ddots & \ddots & \ddots & \ddots & & a_k & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & \ddots & \ddots & a_2 & 0 \\ & & & & \ddots & \ddots & 0 & 0 \\ & & & & & \ddots & a_1 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{pmatrix}. \tag{5.44}$$

Theorem 5.2. *Every nonsplitting MASA of $e(p,q)$ corresponding to the splitting MASA* (5.44) *is $E(p,q)$ conjugate to the following one:*

$$X_e = \begin{pmatrix} 0 & a_1 & 0 & a_2 & \cdots & \cdots & a_k & 0 & \alpha \\ & \ddots & \ddots & \ddots & \ddots & & & a_k & 0 \\ & & \ddots & \ddots & \ddots & \ddots & & 0 & a_k \\ & & & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & & \ddots & \ddots & \ddots & a_2 & 0 \\ & & & & & \ddots & \ddots & 0 & a_2 \\ & & & & & & \ddots & a_1 & 0 \\ & & & & & & & 0 & a_1 \\ & & & & & & & & 0 \end{pmatrix}, \quad K_e = \begin{pmatrix} F_{2k+1} & \\ & 0 \end{pmatrix}, \tag{5.45}$$

where all entries in X_e are free.

Proof. We will construct a nonsplitting MASA from the splitting one, Eq. (5.44),

$$X_e' = \begin{pmatrix} 0 & a_1 & 0 & a_2 & \cdots & & a_k & 0 & \alpha \\ & 0 & a_1 & 0 & a_2 & \cdots & & a_k & \beta_2 \\ & & \ddots & \ddots & \ddots & \ddots & & 0 & \beta_3 \\ & & & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & & \ddots & \ddots & \ddots & a_2 & \beta_{2k-2} \\ & & & & & \ddots & \ddots & 0 & \beta_{2k-1} \\ & & & & & & \ddots & a_1 & \beta_{2k} \\ & & & & & & & 0 & \beta_{2k+1} \\ & & & & & & & & 0 \end{pmatrix}, \tag{5.46}$$

where β's are linearly dependent on a_i's. Before imposing commutation relations we will remove the coboundaries.

Consider one element of the algebra (5.46)

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & & \dots & 0 & 0 \\ & 0 & 1 & 0 & \dots & & \dots & 0 & \alpha_{1,2} \\ & & \ddots & \ddots & \ddots & & & 0 & \alpha_{1,3} \\ & & & \ddots & \ddots & \ddots & & \vdots & \vdots \\ & & & & \ddots & \ddots & \ddots & 0 & \alpha_{1,2k-2} \\ & & & & & \ddots & \ddots & 0 & \alpha_{1,2k-1} \\ & & & & & & \ddots & 1 & \alpha_{1,2k} \\ & & & & & & & 0 & \alpha_{1,2k+1} \\ & & & & & & & & 0 \end{pmatrix}, \tag{5.47}$$

where $\alpha_{1,l}$, $l = 2, \dots, 2k+1$, represent the translations. We note that $\alpha_{1,l}, \dots, \alpha_{1,2k}$ correspond to coboundaries and can be eliminated by conjugation by the translation group. Thus only $\alpha_{1,2k+1}$ is left in A_1.

Now consider an element A_i of algebra (5.46), obtained by setting $a_i = \delta_{ij}$, $j \geqslant 2$

$$A_i = \begin{pmatrix} 0 & 0 & 0 & 1 & \dots & & \dots & 0 & 0 \\ & 0 & 0 & 0 & 1 & & \dots & 0 & \alpha_{i,2} \\ & & \ddots & \ddots & \ddots & \ddots & & 0 & \alpha_{i,3} \\ & & & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & & \ddots & \ddots & \ddots & 1 & \alpha_{i,2k-2} \\ & & & & & \ddots & \ddots & 0 & \alpha_{i,2k-1} \\ & & & & & & \ddots & 0 & \alpha_{i,2k} \\ & & & & & & & 0 & \alpha_{i,2k+1} \\ & & & & & & & & 0 \end{pmatrix}. \tag{5.48}$$

Commuting A_1 with all $A_i, i = 2, \dots, k$, we obtain that $\alpha_{j,2k-2j+3} = \alpha_{1,2k+1}$, $j = 2, \dots, k$ and all other $\alpha_{i,j}$ have to be zero.

Using the normalizer G of the form

$$G = (g_k^k, \dots g_k^2, g_k, 1, g_k^{-1}, \dots, g_k^{-k}) \tag{5.49}$$

we can normalize $\alpha_{1,2k+1}$ to $\alpha_{1,2k+1} = 1$. This leads to the MASA (5.45) and completes the proof of Theorem 5.2. □

2. The series $A(2k+1,1)$ of $o(p,q)$ is represented by the following matrix set:

$$X = \begin{pmatrix} 0 & a_1 & 0 & a_2 & \dots & a_k & 0 & b \\ & \ddots & \ddots & \ddots & \ddots & & a_k & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & \ddots & \ddots & a_2 & 0 \\ & & & & \ddots & \ddots & 0 & 0 \\ & & & & & \ddots & a_1 & 0 \\ & & & & & & 0 & 0 \\ & & & & & 0 & -\epsilon b & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} F_{2k+1} & \\ & 1 \end{pmatrix}, \tag{5.50}$$

where all a_i's and b are free. The corresponding metric is

$$K = \begin{pmatrix} F_{2k+1} & \\ & 1 \end{pmatrix} = \begin{pmatrix} & & & & & \epsilon & 0 \\ & & & & -\epsilon & & 0 \\ & & & \epsilon & & & \vdots \\ & & \iddots & & & & \vdots \\ & -\epsilon & & & & & 0 \\ \epsilon & & & & & & 0 \\ 0 & 0 & \dots & & \dots & 0 & 1 \end{pmatrix}. \tag{5.51}$$

Thus for $\epsilon = 1$ we have

$$M \subset \begin{cases} o(k+1,k+1) & \text{for } k \text{ odd}, \\ o(k+2,k) & \text{for } k \text{ even}, \end{cases}$$

and for $\epsilon = -1$ we have

$$M \subset \begin{cases} o(k+1,k+1) & \text{for } k \text{ even}, \\ o(k+2,k) & \text{for } k \text{ odd}. \end{cases}$$

Theorem 5.3. *Every nonsplitting MASA corresponding to the splitting MASA* (5.50) *is $E(p,q)$ conjugated to the MASA of the form*

$$X_c = \begin{pmatrix} 0 & a_1 & 0 & a_2 & \cdots & & a_k & 0 & b & \alpha \\ & \ddots & \ddots & \ddots & & & a_k & 0 & \lambda b \\ & & \ddots & \ddots & \ddots & & 0 & 0 & 0 \\ & & & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ & & & & \ddots & \ddots & a_2 & 0 & 0 \\ & & & & & \ddots & 0 & 0 & 0 \\ & & & & & \ddots & a_1 & 0 & 0 \\ & & & & & & 0 & 0 & 0 \\ & & & & & & -\varepsilon b & 0 & \lambda a_1 + \mu b \\ & & & & & & 0 & 0 & 0_1 \end{pmatrix} \tag{5.52}$$

with the metric as in Eq. (5.51). The entries a_i, b and α are free. Parameters λ and μ are one of the following sets:

$$(\lambda, \mu) = \begin{cases} (0, 1), \\ (0, -1), \\ (1, \mu), \; \mu \in \mathbb{R}. \end{cases} \tag{5.53}$$

Proof. The proof is similar to that of Theorem 5.2 and we omit it here. □

6. Decomposition properties of MASAs of $e(p, q)$

The results of Sections 4 and 5 can be formulated in terms of a decomposition of the underlying pseudoeuclidean space $S(p, q)$. Both splitting and nonsplitting MASAs have been represented by matrix sets $\{X_c, K_c\}$ as in Eqs. (5.4) and (4.2). We shall call a MASA of $e(p, q)$ *decomposable* if the metric K_c in Eq. (4.2) consists of two or more blocks. The projection of such a MASA onto the $o(p, q)$ subalgebra is then an orthogonally decomposable MASA of $o(p, q)$. Let $M_c(p, q)$ be a decomposable MASA of $e(p, q)$. The space $S(p, q)$ then splits into a direct sum of subspaces

$$S(p,q) = \bigoplus_{i=1}^{l} S(p_i, q_i), \qquad \sum_{i=1}^{l} p_i = p, \qquad \sum_{i=1}^{l} q_i = q \tag{6.1}$$

and each indecomposable component of the decomposable MASA of $e(p, q)$ acts independently in one of the spaces $S(p_i, q_i)$. We shall write

$$M_e(p,q) = \bigoplus_{i=1}^{l} M_e(p_i,q_i). \tag{6.2}$$

Each individual indecomposable MASA $M_e(p_i,q_i) \subset e(p_i,q_i)$ can then be considered separately.

Consider the matrix set $\{X_e, K_e\}$, X_e given by Eq. (5.4), K_e as in Eq. (4.2), where each block is indecomposable. The blocks to be considered consist of a block on the diagonal in X_e, plus an entry from the right hand column in X_e.

The following types of indecomposable MASAs $M_e(p_i,q_i) \subset e(p_i,q_i)$ exist.

- dim $S = 1$. The MASAs are pure positive or negative length translations.

$$M_e(1,0) = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix},\ x \in \mathbb{R},\ K_e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}. \tag{6.3}$$

$$M_e(0,1) = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix},\ y \in \mathbb{R},\ K_e = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \right\}. \tag{6.4}$$

  A MASA $M_e(p,q)$ of $e(p,q)$ contains k_+ of the first ones and k_- of the second.

- dim $S = 2$. The MASAs are $o(2)$ rotations in a $(++)$, or $(--)$ type subspace, or $o(1,1)$ pseudorotations in a $(+-)$ space:

$$M_e(2,0) = \left\{ \begin{pmatrix} 0 & x & 0 \\ -x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ K_e = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \tag{6.5}$$

$$M_e(0,2) = \left\{ \begin{pmatrix} 0 & x & 0 \\ -x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ K_e = \begin{pmatrix} -I_2 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \tag{6.6}$$

$$M_e(1,1) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0 \end{pmatrix},\ K_e = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}. \tag{6.7}$$

- dim $S = k \geqslant 3$. There are two possible types of indecomposable MASAs of $e(p,q)$ for $p+q \geqslant 3$. Both of them have $k_+ = k_- = 0$ (no nonisotropic translations).

  (i) $M_e(p,q)$ contains k_0 isotropic translations with $k_0 \geqslant 1$. The projection of $M_e(p,q)$ onto $o(p,q)$ is then a MANS of $o(p,q)$ with Kravchuk signature $(k_0, p+q-2k_0, k_0)$. The MANS can be free-rowed or non-free-rowed. The MASA of $e(p,q)$ can be splitting , or nonsplitting. Such MASAs exist for any $p+q \geqslant 3$, $\min(p,q) \geqslant 1$. They were treated in Sections 4 and 5.

(ii) $M(p,q)$ is an orthogonally indecomposable MASA of $o(p,q)$ that is not a MANS. It gives rise to a splitting MASA of $e(p,q)$ which contains no translations ($k_0 = 0$). As reviewed in Section 3 such MASAs of $o(p,q)$ exist only for $p+q$ even.

7. A special case: MASAs of $e(p,2)$

The case $q=2$, like $q=1$ and $q=0$, presented earlier [20] is simpler than that of $q \geqslant 3$. All MASAs can be presented explicitly, in particular those involving non-free-rowed MANS of $o(p,2)$.

The possible decomposition patterns, Eq. (6.2), for MASAs of $e(p,2)$ are

$$\begin{aligned} &M_e(p,2) = M_e(p_1,2) \oplus l_+ M_e(2,0) + k_+ M_e(1,0), \\ &\quad p_1 = 1, \quad \text{or} \quad p_1 \geqslant 2, \qquad p_1 + 2l_+ + k_+ = p, \end{aligned} \tag{7.1}$$

$$\begin{aligned} &M_e(p,2) = M_e(p_1,1) \oplus M_e(p_2,1) \oplus l_+ M_e(2,0) + k_+ M_e(1,0), \\ &\quad p_1 + p_2 + 2l_+ + k_+ = p, \end{aligned} \tag{7.2}$$

$$\begin{aligned} &M_e(p,2) = M_e(0,2) \oplus l_+ M_e(2,0) + k_+ M_e(1,0), \\ &\quad 2l_+ + k_+ = p. \end{aligned} \tag{7.3}$$

The algebras $M_e(2,0)$, $M_e(0,2)$ and $M_e(1,0)$ are already abelian (and one dimensional) as are $M_e(0,1)$ and $M_e(1,1)$. The MASAs $M_e(p,1)$ of $e(p,1)$, $p \geqslant 2$ were studied in our earlier paper [20].

Thus, we need to treat only indecomposable MASAs of $e(p,2)$. As was stated in Section 6 for general $e(p,q)$, two cases arise, namely $k_0 = 0$ and $1 \leqslant k_0 \leqslant \min(p,q)$, where k_0 is the number of linearly independent translation generators present.

1. $k_0 = 0$. $M(p,2)$ is an orthogonally indecomposable MASA of $o(p,2)$ that is not a MANS. These exist only when p is even ($p \geqslant 2$).

For $p=2$ three inequivalent OID MASAs that are not MANS exist and the corresponding splitting MASAs of $e(p,2)$ are given by the following matrix sets:

(i) $M(2,2)$ is AOID but D

$$X_e = \begin{pmatrix} a & b & & & 0 \\ 0 & a & & & 0 \\ & & -a & 0 & 0 \\ & & -b & -a & 0 \\ & & & & 0_1 \end{pmatrix}, \quad K_e = \begin{pmatrix} & I_2 & \\ I_2 & & \\ & & 0_1 \end{pmatrix}. \tag{7.4}$$

(ii) $M(2,2)$ is AOID, ID but NAID

$$X_e = \begin{pmatrix} 0 & a & 0 & b & 0 \\ -a & 0 & -b & 0 & 0 \\ & & 0 & a & 0 \\ & & -a & 0 & 0 \\ & & & & 0_1 \end{pmatrix} \tag{7.5}$$

with K_e same as in (i).

(iii) $M(2,2)$ is NAOID but D

$$X_e = \begin{pmatrix} a & b & & & 0 \\ -b & a & & & 0 \\ & & -a & b & 0 \\ & & -b & -a & 0 \\ & & & & 0_1 \end{pmatrix} \tag{7.6}$$

with K_e same as in (i).

For $p = 2l$, $l \geqslant 2$, we have just one OID MASA of $o(p,2)$ (NAOID, ID but NAID), namely $M = RQ \oplus$ MANS of $\mathrm{su}(l,1)$. The corresponding splitting MASA of $e(p,2)$ is represented as following matrix set:

$$X_e = \begin{pmatrix} 0 & b & a_1 & 0 & \dots & a_{l-1} & 0 & 0 & c & 0 \\ -b & 0 & 0 & a_1 & \dots & 0 & a_{l-1} & -c & 0 & 0 \\ & & 0 & b & & & & -a_1 & 0 & 0 \\ & & -b & 0 & & & & 0 & -a_1 & 0 \\ & & & & \ddots & & & \vdots & & \vdots \\ & & & & & 0 & b & -a_{l-1} & 0 & \vdots \\ & & & & & -b & 0 & 0 & -a_{l-1} & 0 \\ & & & & & & & 0 & b & 0 \\ & & & & & & & -b & 0 & 0 \\ & & & & & & & & & 0_1 \end{pmatrix},$$

$$K_e = \begin{pmatrix} & & I_2 & \\ & I_{2l-2} & & \\ I_2 & & & \\ & & & 0_1 \end{pmatrix}. \tag{7.7}$$

2. $k_0 = 1$. The projection of $M_e(p,2)$ onto $o(p,2)$ will be a MANS of $o(p,2)$ with Kravchuk signature $(1\ p\ 1)$. This MANS can be free-rowed, or non-free-

rowed, so we obtain two splitting MASAs of $o(p,2)$ represented, respectively, by

(i) free-rowed

$$X_e = \begin{pmatrix} 0 & \alpha & 0 & z & \\ 0 & 0 & -K_0\alpha^{\mathrm{T}} & 0 & \\ 0 & 0 & 0 & 0 & \\ & & & & 0_1 \end{pmatrix}, \quad K_e = \begin{pmatrix} 0 & 0 & 1 \\ 0 & K_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{7.8}$$

where K_0 has signature $(p-1,1)$, $\alpha \in \mathbb{R}^{1\times p}$, $1 \leqslant p$,

(ii) non-free rowed

$$X_e = \begin{pmatrix} 0 & a & \alpha & 0 & b & 0 & z \\ & 0 & 0 & a & 0 & -b & 0 \\ & & 0 & 0 & 0 & -\alpha^{\mathrm{T}} & \vdots \\ & & & 0 & -a & 0 & 0 \\ & & & & 0 & -a & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0_1 \end{pmatrix},$$

$$K_e = \begin{pmatrix} & & & & 1 & 0 \\ & & & 1 & 0 & 0 \\ & & I_{\nu+1} & 0 & 0 & \vdots \\ & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0_1 \end{pmatrix}. \tag{7.9}$$

$\alpha \in \mathbb{R}^{1\times\nu}$, $1 \leqslant \nu$ and $\nu = p-3$.

The MASA (7.8) gives rise to three different nonsplitting MASAs for $p \geqslant 2$ which can be expressed as

$$X_e = \begin{pmatrix} 0 & \alpha & 0 & z \\ 0 & 0 & -K_0\alpha^{\mathrm{T}} & BK_0\alpha^{\mathrm{T}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_e = \begin{pmatrix} & & 1 & \\ & K_0 & & \\ 1 & & & \\ & & & 0_1 \end{pmatrix}. \tag{7.10}$$

K_0 is the same as in Eq. (7.8) and B satisfies the condition $BK_0 = K_0B^{\mathrm{T}}$, i.e. B is an element of the Jordan algebra $\mathrm{jo}(p-1,1)$. A classification of the elements of Jordan algebras was performed in the paper by Djokovic et al. [23] and the couple $\{B, K_0\}$ can have one of the three different following forms (keeping in mind the signature of K_0):

(i)

$$B = \begin{pmatrix} a & \\ & B_0 \end{pmatrix}, \quad K_0 = \begin{pmatrix} -1 & \\ & I \end{pmatrix}. \tag{7.11}$$

(ii)

$$B = \begin{pmatrix} a & 0 & \\ 1 & a & \\ & & B_0 \end{pmatrix}, \quad K_0 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & I \end{pmatrix}. \tag{7.12}$$

(iii)

$$B = \begin{pmatrix} a & 0 & 0 & \\ 1 & a & 0 & \\ 0 & 1 & a & \\ & & & B_0 \end{pmatrix}, \quad K_0 = \begin{pmatrix} 0 & 0 & 1 & \\ 0 & 1 & 0 & \\ 1 & 0 & 0 & \\ & & & I \end{pmatrix}, \tag{7.13}$$

where B_0 is a diagonal matrix.

For $p = 1$ the nonsplitting MASA corresponding to Eq. (7.15) is

$$X_c = \begin{pmatrix} 0 & a & 0 & z \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_c = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0_1 \end{pmatrix}. \tag{7.14}$$

The MASA (7.9) for $v \geqslant 2$ gives rise to one type of nonsplitting MASA that can be represented as

$$X_c = \begin{pmatrix} 0 & a & \gamma & 0 & b & 0 & z \\ & 0 & 0 & a & 0 & -b & \alpha\rho^{\mathrm{T}} \\ & & 0 & 0 & 0 & -\alpha^{\mathrm{T}} & a\rho^{\mathrm{T}} + \Lambda\alpha^{\mathrm{T}} \\ & & & 0 & -a & 0 & 0 \\ & & & & 0 & -a & 0 \\ 0 & & & & & 0 & 0 \\ & & & & & & 0_1 \end{pmatrix} \tag{7.15}$$

with $\Lambda = \Lambda^{\mathrm{T}}$. Using the normalizer $G = \mathrm{diag}(g, g_1, G_2, g_3, 1/g_1, g, 1)$, $G_2 \in \mathbb{R}^{v \times v}$, $g, g_1, g_3 \in \mathbb{R}$, satisfying $G_2 G_2^{\mathrm{T}} = I_v$, $g^2 = g_3^2 = 1$ we can transform Λ, ρ into

$$\Lambda' = \frac{1}{g} G_2 \Lambda G_2^{\mathrm{T}}, \qquad \rho' = \frac{1}{g_1 g_3} G_2 \rho. \tag{7.16}$$

We can use G_2 either to diagonalize Λ, or to rotate ρ into e.g. $\rho = (\rho_1, 0, \ldots, 0)$.

3. $k_0 = 2$. The projection of $M_e(p,2)$ onto $o(p,2)$ is a free-rowed MANS with Kravchuk signature $(2\ p-2\ 2)$. The corresponding splitting MASA of $e(p,2)$ is given in Theorem 5.1 with $q = k_0 = 2$ and $K_{p_0,q_0} = I_{p-2}$. In this case Q_2 can be chosen as $Q_2 = \mathrm{diag}(1, q_2, \ldots, q_\mu)$, $q_1 = 1 \geqslant |q_2| \geqslant \cdots \geqslant |q_\mu|$.This MASA in turn gives rise to the following nonsplitting MASAs.

$$X_e = \begin{pmatrix} 0 & 0 & \alpha & 0 & y & z_1 \\ 0 & 0 & \alpha Q & -y & 0 & z_2 \\ 0 & 0 & 0 & -\alpha^{\mathrm{T}} & -Q\alpha^{\mathrm{T}} & A^{\mathrm{T}}\alpha^{\mathrm{T}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_1 \end{pmatrix}. \tag{7.17}$$

Here A is a diagonal matrix, $\mathrm{Tr}\,A = 0$ and K_e is same as in Eq. (5.5).

8. Conclusions

The main conclusion is that we have presented guidelines for constructing all MASAs of $e(p,q)$ for any fixed values of p and q. Some of the results are entirely explicit, such as Theorem 4.1 describing all splitting MASAs of $e(p,q)$, and Theorem 5.1 presenting nonsplitting MASAs containing a free-rowed MANS of $o(p_0 + k_0, q_0 + k_0) \subset o(p,q)$. The results on MASAs of $e(p,q)$ involving non-free-rowed MANS of $o(p_0 + k_0, q_0 + k_0)$ are less complete and amount to specific examples (see Theorems 5.2 and 5.3). The decomposition results of Section 6 allow us to restrict all considerations to indecomposable MASAs of $e(p,q)$. both splitting and nonsplitting ones. The results for $e(p,2)$ presented in Section 7 are complete and explicit, like those given earlier for $e(p,0)$ and $e(p,1)$ [20]. In particular we have constructed all MASAs related to non-free-rowed MANSs.

Work concerning the application of MASAs of $e(p,q)$ is in progress. In particular, we use MASAs of $e(p,q)$ to construct the coordinate systems in which certain partial differential equations (Laplace–Beltrami, Hamilton–Jacobi) allow the separation of variables.

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