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On end extensions of models of subsystems of peano arithmetic

C. Dimitracopoulos

Department of History and Philosophy of Science, University of Athens, 37, J. Kennedy Street, 16121 Kaisariani, Greece

Abstract

We survey results and problems concerning subsystems of Peano Arithmetic. In particular, we deal with end extensions of models of such theories. First, we discuss the results of Paris and Kirby (Logic Colloquium '77, North-Holland, Amsterdam, 1978, pp. 199–209) and of Clote (Fund. Math. 127 (1986) 163; Fund. Math. 158 (1998) 301), which generalize the MacDowell and Specker theorem (Proc. Symp. on Foundation of Mathematics, Warsaw, 1959, Pergamon Press, Oxford, 1961, pp. 257–263) we also discuss a related problem of Kaufmann (On existence of Σ_n end extensions, Lecture Notes in Mathematics, Vol. 859, Springer, Berlin, 1980, p. 92). Then we sketch an alternative proof of Clote's theorem, using the arithmetized completeness theorem in the spirit of McAloon (Trans. Amer. Math. Soc. 239 (1978) 253) and Paris (Some conservation results for fragments of arithmetic, Lecture Notes in Mathematics, Vol. 890, Springer, Berlin, 1981, p. 251). © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Σ_n-induction schema; Σ_n-collection schema; End extension; Σ_n-ultrafilter; Arithmetized completeness theorem

1. Introduction

Our aim is to survey some results and problems concerning theories in the usual first-order language of arithmetic LA. Our notation is standard, so let P^- denote the set of axioms for non-negative parts of discretely ordered rings, $I\Sigma_n$ denote P^- plus the induction schema for Σ_n formulas of LA, $L\Sigma_n$ denote P^- plus the least number schema for Σ_n formulas and $B\Sigma_n$ denote $I\Sigma_0$ plus the collection schema for Σ_n formulas; $I\Pi_n$, $L\Pi_n$ and $B\Pi_n$ are defined similarly.

The following theorem summarizes the relationships among the theories $I\Sigma_n$, $L\Sigma_n$, etc.

E-mail address: cdmitr@cc.uoa.gr (C. Dimitracopoulos).

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Theorem 1. *For all* $n \geqslant 0$:

$$\begin{array}{l} I\Sigma_{n+1} \\ \Downarrow \\ B\Sigma_{n+1} \Leftrightarrow B\Pi_n \\ \Downarrow \\ I\Sigma_n \Leftrightarrow I\Pi_n \Leftrightarrow L\Sigma_n \Leftrightarrow L\Pi_n. \end{array}$$

Furthermore, the converses to the two vertical arrows are false.

The implication $I\Sigma_{n+1} \Rightarrow B\Sigma_{n+1}$ and the non-implication $I\Sigma_n \not\Rightarrow B\Sigma_{n+1}$ were proved by Parsons [14], while the rest of the theorem is due to Paris and Kirby [13].

The study of subsystems of PA (=Peano Arithmetic) was continued towards achieving two objectives, namely

(a) finding a fragment strong enough to serve as a basis for elementary number theory and combinatorics, and
(b) obtaining independence results for central open problems in complexity theory, by relating them to the strength of fragments of PA.

Work concerning objective (a) concentrated on the study of subsystems whose strength is strictly below that of the theory $I\Sigma_1$. The following fundamental problems were posed, by Wilkie and Macintyre, respectively, in the late 1970s and still remain open.

Problem 1. *Does* $I\Sigma_0$ *prove that the set of primes is unbounded*?

Problem 2. *Does* $I\Sigma_0$ *prove the pigeonhole principle for* Σ_0 *definable maps*?

An excellent source of related information is the survey paper [9] of Macintyre.

The impetus for work-concerning objective (b) was given by Hartmanis and Hopcroft, who proved [5] that for a certain recursive set A the sentence $P^A = NP^A$ is independent of the axioms of set theory. Later a surprising connection was observed between the question whether or not $P = co\,NP$ and the following problem of Paris.

Problem 3. *Does* $I\Sigma_0$ *prove the Davis–Matiyasevich–Putnam–Robinson theorem*? *I.e., is it true that for every* Σ_1 *formula* $\varphi(\vec{x})$ *we can effectively find a polynomial with integer coefficients* $p(\vec{x}, \vec{y})$ *such that*

$$I\Sigma_0 \vdash \forall\vec{x}(\varphi(\vec{x}) \leftrightarrow \exists\vec{y}\, p(\vec{x}, \vec{y}) = 0)?$$

The connection was discovered by Wilkie [15] and is as follows.

If Problem 3 has a positive solution, then $NP = co\,NP$.

As a consequence, there was great interest in obtaining independence results for fragments of PA, hoping that they would lead to such results for PA itself. A thorough survey of related work is due to Joseph and Young [7].

The problem of the existence of end extensions of models of theories like $B\Sigma_n$ attracted attention in the early days of work on fragments of PA. The aim has been to prove miniaturizations of the following prototypical result of MacDowell and Specker [10].

Theorem 2. *Every model of PA has a proper elementary end extension of the same cardinality.*

The next section is devoted to discussing some results and problems in this direction.

2. On end extensions

Let us start with the main idea behind the proof of Theorem 2.

Idea of proof. Assume that $M \models PA$. Construct an ultrafilter $\mathscr{U}$ on the set of definable subsets of M and then let K be the ultrapower $\mathscr{F}/\mathscr{U}$, where $\mathscr{F}$ is the set of definable functions from M into M. It can be verified that $M <_e K$.

By modifying this proof, Paris and Kirby obtained the following miniaturization of the MacDowell–Specker result [13].

Theorem 3. *For any countable structure M and $n \geqslant 2$, if $M \models B\Sigma_n$, then there exists a proper Σ_n-elementary end extension of M.*

Idea of proof. Let M be a countable model of $B\Sigma_n$, $n \geqslant 2$ and $(F_m)_{m \in \mathbf{N}}$ be an enumeration of all functions Σ_{n-1} definable in M with unbounded domain and bounded range. The authors define a chain $M = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_m \supseteq \cdots$ such that, for every m, if $Y_m = X_m \cap dom(F_m)$ is unbounded and $G_m = F_m \lceil Y_m$, then $X_{m+1} = G_m^{-1}(\{i\})$, for some i such that $G_m^{-1}(\{i\})$ is unbounded in M. The sequence $(X_m)_{m \in \mathbf{N}}$ leads to a complete Σ_{n-1} ultrafilter on M, i.e. a collection $\mathscr{U}$ of Σ_{n-1} definable subsets of M such that

(i) $X \in \mathscr{U} \Rightarrow X$ is unbounded in M,

(ii) if $X \subseteq M$ is Σ_{n-1} definable in M, then either $X \in \mathscr{U}$ or $X \cap Y = \emptyset$ for some $Y \in \mathscr{U}$,

(iii) if f is a Σ_{n-1} definable function from M to M and the range of f is bounded by $a \in M$, then there exists $i < a$ such that $f^{-1}(\{i\}) \in \mathscr{U}$.

Then the authors let K be the ultrapower $\mathscr{F}/\mathscr{U}$, where $\mathscr{F}$ is the set of Σ_{n-1} definable functions from M to M and show that $M <_{n,e} K$.

Remark 1. Paris and Kirby also proved that

(a) For any structure M and any $n \geqslant 2$, if M has a proper Σ_n-elementary end extension satisfying $I\Delta_0$, then $M \models B\Sigma_n$.

(b) For any structure M, if M has a proper Σ_1-elementary end extension, then $M \models B\Sigma_2$.

In view of the fact that the MacDowell–Specker theorem holds for any M, the following problem arises naturally.

Problem 4. *Does every model M* of $B\Sigma_n$, $n \geqslant 2$, *have a proper Σ_n-elementary end extension*?

Clote attacked this problem and showed, using formalized recursion theoretic arguments, that it has a positive answer, provided that M satisfies a stronger theory [1, 2].

Theorem 4. *For any $n \geqslant 2$, if M satisfies $I\Sigma_n$, then there exists a proper Σ_n-elementary end extension of M.*

Idea of proof. Let M be a model of $I\Sigma_n$, $n \geqslant 2$, and let $\psi_i(x)$, $i \in M$, be an enumeration of all Σ_{n-1} definable subsets of M. Clote considered the binary tree T defined as follows:

$$\sigma \in T \text{ iff } M \models \text{“} \bigcap_{i < lh(\sigma),\ \sigma(i)=0} \{x \mid \psi_i(x)\} \text{ contains at least } lh(\sigma) \text{ many elements”.}$$

By modifying the proof of the low basis theorem, he then showed that T has an infinite branch Δ_n definable in M, which can be used to define a complete Σ_{n-1} ultrafilter on M. The rest of the proof follows that of Paris–Kirby.

A related problem, the arithmetical version of a question of Kaufmann [8], is the following.

Problem 5. *Does every countable model of $B\Sigma_n$, $n \geqslant 2$, have a proper Σ_n-elementary end extension satisfying $B\Sigma_{n-1}$*?

Remark 2. (a) By the Paris–Kirby result, if $M <_{n,e} K$ and $M \models B\Sigma_n$, $n \geqslant 2$, then K satisfies $I\Sigma_{n-2}$, a theory weaker than $B\Sigma_{n-1}$.

(b) By another result of Paris–Kirby [13], for any $n \geqslant 2$, there exists a countable $M \models B\Sigma_n$ with no proper Σ_n-elementary end extension satisfying $I\Sigma_{n-1}$.

We turn now to an alternative approach for handling end extendibility problems, namely that using the proof of the Arithmetized Completeness Theorem ($=ACT$), attributed to Hilbert–Bernays (see [6]).

Theorem 5. *Let M be a model of PA and T be a set of sentences in M such that $M \models Con(T)$. Then there exists $K \models T$ such that K is definable in M and M is isomorphically embedded onto a proper initial segment of K.*

The definability of K in M prevents the isomorphic image of M from being an elementary substructure of K. Indeed, let $S(x)$ be a formula defining truth for K in M. By the fixed-point theorem, there exists a sentence φ such that $PA \vdash \varphi \leftrightarrow S(\ulcorner \underline{\varphi} \urcorner)$. Clearly then, $M \models \varphi$ iff $K \not\models \varphi$, so M cannot be isomorphic to an elementary substructure of K.

However, the above theorem can be used to obtain proper Σ_n-elementary end extensions, as the following result, first stated explicitly by McAloon (see [11]), showed.

Theorem 6. *Let M be a model of PA, $n \geqslant 0$ and T be a set of sentences in M such that $M \models Con(T + \Pi_n - Th)$, where $\Pi_n - Th$ denotes the set of true Π_n sentences. Then there exists $K \models T$ such that K is definable in M and M is isomorphically embedded onto a proper Σ_n-elementary initial segment of K.*

The proof of this result is essentially the same as that of the ACT, the only difference being that the set of true Π_n sentences is added to the original theory T so that the resulting model is a Σ_n-elementary extension of an isomorphic image of M.

In order to obtain a counterpart of the previous theorem for models of fragments of PA, we need a miniaturization of the syntactic part of the proof of the ACT, i.e. the extension of the theory T to a complete consistent theory Σ. Such a result was proved by Paris (see [12]).

Theorem 7. *Let $M \models B\Sigma_n$, $n \geqslant 2$, and L be a recursive language extending LA in M. If T is a Δ_{n-1} definable set of sentences of L such that $M \models Con(T)$, then there exists a set Σ of sentences of L such that*
(a) *Σ is Δ_n definable in M,*
(b) *Σ is a maximal consistent extension of T, and*
(c) *the set of formulas $\theta(x)$ of L such that $M \models \forall a(\theta(\underline{a}) \in \Sigma)$ is Δ_n definable in M.*

Paris used this theorem to obtain proper end extensions of models of $B\Sigma_n$ or $I\Sigma_n$ (for $n \geqslant 2$) so that the end extensions would satisfy the fragment true in the original model. In view of Remark 2(b), it is impossible to demand at the same time that M be (isomorphic to) a Σ_n-elementary extension of K.

If one tries to prove a result that bears the same relation to Theorem 7 as Theorem 6 to the syntactic part of the ACT, one faces the following technical problem: the addition of the set of true Π_n sentences to (a suitable theory) T, which is needed to guarantee the Σ_n-elementary extendibility of M, increases the complexity of the base theory so that it is not possible to extend it in M to a maximal consistent theory Σ. However, this obstacle is not as serious as it seems.

First we need to modify slightly Theorem 7 as follows.

Theorem 8. *Let $M \models I\Sigma_n$, $n \geqslant 2$, and L be a recursive language extending LA in M. If T_1 is a Σ_{n-1} definable set of sentences of L and T_2 is a Π_{n-1} set of sentences of L such that $M \models Con(T_1 + T_2)$, then there exists a set Σ of sentences of L such that*
(a) *Σ is $\Sigma_0(\Sigma_n)$ definable in M,*
(b) *Σ is a maximal consistent extension of T, and*
(c) *the set of formulas $\theta(x)$ of L such that $M \models \forall a(\theta(\underline{a}) \in \Sigma)$ is $\Sigma_0(\Sigma_n)$ definable in M.*

($\Sigma_0(\Sigma_n)$ formulas are obtained from atomic formulas and instances of Σ_n formulas through the use of connectives and bounded quantifiers – see in [4, Section 2.2, Chapter I]).

Using this result, we can now give an alternative proof of Theorem 4.

Indeed, assume that $M \models I\Sigma_n$, $n \geqslant 2$, and let T be $I\Delta_0 + \Sigma_{n-1} - Th(M) + \Pi_{n-1} - Th(M)$. By a standard result (see, e.g., in [4, Theorem 4.33, Chapter I]), $M \models Con(T)$. Therefore, T can be extended to a theory Σ in M satisfying (a)–(c) of Theorem 8. This theory Σ can be used to transfer the Paris–Kirby construction "inside M". Property (c) is crucial, since it implies that, for any Σ_{n-1} formula $\theta(x)$, the formula expressing "the set $\{x \in M \mid \theta(x)\}$ is unbounded" is (equivalent to) a $\Sigma_0(\Sigma_n)$ formula in M.

Let F_i, $i \in M$, be an enumeration of all functions Δ_{n-1} definable in M with unbounded domain and bounded range. Define a chain $M = X_0 \supseteq X_0 \supseteq X_1 \supseteq \cdots \supseteq X_m \supseteq \cdots$ of Δ_{n-1} definable sets as follows:

$$X_0 = M,$$

$$X_{i+1} = \begin{cases} X_i \cap (M - dom(F_i)) & \text{if } X_i \cap dom(F_i) \text{ is unbounded in } M, \\ G_i^{-1}(\{k\}) & \text{otherwise,} \end{cases}$$

where $G_i^{-1} = F_i \lceil (M - dom(F_i))$ and k is the least j such that $G_i^{-1}(\{j\})$ is unbounded in M – such a j always exists by $B\Sigma_n$.

Note that, by property (c) of Σ, $X = X_m$ is (equivalent to) a $\Sigma_0(\Sigma_n)$ formula. Hence, we can use induction to prove that $M \models \forall m \exists X (X = X_m)$ (for details, see [3]). As in the proofs of Theorems 3 and 4, the sequence $(X_m)_{m \in M}$ leads to a complete Δ_{n-1} ultrafilter on M and hence to a proper Σ_n-elementary end extension of M.

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