Polygonal radix representations of complex numbers

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Abstract

Complex numbers can be represented in positional notation using certain digit sets. In this paper, we present the polygonal representation which uses zero and the n-roots of unity as digits. We give conditions on the base in order that every complex number be representable in such a system. We totally characterize complete polygonal numeration systems in imaginary quadratic fields. © 1999 Published by Elsevier Science B.V. All rights reserved

Keywords: Radix representations of complex numbers; Number systems; Polygonal numeration systems; Complete numeration systems.

1. Introduction

Within the field of computer arithmetic, choosing the right way to represent real or complex numbers and to execute arithmetic computation can be crucial for execution time and complexity. Various bases and digit sets have been proposed for writing complex numbers in positional notation as a single string of digits, without separating the real and imaginary parts. Such representations are analogous to the binary and decimal radix representations of the positive real numbers using base two and ten, respectively.

A numeration system is a couple (β, A) where the base β is a complex number and the digit set A is a finite set of complex numbers.

A representation of a complex number z in (β, A) is a sequence (a_i)_{i \leq m}, with a_i ∈ A and m ∈ Z, such that

\[ z = \sum_{i=-\infty}^{m} a_i \beta^i. \]

Such a representation is denoted by

\[ (a_m a_{m-1} \cdots a_0 a_{-1} \cdots)_{\beta}. \]

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PII: S0304-3975(98)00100-5
For instance, it is known that every complex number can be written as
\[ \sum_{k=-\infty}^{m} a_k (-1 + i)^k, \]
where \( a_k = 0 \) or 1 [7, 9, 12].

I. Kátaí and J. Szabó [9] showed that Gaussian integers correspond to the numbers that are integral for this representation (in which \( a_k = 0 \) for all \( k < 0 \)). For example,
\[ 3 + 2i = (1001)_{-1+i} \quad \text{and} \quad \frac{-1 + 2i}{5} = (0.01010\cdots)_{-1+i}. \]

The fact that the set \{0, 1\} is stable under multiplication makes doing multiplication in the numeration system \((-1 + i, \{0, 1\})\) not harder than doing addition.

A system in which every complex number has at least one representation is said to be complete. Given a \( \beta \) and \( A \), it is in general difficult to decide whether every complex number can be represented in the numeration system \((\beta, A)\). This question has been studied by Thurston [17], Daróczy and Kátaí [2] in the case where the base is fixed.

The problem of being able to perform fast computation is also of importance. In [3] it is shown that the numeration system \((2, A)\), where \( A = \{0, 1, \xi, \xi^2, \cdots, \xi^5\} \) and \( \xi = e^{i\pi/3} = (1 + i\sqrt{3})/2 \), enables fast carry-free addition and an on-line multiplication (in a way quite similar to carry-free addition in signed-digit numeration systems [1]).

Systems where the digit set is of the form \( A_n = \{0, 1, \xi, \xi^2, \cdots, \xi^{n-1}\} \), where \( \xi = e^{2i\pi/n} \) is the \( n \)-root of unity, are called polygonal numeration systems. In such a system, the product of two digits is still a digit, which imply that multiplication is "simple".

The polygonal numeration system \((\pm i\sqrt{2}, \{-1, 0, 1\})\) is one in which a limited carry propagation addition is possible [6, 13]. As a matter of fact, addition in this system is computable by an on-line finite state automaton [6].

Another example is given by base \( i\sqrt{3} \). Robert [15] proved that every complex number can be represented in the number system \((i\sqrt{3}, \{0, 1, \xi\})\) and that the integral numbers correspond to elements of \( Z[(1 + i\sqrt{3})/2] \). With digits in \( A_6 \), we obtain a redundant polygonal numeration system \((i\sqrt{3}, A_6)\), where complex numbers can have multiple representations.

So, choosing an appropriate numeration system can lead to a "simple" computation, faster and more efficient.

In this paper we study the problem of the completeness of polygonal numeration systems. We give sufficient conditions on the base which imply that the system is complete or not (Theorem 2 and Proposition 1).

We then turn to polygonal representations in imaginary quadratic number fields, that are fields of the form \( \mathbb{Q}[\sqrt{d}] \) where \( d \) is a negative integer. We have proved in [16] that if the base and the digits are algebraic integers of \( \mathbb{Q}[\sqrt{d}] \), then addition is computable by a finite state automaton (as it is the case when \( \beta \) is an integer). Here we totally characterize complete polygonal numeration systems of imaginary quadratic fields (Proposition 2).
2. Representability of complex numbers

2.1. Definitions and previous results

2.1.1. Definitions

A numeration system is a couple \((\alpha, A)\) where the base \(\alpha\) is a complex number of modulus \(> 1\), and the digit set \(A = \{d_0 = 0, d_1, \ldots, d_n\}\) is a finite set of complex numbers.

A representation of a complex number \(z\) in the numeration system \((\alpha, A)\) is a sequence \((a_i)_{i \leq m}\), with \(a_i \in A\) and \(m \in \mathbb{Z}\), such that

\[
z = \sum_{i=-\infty}^{m} a_i \alpha^i.
\]

We write \(E\) for the set of "integral numbers"; i.e. numbers of the form \(\sum_{i=0}^{m} a_i \alpha^i\), where \(m \in \mathbb{N}\), and \(a_i \in A\) for \(0 \leq i \leq m\). And we write \(W\) for the set of "fractions"; i.e. numbers of the form \(\sum_{i=-\infty}^{-1} a_i \alpha^i\), where \((a_i)_{i \leq -1} \in A^\mathbb{N}\).

A complete numeration system is one for which all complex numbers have at least one representation. A polygonal numeration system \((\alpha, A)\) is a system where digits are zero and the \(n\)-roots of unity \((n \in \mathbb{N} \setminus \{0\})\), that i.e.

\[
A_n = \{0, 1, \zeta, \zeta^2, \ldots, \zeta^{n-1}\}, \text{ where } \zeta = e^{2\pi i/n}.
\]

We are looking for conditions on the base and on the digit set to have a complete polygonal numeration system. But first, let us give some previous results on complete numeration systems.

2.1.2. Previous results

The first result we give here is the sufficient condition to have a complete numeration system given by Thurston [17, 14] as follows:

**Theorem 1.** Let \(\alpha\) be a complex number of modulus \(> 1\), and let \(A\) be a finite set of complex numbers with \(0 \in A\). If there exists a neighborhood \(U\) of the origin such that \(\alpha U \subseteq U + A\), then all complex numbers have a representation in the numeration system \((\alpha, A)\).

**Remark 1.** It is not easy to show the existence of a neighborhood \(U\) of zero satisfying the condition \(\alpha U \subseteq U + A\) given by Theorem 1. A sufficient and more practical one is that the convex hull \(\hat{A}\) of the digit set \(A\) satisfies this condition [13]. That is if \(\alpha \hat{A} \subseteq \hat{A} + A\), then \((\alpha, A)\) is a complete numeration system. This condition is not a necessary one. For example, if \(\alpha = -1 + i\) and \(A = \{0, 1\}\) (here \(\hat{A} = [0, 1]\)), the numeration system \((\alpha, A)\) is complete as the set \(W\) of fraction satisfies Theorem 1 [7, 9, 12], and we do not have \(\alpha \hat{A} \subseteq \hat{A} + A\).

We are now looking for conditions on the digit set \(A\) and on the base \(\alpha\) to have a complete numeration system \((\alpha, A)\). More precisely, we fix \(A\) (resp. \(\alpha\)) and we try to
find conditions on \( \beta \) (resp. \( A \)) under which all complex numbers can be represented in the numeration system \((\beta, A)\).

Let \( \beta \) be a fixed complex number of modulus \( > 1 \). It can be seen that if \( \beta \) is not real then it is always possible to choose \( A \) in such a way that \( W \) is a neighborhood of zero \([14, 17]\). Moreover, Daróczy and Kátai [2] have shown that there always exists an integer \( k \geq 1 \) such that every complex number is represented in base \( \beta \) with digits in \( \{0, 1, 2, \ldots, k\} \).

On the other hand, let the digit set \( A \) be fixed, containing 0. In this case, two strategies can be adopted to determine a base \( \beta \) giving a complete system \((\beta, A)\). The first one consists in searching the maximal value \( s_{\max} \) to have \((\beta, A)\) a complete numeration system when \( \beta = se^{i\theta} \) with \( 1 < s < s_{\max} \) and \( \theta \) any element of \([0, 2\pi]\). A result of Daróczy and Kátai [2] states that if

\[
\hbar = \min_{|z|=1} \max_{a \in A} \cos(\arg a - \arg z) > \frac{1}{2},
\]

then

\[
\sqrt{h + \frac{1}{2} + \sqrt{h(1 + h)}} \leq s_{\max}.
\]

It is easy to see that \( s_{\max} \leq \sqrt{\text{card}(A)} \), where \( \text{card}(A) \) denotes the cardinal of \( A \) (see [2] for instance).

The second strategy consists in fixing the argument \( \theta \) and searching the maximal value \( s_{\max}(\theta) \) to have a complete numeration system \((\beta, A)\) when \( \beta = se^{i\theta} \) and \( 1 < s < s_{\max}(\theta) \). Another result of Daróczy and Kátai [2] uses this strategy. It states that, when \( A = \{0, 1\} \) and \( \theta \neq 0 \) modulo \( \pi \), there always exists a real number \( r_\theta > 1 \) such that if \( 1 < s < r_\theta \) then \((\beta, A)\) is a complete numeration system. In this case, we have \( r_\theta < s_{\max}(\theta) \).

2.2. Polygonal numeration systems

The main results of this paper are the sufficient conditions on the completeness or not of a polygonal numeration system.

**Theorem 2.** Let \( \beta = se^{i\theta} \in \mathbb{C} \) and let \( A_n = \{0, 1, \xi, \xi^2, \ldots, \xi^{n-1}\} \), with \( n \geq 4 \) and \( \xi = e^{2i\pi/n} \). If \( 1 < s \leq 1 + 2 \cos(2\pi/n) \), then the polygonal numeration system \((\beta, A_n)\) is complete, and if \( s > 1 + 2 \cos(\pi/n) \), then there exist complex numbers with no representation in \((\beta, A_n)\).

**Remark 2.** This theorem had been proved in the case where the base \( \beta \) is a real number by Herreros [8].

The proof of Theorem 2 will be given in two steps. It uses the next lemma.

**Lemma 1.** If \( 1 < s \leq 1 + 2 \cos(2\pi/n) \), the disc \( D(0, sr) \) centered in zero and of radius \( sr \), where \( r = 1/2 \cos(\pi/n) \), can be covered by discs \( D(\xi_j, r) \), where \( \xi_j \in A \).
Proof. Denote by $H$ the intersection point of the medians of the triangle $(0, 1, \xi)$ and by $(x, y)$ its coordinates (see Fig. 1). Its modulus $h$ satisfies $h^2 = |H|^2 = |H - 1|^2 = |H - \xi|^2$. As $H$ belongs to the median of 0 and 1, then $x = \frac{1}{2}$ and $h^2 = (\frac{1}{2})^2 + y^2 = (\frac{1}{2} - \cos (2\pi/n))^2 + (y - \sin (2\pi/n))^2$. So, we have

$$y = \frac{1 - \cos \frac{2\pi}{n}}{2 \sin \frac{2\pi}{n}} \quad \text{and} \quad h = \frac{1}{2 \cos \frac{\pi}{n}} = r.$$

Now, let $H'$ be the point symmetric to $H$ with respect to the line $(1, \xi)$. Its modulus is $h' = |H'| = r(1 + 2 \cos (2\pi/n))$. To cover disc $D(0, sr)$ by discs $D(\xi_j, r)$, with $\xi_j \in A_n$, we must have $sr \leq h'$ and this is true from the hypothesis $s \leq 1 + 2 \cos (2\pi/n)$. So, Lemma 1 is proved. □

Proof of Theorem 2. Step 1: Assume $1 < s \leq 1 + 2 \cos (2\pi/n)$. To see that $(\beta, A_n)$ is a complete system, it is enough to prove that any element $x$ of the disc $D(0, r)$
has a representation in this system. Put \( z_0 = z \). From Lemma 1, the disc \( D(0,r) \) is a neighborhood of zero and satisfies \( \beta D(0,r) \subset D(0,r) + A_n \). By direct computation, a sequence \( z_{j+1} = \beta z_j - \xi_{j+1} \in D(0,r) \), where \( \xi_{j+1} \in A_n \), can be constructed and we have

\[
z = \sum_{j=1}^{k} \xi_j \beta^{-j} + z_{k+1} \beta^{-(k+1)} \longrightarrow \sum_{j=1}^{\infty} \xi_j \beta^{-j}, \quad \text{as} \ k \to +\infty.
\]

**Step 2:** Assume \( s > 1 + 2 \cos(\pi/n) \). Then the point \( H \) defined above has no representation in the numeration system \( (\beta, A_n) \). To prove it, put

\[
E_k = \left\{ \sum_{j<k} a_j \beta^j \mid a_j \in A_n \right\}.
\]

We show by computation that \( H \notin E_k \) for all \( k \in \mathbb{N} \). First, remark that \( E_k \subset D(0, s^k/(s-1)) \), for all \( k \). Now, as \( |H| = 1/2 \cos(\pi/n) > 1/(s-1) \), we have \( H \notin E_0 = W \). Suppose now \( H \notin E_k \) for some \( k \geq 0 \). As \( E_{k+1} = \bigcup_{0 \leq j \leq n-1} (E_k + \xi_j \beta^k) \cup E_k \), it is enough to show that \( (H - \xi_j \beta^k) \notin E_k \), for all \( 0 \leq j \leq n-1 \). As \( \beta = se^{i\theta} \) and the coordinates of \( H \) are \( x = \frac{1}{2} \) and

\[
y = \frac{1 - \cos \frac{2\pi}{n}}{2 \sin \frac{2\pi}{n}},
\]

we have

\[
|\xi_j \beta^k - H|^2 = s^k \left( \cos \left( \frac{j \pi}{n} \right) + is \left( \frac{j \pi}{n} \right) \right) \left( \cos(\theta k) + is \left( \theta k \right) \right)
- \left( \frac{1}{2} + \frac{i}{2} \tan \frac{\pi}{n} \right)^2
= s^{2k} + \frac{1}{4 \cos^2 \frac{\pi}{n}} - s^k \cos \left( j - \frac{1}{2} \right) \frac{2\pi}{n} + k\theta \cos \frac{\pi}{n}
\]

\[
> s^{2k} + \frac{1}{4 \cos^2 \frac{\pi}{n}} - s^k \frac{1}{\cos \frac{\pi}{n}} = \left( s^k - \frac{1}{2 \cos \frac{\pi}{n}} \right)^2.
\]

Finally, we obtain

\[
|\xi_j \beta^k - H| - \frac{s^k}{s-1} > s^k - \frac{1}{2 \cos \frac{\pi}{n}} \frac{s^k}{s-1} = s^k \frac{s-2}{s-1} - \frac{1}{2 \cos \frac{\pi}{n}}.
\]

Since by hypothesis, \( s > 1 + 2 \cos(\pi/n) > 1 \) and because the function \( f(x) = (x-2)/(x-1) \) is increasing for \( x > 1 \), we have for \( n \geq 4 \)

\[
|\xi_j \beta^k - H| - \frac{s^k}{s-1} > \left( 1 + 2 \cos \frac{\pi}{n} \right) \left( \frac{2 \cos \frac{\pi}{n} - 1}{2 \cos \frac{\pi}{n}} \right) - \frac{1}{2 \cos \frac{\pi}{n}} = \frac{\cos \frac{2\pi}{n}}{\cos \frac{\pi}{n}} > 0. \quad \Box
\]

From Theorem 2, we deduce
Corollary 1. Let the set of digits $A_n = \{0, 1, \xi, \xi^2, \ldots, \xi^{n-1}\}$ be fixed, with $n \geq 4$. Let $s_{\text{max}}$ denote the maximum of value of $|\beta|$, for all $\beta$'s such that $|\beta| > 1$ and $(\beta, A_n)$ is a complete numeration system. Then, we have

$$1 + 2 \cos \frac{2\pi}{n} \leq s_{\text{max}} \leq 1 + 2 \cos \frac{\pi}{n}.$$ 

A better majoration of $s_{\text{max}}$ than that of Corollary 1 can be obtained using the second strategy, that is following a fixed direction $\theta$. In the following, only case $\theta = 2\pi t/n$ where $t \in \{0, 1, \ldots, n - 1\}$ will be considered.

Proposition 1. Let $n$ be an integer $\geq 5$ and let $\theta = 2\pi t/n$, with $t \in \{0, 1, \ldots, n - 1\}$. Put $\beta = se^{i\theta}$ and $A_n = \{0, 1, \xi, \xi^2, \ldots, \xi^{n-1}\}$, where $\xi = e^{2i\pi/n}$. Then $(\beta, A_n)$ is not a complete numeration system if $n$ is even and 

$$s > 2 + \cos \left(\frac{2\pi}{n}\right),$$

or if $n$ is odd and 

$$s > 1 + \cos \left(\frac{\pi}{n}\right) + \cos^2 \left(\frac{\pi}{n}\right).$$

Proof. As for Theorem 2, it is enough to give a point that does not belong to any $E_k$, for $k \in \mathbb{N}$.

(1) Suppose $n$ even and let $s > 2 + \cos (2\pi/n)$. The side of the polygon enveloping $E_0$ that is closest to $H$ cuts the $x$-axis at $x = 1/(s - 1)$, while the line parallel to this side containing $H$ cuts the same axis at $x = \frac{1}{2\cos^2 \frac{\pi}{n}}$ (see Fig. 2).

So, $H \notin E_0 \subset D(0, 1/(s - 1))$.

Suppose now that $H \notin E_k$ for some $k \geq 1$. As $E_{k+1} = \bigcup(E_k + z^j\beta^k) \bigcup E_k$, it is enough to show that for all $j \in \mathbb{N}$, we have $H - \xi^j\beta^k \notin E_k$. The fact that $|1 - H| = |\xi - H| \leq |\xi^j - H|$, for all $j \in \mathbb{N}$, and the symmetry of $(E_k + s^k)$ and $(E_k + \xi s^k)$ with respect to the axis $(0H)$, make enough to show that $H \notin E_k + s^k$. From $E_k = \beta^k E_0 = s^k \tilde{e}^k E_0$ and as $E_0$ is invariant under rotation of angle $2\pi/n$, we obtain $E_k = s^k E_0$. By hypothesis $s > 2 + \cos (2\pi/n)$, so

$$\frac{s^k s - 2}{s - 1} > \frac{s - 2}{s - 1} > 1 - \frac{1}{2\cos^2 \left(\frac{\pi}{n}\right)}.$$ 

Denote by $A$ the side of the polygon enveloping $E_k + s^k$ that is closest to $H$. As $n$ is even, $A$ cuts the $x$-axis at

$$x = \frac{s^k s - 2}{s - 1}.$$
while the line parallel to $\Delta$ containing $H$ cuts the same axis at

$$x = 1 - \frac{1}{2 \cos^2 \left( \frac{\pi}{n} \right)} \quad \text{(see Fig. 2).}$$

We deduce that $H \notin E_k + s^k$.

(2) Suppose $n$ odd and

$$s > 1 + \cos \frac{\pi}{n} + \cos^2 \left( \frac{\pi}{n} \right).$$

The same method can be applied to the point $P = x_P + iy_P$ belonging to the median of the segment $[1, \xi]$ the modulus of which verifies $|P|^2 = x_P^2 + y_P^2 = (1 - x_P)^2$. Then $y_P = \tan(\pi/n)x_P$, and $x_P^2 + y_P^2 = (1 - x_P)^2$. We deduce that $x_P = \cos(\pi/n)/1 + \cos(\pi/n)$. As

$$\frac{\cos \frac{\pi}{n}}{1 + \cos \frac{\pi}{n}} > \frac{\cos^2 \frac{\pi}{n}}{s - 1},$$

the line $(OP)$ intersects the point $P$ outside of the polygon enveloping $E_0$ (see Fig. 3). So $P \notin E_0$. 

---

Fig. 2. Upper bound of $s_{\text{max}}(\theta)$ for $\theta = 2\pi/n$ and $n$ even.
Suppose now $P \notin E_k$ for some $k \geq 0$. The same arguments as in the first case are still true. We use this time the symmetry of $(E_k + s^k)$ and $(E_k + \xi s^k)$ with respect to the axis $(0P)$ and the fact that $s > 1 + \cos(\pi/n) + \cos^2(\pi/n)$ which implies

$$s^k - \frac{s^k}{s-1} \cos \frac{\pi}{n} > 1 - \frac{\cos \frac{\pi}{n}}{s-1} > \frac{\cos \frac{\pi}{n}}{1 + \cos \frac{\pi}{n}}.$$  

As $n$ is odd, the side of the polygon enveloping $E_k + s^{k+1}$ that is closest to $P$ has equation

$$x - s^k - \frac{s^k}{s-1} \cos \frac{\pi}{n} \quad \text{(see Fig. 3)}.$$  

We deduce that $P \notin E_k + s^{k+1}$. □

2.3. Polygonal representations in imaginary quadratic fields

Here we totally characterize complete polygonal numeration systems of imaginary quadratic fields. First, let us give some definitions and notations. It is well known that a quadratic number field is of the form $\mathbb{Q}[\sqrt{d}] = \{r + s\sqrt{d} \mid (r, s) \in \mathbb{Q}\}$, where $d$ is a square-free integer. A complex number $z$ is an algebraic integer if it is a root of some polynomial $P(X) = X^m + p_{m-1}X^{m-1} + \cdots + p_1X + p_0$, where $p_0, p_1, \ldots, p_{m-1}$ are integers and $m \in \mathbb{N} \setminus \{0\}$. Denote by $L_d$ the set of all algebraic integers of the quadratic field $\mathbb{Q}[\sqrt{d}]$. 
In what follows, only the negative integers $d$ are considered, and complete polygonal numeration systems $(\beta, A_n)$ such that $\beta \in L_d$ and $A_n = \{0, 1, \xi, \xi^2, \ldots, \xi^{n-1}\} \subset L_d$ are looked for. Our interest in these numeration systems comes from a result we have proved in [16]. It states in particular that in these numeration systems, addition is computable by a finite state automaton. This fact generalizes a well-known one on classical numeration systems [4].

**Proposition 2.** *The only complete polygonal numeration systems $(\beta, A_n)$ of imaginary quadratic fields $\mathbb{Q}[\sqrt{d}]$, that satisfy $\beta \in L_d$, and $A_n \subset L_d$, where $d < 0$, are given in Table 1.*

To prove Proposition 2, we will use the three following lemmas.
Table 1

<table>
<thead>
<tr>
<th>$d$</th>
<th>$L_d = \mathbb{Z} [\rho]$</th>
<th>$n$</th>
<th>$A_n$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>$\mathbb{Z} [i]$</td>
<td>1</td>
<td>${0, 1}$</td>
<td>$\pm 1 \pm i$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>${0, \pm 1}$</td>
<td>$\pm 1 \pm i$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>${0, \pm 1, \pm i}$</td>
<td>$\pm 1 \pm i, \pm 2 \pm 2i, \pm 1 \pm 2i, \pm 2$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$\mathbb{Z} [i\sqrt{2}]$</td>
<td>1</td>
<td>${0, 1}$</td>
<td>$\pm i\sqrt{2}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
<td>${0, \pm 1}$</td>
<td>$\pm i\sqrt{2}, \pm 1 \pm i\sqrt{2}$</td>
</tr>
<tr>
<td>$-3$</td>
<td>$\mathbb{Z} \left[\frac{1+i\sqrt{3}}{2}\right]$</td>
<td>2</td>
<td>${0, \pm 1}$</td>
<td>$\pm i\sqrt{3}, \pm \frac{1+i\sqrt{3}}{2}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>${0, 1, \rho^{2}, \rho^{3}}$</td>
<td>$\pm i\sqrt{3}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6</td>
<td>${0, 1, \rho, \rho^{2}, \rho^{3}, \rho^{4}}$</td>
<td>$\pm 2, \pm 2(1 + i\sqrt{3}), \pm (2 + i\sqrt{3})$</td>
</tr>
<tr>
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<td>$\mathbb{Z} \left[\frac{1+i\sqrt{7}}{2}\right]$</td>
<td>1</td>
<td>${0, 1}$</td>
<td>$\pm i\sqrt{7}$</td>
</tr>
<tr>
<td></td>
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<td>2</td>
<td>${0, \pm 1}$</td>
<td>$\pm \frac{1+i\sqrt{7}}{2}$</td>
</tr>
<tr>
<td>$-11$</td>
<td>$\mathbb{Z} \left[\frac{1-i\sqrt{31}}{2}\right]$</td>
<td>2</td>
<td>${0, \pm 1}$</td>
<td>$\pm \frac{1-i\sqrt{31}}{2}$</td>
</tr>
</tbody>
</table>

Lemma 2. Let $d$ be a negative square-free integer and let $L_d$ and $A_n$ be defined as above ($n \in \mathbb{N} \setminus \{0\}$). Then, $A_1 = \{0, 1\}$ and $A_2 = \{-1, 0, 1\}$ are subsets of $L_d$ for all values of $d$, and for $n > 2$, we have

- if $d \equiv 2$ or $3$ modulo $4$, then $A_n \subset L_d$ if and only if $d = -1$ and $n = 4$,
- if $d \equiv 1$ modulo $4$, then $A_n \subset L_d$ if and only if $d = -3$ and $n \in \{3, 6\}$.

Proof. This can easily be deduced from the following classical result of algebra (see [5, p. 176]): if $d$ is a square-free integer, then $L_d = \mathbb{Z} [\rho]$, where

$$\rho = \begin{cases} 
\sqrt{d} & \text{if } d \equiv 2 \text{ or } 3 \text{ modulo } 4, \\
\frac{1 + \sqrt{d}}{2} & \text{if } d \equiv 1 \text{ modulo } 4.
\end{cases}$$

Lemma 3 (Kenyon [10, Theorem 2.10]). Let $\beta$ be a quadratic integer of modulus $|\beta| > 1$ and $A$ be a subset of $\mathbb{Q} [\beta]$ with $|\beta|^2$ elements, with $0 \in A$. Then, the set of fractions $W = \left\{ \sum_{i=-\infty}^{-1} a_i \beta^i \mid \{a_i\}_{i=-1} \in A^{\mathbb{N}} \right\}$ contains an open set if and only if there are no elements of $\mathbb{Q} [\beta]$ with two distinct finite representations.

In our case, Lemma 3 implies that: if there is a subset $A$ of the digit set $A_n \subset L_d$ with $|\beta|^2$ elements which is a complete residue system of $\mathbb{Z} [\beta]$ modulo $\beta$, i.e. each element of $\mathbb{Z} [\beta]$ is congruent to one and only one element of $A$, then $(\beta, A)$ is a complete numeration system. In such a case, $(\beta, A_n)$ is a complete polygonal numeration system.

Lemma 4. For all integers $n \geq 1$, $A_n$ is stable under multiplication by $\xi^k$ for all $k \in \mathbb{N}$, and by conjugation, i.e. $\xi^k A = A$ and $\bar{A} = A$. So, if $(\beta, A_n)$ is a complete numeration system, then so are $(\bar{\beta}, A_n)$ and $(\xi^k \beta, A_n)$, for all $k \in \mathbb{N}$.

Proof of Proposition 2. Now, set $\beta = a + \rho b$, where $a$ and $b \in \mathbb{Z}$. 


Case 1: $d \equiv 2$ or $3$ modulo 4. Then $\rho = \sqrt{d}$ and $|\beta|^2 = a^2 - db^2$.

1. If $n = 1$, then we must have $1 < |\beta|^2 \leq \text{card}(A_1) = 2$. So, $\beta \in \{\pm 1 \pm i, \pm i\sqrt{2}\}$ and in all these cases, $(\beta, A_1)$ is a complete numeration system (Lemma 3).

2. If $n = 2$, then we must have $1 < |\beta|^2 \leq \text{card}(A_2) = 3$. So, $\beta \in \{\pm 1 \pm i, \pm i\sqrt{2}, \pm 1 \pm i\sqrt{2}\}$ and in all these cases, $(\beta, A_2)$ is a complete numeration system too.

3. If $n = 4$, then $d = -1$. From Theorem 2, we must have $1 < s \leq 1 + 2\cos \left(\frac{\pi}{4}\right) = 1 + \sqrt{2}$. So, $\beta \in \{\pm 1 \pm i, \pm 2i, \pm 1 \pm 2i, \pm 2 \pm i\}$ and all these cases give a complete numeration system $(\beta, A_4)$.

Case 2: $d \equiv 1$ modulo 4. Then $\rho = (1 + \sqrt{d})/2$ and $|\beta|^2 = a^2 + ab + ((1 - d)/4)b^2$.

1. If $n = 1$, then we must have $1 < |\beta|^2 \leq \text{card}(A_1) = 2$. So, $d = -7$ and $\beta = (\pm 1 \pm \sqrt{7})/2$. In all these cases, $(\beta, A_1)$ is a complete numeration system (use Lemma 3).

2. If $n = 2$, then we must have $1 < |\beta|^2 \leq \text{card}(A_2) = 3$. So, $d = -3$ and $\beta \in \{\pm i\sqrt{3}, (\pm 3 \pm i\sqrt{3})/2\}$, or $d = -7$ and $\beta = (\pm 1 \pm i\sqrt{7})/2$. Or $d = -11$ and $\beta = (\pm 1 \pm i\sqrt{11})/2$. In all these cases, $(\beta, A_2)$ is a complete numeration system (use Lemma 3).

3. If $n = 3$, then $d = -3$ and we must have $1 < s = \sqrt{a^2 + ab + b^2} \leq \sqrt{\text{card}(A_3)} = 2$, that is $\beta \in \{\pm 2, \pm i\sqrt{3}\}$. Herreros [8, p. 82] has shown that in the numeration systems $(\pm 2, A_3)$, no disc centered in zero can be represented. So, the numeration systems $(\pm 2, A_3)$ are not complete. But the bases $\pm i\sqrt{3}$ give a complete ones with $A_3$ as a digit set (use Lemmas 3 and 4). Note that $\rho^2 = i\sqrt{3} + \rho^4$ and that $(i\sqrt{3}, \{0, 1, \rho\})$ is a complete numeration system. This numeration system have been studied by Robert [15].

4. If $n = 6$, then $d = -3$ and from Theorem 2 we must have $1 < s = \sqrt{a^2 + ab + b^2} \leq 1 + 2\cos(2\pi/6) = 2$, that is $\beta \in \{\pm 2, \pm 2\rho, \pm 1 \pm 2\rho, \pm i\sqrt{3}, (\pm 3 \pm i\sqrt{3})/2\}$. In all these cases, $(\beta, A_6)$ is a complete numeration system (use Remark 1 for $\beta = \pm 2$). The numeration system $(2, A_6)$ has been studied by Herreros [3, 8].

Acknowledgements

I would like to thank the referees for their careful reading, which greatly improved the style and content of the paper.

References