3-Blocks and 3-Modular Characters of $G_2(q)$

GERHARD HISS

Rheinisch-Westfälische Technische Hochschule Aachen,
D-5100 Aachen, West Germany

AND

JOSEPHINE SHAMASH

Department of Theoretical Mathematics,
The Weizmann Institute of Science,
Rehovot 76100, Israel

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1. Introduction

This paper continues the investigations of the modular characters of the finite Chevalley groups $G = G_2(q)$. In a series of papers [7–10, 5] the authors have investigated the $p$-blocks, the Brauer trees, and the $p$-modular characters in characteristics $p$ different from 2 and 3. In the present paper we deal with the case $p = 3$ and $q$ not divisible by 3. The case of $p = 2$ and odd $q$ will be considered in a subsequent paper.

We determine the distribution of the ordinary characters of $G$ into 3-blocks, the defect groups, and for blocks with cyclic defect groups, the Brauer trees. For blocks with non-cyclic defect groups all but two of the irreducible Brauer characters are determined. The results are sufficient to find the minimal degree of a faithful 3-modular representation of $G$.

Throughout our paper we have to distinguish between the two cases $q \equiv 1 \pmod{3}$ and $q \equiv -1 \pmod{3}$. The ordinary characters of $G$ are taken from [4, 2]. Our notation is that of Chang and Ree in [2]. The blocks are determined by using the method of central characters. Using lemmas from [8, 9] the distribution into blocks and the exceptional characters are calculated. The Brauer trees for blocks with cyclic defect groups are then determined. It turns out that there is a unique block of maximal defect, the principal block. The defect groups of all the remaining blocks are abelian.
Their decomposition matrices are exactly the same as those for the corresponding blocks in case $p > 3$. Since those are already given in [5], we do not repeat them here. However, the proofs in these cases have to be slightly modified.

The methods for finding the decomposition matrices for the principal block are much the same as those used in [5]. We determine a basic set of Brauer characters, consisting of some—but not all—of the unipotent characters in the block. Then we produce a set of projective characters by inducing projectives from subgroups or tensoring with defect 0 characters. The projectives are selected as to approximate the projective indecomposables as closely as possible. In determining these characters and some scalar products with other characters in the block, the computer algebra system MAPLE was used.

The remaining problems appear to be very hard to solve. Roughly speaking it comes down to finding all modular constituents of the Steinberg character including their multiplicity. Certainly none of the methods used in this paper will solve this problem.

2. Blocks and Brauer Trees

2.1. Preliminaries

We recall some facts and notation used in [8–10]:

$$|G_2(q)| = q^6(q^6 - 1)(q^2 - 1).$$

The following subgroups are maximal tori of $G_2(q)$:

$$H_1 \cong C_{q-1} \times C_{q-1}, \quad H_3 \cong C_{q^2+q+1},$$

$$H_2 \cong C_{q+1} \times C_{q+1}, \quad H_6 \cong C_{q^2-q+1},$$

$$H_a \cong C_{q^2-1} \cong H_b.$$ 

For $a \in \{1, 2, a, b, 3, 6\}$, we denote elements of $H_x$ by $h_a$, complex linear characters of $H_x$ by $\pi_x$, $\hat{\pi}_x(h_a)$ is the sum of the images under $\pi_x$ of the conjugates of $h_a$ in $H_x$.

As in the previous papers, $\pi_x$ will usually denote a character such that $X_x(\pi_x)$ is irreducible, $\pi_x^+$ are of order 3, $\pi_x^*$ are of order 2, $I_x$ denotes the trivial character on $H_x$.

For $\alpha = a$ or $b$, $\pi_x^{\#}$ and $\pi_x^{\ast}$ are such that $(\pi_x^{\#})^{q+1} = I_x = (\pi_x^{\ast})^{q-1}$, and $\pi_x^{\ast}, \pi_x^{\#}$ are of order $> 3$.

We recall that if $\omega_x$ is the central character determined by the character
\( \chi, \ P \) the maximal ideal in a valuation ring \( R \), that contains all the values of the irreducible characters of \( G_2(q) \), \( pR \subseteq P \),

\[ \omega_\chi \equiv \omega_\chi (\text{mod } P) \Rightarrow \chi \quad \text{and} \quad \chi' \text{ are in same } p\text{-block} \]

so that the blocks are determined by inspection of the central character tables \( \text{(mod 3)} \). These appear in the Appendix. As is apparent, the distribution of characters into the blocks depends on knowing when we have \( \pi_a(h) = \pi_a'(h) \text{ (mod 3)} \), \( h \in H_3 \), for some \( a \in \{1, 2, a, b\} \), for different \( \pi_a, \pi_a' \) linear characters of \( H_3 \). Criteria for this were given in [9] in Lemmas (3.4), (5.3), and (6.3) to which we refer the reader. It is easy to verify that these hold also for \( p = 3 \). For the noncyclic blocks we count exceptional characters by repeating arguments used in [9, 10], modifying wherever necessary.

2.2. The case \( 3 \mid q - 1 \)

We write \( q - 1 = 3^d \cdot r, \ 3 \nmid r \). So

\[ |G_2(q)|_3 = 2d + 1. \]

If \( q = 2^k \) then \( k \) is even.

*Defect 0.* All characters of types \( \{X_2\} \) and \( \{X_6\} \) and the unipotent character \( X_{17} \) are of defect 0.

*Defect 1.* There are \( \frac{1}{18}(q^2 + q - 2) \) blocks of cyclic defect 1. Each contains three characters of type \( X_3(\pi_3) \), has Brauer tree

\[ \begin{array}{c}
\text{ex} \\
\downarrow \\
1 - \quad 2
\end{array} \]

and corresponds to a choice of \( \pi_3 \) restricted to 3-regular elements of \( H_3 \).

*Defect \( d \).* We check (as in [9, 10]) that all Sylow 3-subgroups of order \( 3^d \) of centralizers of 3-regular elements are indeed cyclic. This implies that the blocks of defect \( d \) have cyclic defect.

We have the following blocks with Brauer trees:

\[ \begin{array}{c}
\text{ex} \\
\downarrow \\
X_{2a} - X_a - X'_{2a}
\end{array} \quad \begin{array}{c}
\text{ex} \\
\downarrow \\
X_{2b} - X_b - X'_{2b}
\end{array} \]

\[ \begin{array}{c}
\text{ex} \\
\downarrow \\
X_a(3^d - 1) - 1
\end{array} \]

\( \alpha = a, b \) each containing \( 3^d \{X_\alpha\} \).
The determination of the trees is as for \( p \mid q - 1, p \neq 2, 3 \). See [9, (3.13), (3.16)].

Hence the numbers of blocks are

<table>
<thead>
<tr>
<th>Blocks</th>
<th>( G_2(q), 2, 3 \n q )</th>
<th>( G_2(2^k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_2 )</td>
<td>( \frac{1}{2}(q - 1) )</td>
<td>( \frac{1}{2}q )</td>
</tr>
<tr>
<td>3(d-1)</td>
<td>( \frac{1}{2}(rq - q + 1) )</td>
<td>( \frac{1}{4}q(r - 1) )</td>
</tr>
</tbody>
</table>

**Blocks of defect 2d.** These have noncyclic defect groups isomorphic to

\[
C_{3d} \times C_{3d} \subseteq H_1 \cong C_{q - 1} \times C_{q - 1}.
\]

The situation is similar to the case \( p \mid q - 1, p \neq 2, 3 \) for nonprincipal blocks of maximal \( p \)-defect. See [9, 10] for details of the determination of the exceptional characters and calculations and [5] for the decomposition matrices. We note that if \( q = 2^k \) there is no block \( B_2 \).

**Exceptional families**

<table>
<thead>
<tr>
<th>Number of blocks for odd ( q )</th>
<th>Number of blocks if ( q = 2^k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B_2 ) ( X_1, X_7, X_9, X_{2a} ) {( X_{1a}}{( X_{1b}}{( X_{1b}}{( X_{1}}} {( X_{1}}}</td>
<td>1</td>
</tr>
<tr>
<td>( B_a ) ( X_{1a}, X_{1b} ) {( X_{1b}}} {( X_{1}}}</td>
<td>( \frac{1}{2}(r - 2) )</td>
</tr>
<tr>
<td>( B_b ) ( X_{1b} ) {( X_{1b}}} {( X_{1}}}</td>
<td>( \frac{1}{2}(r - 2) )</td>
</tr>
<tr>
<td>( B_{X_1} ) {( X_{1}}}</td>
<td>( \frac{1}{2}(r^2 - 6r + 8) )</td>
</tr>
</tbody>
</table>

**Note.** When \( q = 2^k \) the number of blocks \( B_{X_1} \) is as for \( p \mid q - 1, q \equiv -1 \pmod{3}, p > 3 \).

**Number of exceptionals.**

<table>
<thead>
<tr>
<th>( X_{1a}, X_{1b} )</th>
<th>( X_{1b}, X_{1b} )</th>
<th>( X_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B_2 ) (when ( q ) is odd)</td>
<td>( \frac{1}{2}(3^d - 1) )</td>
<td>( \frac{1}{2}(3^d - 1) )</td>
</tr>
<tr>
<td>( B_a )</td>
<td>( 3^d )</td>
<td>( 3^d )</td>
</tr>
<tr>
<td>( B_b )</td>
<td>( - )</td>
<td>( 3^d )</td>
</tr>
<tr>
<td>( B_{X_1} )</td>
<td>( - )</td>
<td>( - )</td>
</tr>
</tbody>
</table>

**Principal block \( B_1 \).** This is the unique block of maximal defect. We note that the Sylow 3-subgroup of \( G_2(q) \) is isomorphic to the Sylow 3-subgroup of \( SL(3, q) \). The principal block contains the following characters:
(i) $1_G$, $X_{12} = St$, $X_{13}$, $X_{14}$, $X_{15}$, $X_{16}$
$X_{31}$, $X_{32}$, $X_{33}$
all of maximal defect $(2d+1)$.

(ii) $X_{18}$, $X_{19}$, $X_{19}$ of defect 2.

(iii) The exceptional families (all of defect 2d),

<table>
<thead>
<tr>
<th>Number in blocks</th>
<th>$\frac{1}{2}(3d-3)$</th>
<th>$\frac{1}{2}(3d-1)$</th>
<th>$\frac{1}{12}(3d-3)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${X_{1a}} $, ${X_{1a}'} $</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>${X_{1b}} $, ${X_{1b}'} $</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>${X_1} $</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note. Since the only root of unity of order 3 in a field of characteristic 3 is 1, we have $\pi_1^+(h_1) \equiv 1 \pmod{3}$. Hence $X_{31}$, $X_{32}$, $X_{33} \in B_1$ (see tables in Appendix).

Calculation of numbers of exceptionals in $B_1$. By the tables of central characters (mod 3) in the Appendix, we have $X_{1a}$, $X_{1b} \in B_1$, if and only if for the $\pi_1^a$ defining them we have $\pi_1^a(h) \equiv 0$ for all $h \in H_1$, $h \neq 1$.

Hence any $i^a \neq 0$ gives us $X_{1a}$, $X_{1b} \in B_1$ yielding $\frac{1}{2}(3d-1)$ characters, and any $i^a \neq 0$, $3d-1 \cdot r$, $2 \cdot 3d-1 \cdot r$ gives $X_{1a}$, $X_{1b} \in B_1$, yielding $\frac{1}{2}(3d-3)$ characters. ($i = 3d-1 \cdot r$ or $2 \cdot 3d-1 \cdot r$ in the formula for $X_{1a}$, $X_{1b}$ define $\pi_1^a$.)

If $X_1 \in B_1$, then $\pi_1 \sim (i, j)$ such that $(i, j) \equiv (0, 0) \pmod{r}$. We have $(3d-1)^2(i, j)$ s.t. $i \equiv 0 \pmod{r}$, $j \equiv 0 \pmod{r}$ for $1 \leq i, j \leq 3d-1$. (So $(i, j) \neq (k, 0), (0, k) \pmod{q-1}$.) As in [9, (5.13)], we exclude pairs of types:

$$(2k, 3k), \ (k, 3k), \ (k, k), \ (k, 2k)$$

$(k, k) \equiv (sr, tr) \pmod{(q-1)}$ implies $s \equiv t \pmod{3d}$: $3d-1$ possibilities
$(k, 2k) \equiv (sr, tr) \pmod{(q-1)}$ implies $2s \equiv t \pmod{3d}$: $3d-1$ possibilities
$(2k, 3k) \equiv (sr, tr) \pmod{(q-1)}$ implies $2t \equiv 3s \pmod{3d}$: $3d-3$ possibilities
$(k, 3k) \equiv (sr, tr) \pmod{(q-1)}$ implies $3s \equiv t \pmod{3d}$: $3d-3$ possibilities

— as here $s = 3d-1$ or $2 \cdot 3d-1$ gives $t = 0$ as the only solution, which is impossible, so only $3d-3$ solutions to these two congruences,

$$(3d-1)^2 - 2 \cdot (3d - 3) - 2 \cdot (3d - 1) = (3d - 3)^2.$$ 

This yields $\frac{1}{12}(3d-3)^2$ different $X_1(\pi_1)$. 

2.3. The Case $3 \mid q + 1$.

This is analogous to the previous case, so we omit proofs. We write $q + 1 = 3^d \cdot r$, $3 \nmid r$. So $|G_2(q)|_3 = 2d + 1$.

Defect 0. The characters in families $\{X_1\}$, $\{X_3\}$ are of defect 0. Also the unipotent character $X_{15}$. 

Defect 1. There are $\frac{1}{15}(q^2 - q - 2)$ blocks of cyclic defect 1, each containing 3 characters of type $X_6(\pi_a)$, Brauer tree

\[
\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ \\
1 \quad 2
\end{array}
\]

and corresponding to a choice of $\pi_a$ restricted to 3-regular elements of $H_6$.

Defect $d$. There are blocks of cyclic defect $d$ with Brauer trees:

\[
\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ \\
1 \quad 2
\end{array}
\]

and

\[
\begin{array}{c}
\circ \quad \circ \\
\circ \quad \circ \\
3^d - 1 \quad 1
\end{array}
\]

\[\alpha = a, b\] each containing $3^d \{X_\alpha\}$.

We determine these trees as in the case $p \mid q + 1$, $p \neq 2, 3$ (see [9, (4.8), (4.11)]).

The numbers of blocks are:

<table>
<thead>
<tr>
<th>Blocks</th>
<th>$G_2(q)$, $2, 3 \nmid q$</th>
<th>$G_2(2^k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{1a}$</td>
<td>$\frac{1}{2}(q - 3)$</td>
<td>$\frac{1}{2}(q - 2)$</td>
</tr>
<tr>
<td>$X_{1a}$</td>
<td>$\frac{1}{2}(rq - q - 2r + 3)$</td>
<td>$\frac{1}{4}(q - 2)(r - 1)$</td>
</tr>
</tbody>
</table>

Blocks of defect $2d$. The defect groups are isomorphic to

\[C_{3^d} \times C_{3^d} \leq H_2 \cong C_{q + 1} \times C_{q + 1} \]
Here the situation is similar to the case \( p \mid q + 1, p \neq 2, 3 \), for non-principal blocks of maximal defect. Again, if \( q = 2^k \) there is no block \( B_2 \).

### Exceptional families

<table>
<thead>
<tr>
<th>( B_2 )</th>
<th>( X_{2a}X_{2b}X_{23}X_{24} )</th>
<th>{ ( X_{2a} } { X_{2b} } { X_{23} } { X_{24} } { X_2 } )</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B_a )</td>
<td>{ ( X_{2a} } { X_{2b} } { X_2 } )</td>
<td>( \frac{1}{2}(r - 2) )</td>
<td>( \frac{1}{2}(r - 1) )</td>
<td></td>
</tr>
<tr>
<td>( B_b )</td>
<td>{ ( X_{2a} } { X_{2b} } { X_2 } )</td>
<td>( \frac{1}{2}(r - 2) )</td>
<td>( \frac{1}{2}(r - 1) )</td>
<td></td>
</tr>
<tr>
<td>( B_{x1} )</td>
<td>{ ( X_2 } )</td>
<td>( \frac{1}{12}(r^2 - 6r + 8) )</td>
<td>( \frac{1}{12}(r^2 - 6r + 5) )</td>
<td></td>
</tr>
</tbody>
</table>

**Note.** When \( q = 2^k \), the number of blocks \( B_{x2} \) is as \( p \mid q + 1, q \equiv 1 \pmod{3}, p > 3 \).

### Number of exceptionals.

<table>
<thead>
<tr>
<th>( X_{2a}, X_{2a} )</th>
<th>( X_{2b}, X_{2b} )</th>
<th>( X_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B_2 ) (when ( q ) is odd)</td>
<td>( \frac{1}{2}(3^d - 1) )</td>
<td>( \frac{1}{2}(3^d - 1) )</td>
</tr>
<tr>
<td>( B_a )</td>
<td>( \frac{1}{2}(3^d - 1) )</td>
<td>( \frac{1}{2}(3^d - 1)^2 )</td>
</tr>
<tr>
<td>( B_b )</td>
<td>( 3^d )</td>
<td>( 3^d )</td>
</tr>
<tr>
<td>( B_{x2} )</td>
<td>( 3^d )</td>
<td>( \frac{1}{2}3^d(3^d - 1) )</td>
</tr>
</tbody>
</table>

### 3. Decomposition Matrices

#### 3.1. Explanation of the Tables

In this section we present the 3-modular decomposition numbers of \( G = G_2(q) \), \( q \) not divisible by 3, for the principal block. We start with some explanation of the tables. The ordinary characters in a block fall naturally into two distinct sets. The first set consists of what is called the non-exceptional characters in the following. Its members all belong to a fixed geometric conjugacy class of characters (see [1, Section 12.1]). In case of the principal block, the non-exceptional characters are exactly the unipotent characters lying in the block. Their restrictions to the 3-regular conjugacy classes generate the ring of generalized Brauer characters, but are linearly dependent as class functions. However, a basic set can be selected from these. The decomposition of these basic set characters is given in the upper half of the decomposition matrix.

The remaining characters in a block are called the exceptional characters. They fall into families of characters which have the same restriction to 3-regular classes. Only one row is printed for any one family of exceptional
3.2. The case \( 3 \mid q - 1 \)

Let \( q \equiv 1 \pmod{3} \). Then the principal 3-block of \( G \) has the decomposition matrix shown in Table I.

Remarks. (i) \( 0 \leq \alpha \leq 1 \), \( 0 \leq \beta \leq q - 2 \), \( 1 \leq \gamma \leq q + 1 \).

(ii) If \( q \) is a power of 2 we have \( \alpha = \beta = 0 \).

(iii) K. Lux has shown that \( \gamma = 1 \) in case \( q = 4 \). He used a defect 0 character of the (exceptional) double cover of \( G_2(4) \).

(iv) We have \( \varphi_{14}(1) \in \left\{ \frac{1}{6} q(q^2 - q + 1)(q^2 + 4q + 1), \frac{1}{2} q^2(q^2 - q + 1) \right\} \) and \( \varphi_{21}(1) \geq \frac{1}{2}(q - 1)^2(q^4 + 2q^3 + 3q + 2) \), so that \( \varphi_{16}(1) = q^3 \) is the smallest non-trivial 3-modular character degree of \( G_2(q) \) if \( q \equiv 1 \pmod{3} \).

\[
\begin{array}{cccccccc}
\text{Degrees} & \varphi_{11} & \varphi_{18} & \varphi_{19} & \varphi_{14} & \varphi_{15} & \varphi_{16} & \varphi_{12} & \text{No.ofchars.} \\
1 & X_{11} & 1 & & & & & & \\
1/2 q(q - 1)^2(q^2 - q + 1) & X_{18} & 1 & & & & & & \\
1/2 q(q - 1)^2(q + 1)^2 & X_{19} & 1 & & & & & & \\
1/2 q(q^4 + q^2 + 1) & X_{14} & \alpha + 1 & 1 & & & & & \\
1/2 q(q + 1)^2(q^2 - q + 1) & X_{15} & 1 & & & & & & \\
1/2 q(q + 1)^2(q^2 + q + 1) & X_{16} & \alpha & 1 & 1 & & & & \\
q^6 & X_{17} & \beta & \gamma & 1 & 1 & 1 & & \\
1/2 q(q^4 + q^2 + 1) & X_{13} & 1 & 1 & 1 & & & & \\
1/2 q(q - 1)^2(q + 1)^2 & X_{19} & 1 & & & & & & \\
q^3 + 1 & X_{32} & 1 & & & & & & \\
q(q + 1)(q^3 + 1) & X_{33} & \alpha & 1 & 1 & 1 & 1 & 1 & \\
q^3(q^3 + 1) & X_{31} & \alpha + \beta + 1 & \gamma - 1 & 1 & 1 & 1 & 1 & \\
(q + 1)(q^4 + q^2 + 1) & \{X_{1a} \} & 2 & \alpha & 1 & 1 & 1 & 2 & \frac{1}{2}(3^d - 3) \\
q(q + 1)(q^4 + q^2 + 1) & \{X_{1c} \} & 1 & 2\alpha + \beta + 1 & \gamma & 2 & 2 & 1 & 1 & \frac{1}{2}(3^d - 3) \\
(q + 1)(q^4 + q^3 + 1) & \{X_{1b} \} & 2 & 2\alpha + 1 & \gamma & 2 & 1 & 1 & \frac{1}{2}(3^d - 1) \\
q(q + 1)(q^4 + q^2 + 1) & \{X_{1b} \} & 1 & \alpha + \beta & \gamma + 1 & 1 & 2 & 2 & 1 & \frac{1}{2}(3^d - 1) \\
(q + 1)(q^4 + q^3 + 1) & \{X_{1a} \} & 3 & 3\gamma + \beta + 1 & \gamma + 1 & 3 & 3 & 3 & 1 & \frac{1}{2}(3^d - 3)^2 \\
\end{array}
\]
The degrees of the irreducible Brauer characters are:

\[
\begin{align*}
\varphi_{11} & : q(q - 1)^2(q^2 - q + 1) \\
\varphi_{18} & : \frac{1}{2}q(q - 1)^2(q^2 - q + 1) \\
\varphi_{19} & : \frac{1}{2}q(q - 1)^2(q + 1)^2 \\
\varphi_{14} & : \frac{1}{2}q(q^2 - q + 1)[(1 - \alpha)q^2 + (4 + 2\alpha)q + (1 - \alpha)] \\
\varphi_{15} & : \frac{1}{2}(q^3 + q^4 + q^2 + q - 2) \\
\varphi_{16} & : q^3 \\
\varphi_{12} & : \frac{1}{2}(q - 1)^2[6q^4 + (9 - \beta - 2\gamma)q^3 + (9 + \beta - 4\gamma)q^2 + (9 - \beta - 2\gamma)q + 6]
\end{align*}
\]

3.3. \textit{The Case }3|q + 1

Let \(q \equiv -1 \pmod{3}\). Then the principal 3-block of \(G\) has the decomposition matrix shown in Table II

**Remarks.**

(i) \(1 \leq \alpha \leq q + 1, 1 \leq \beta \leq q - 1, 1 \leq \gamma \leq \frac{1}{2}q\).

(ii) In case \(q = 2\), \(G_2(2) \cong U_3(3) \cdot 2\) and we have \(\alpha = \beta = \gamma = 1\).

(iii) We have \(\varphi_{12}(1) \geq \frac{1}{2}(q - 1)^2(q + 2)^2(q^2 + q + 1)\), so that \(\varphi_{16}(1) = q^3 - 1\) is the smallest non-trivial 3-modular character degree of \(G_2(q)\) if

**TABLE II**

<table>
<thead>
<tr>
<th>Degrees</th>
<th>(\varphi_{11})</th>
<th>(\varphi_{18})</th>
<th>(\varphi_{19})</th>
<th>(\varphi_{14})</th>
<th>(\varphi_{15})</th>
<th>(\varphi_{16})</th>
<th>(\varphi_{12})</th>
<th>No. of chars</th>
</tr>
</thead>
<tbody>
<tr>
<td>(q^3 - 1)</td>
<td>(X_{31})</td>
<td>(X_{32})</td>
<td>(X_{33})</td>
<td>(X_{34})</td>
<td>(X_{35})</td>
<td>(X_{36})</td>
<td>(X_{37})</td>
<td>(X_{38})</td>
</tr>
<tr>
<td>((q - 1)(q^4 + q^2 + 1))</td>
<td>({X_{2a}})</td>
<td>({X_{2b}})</td>
<td>({X_{2c}})</td>
<td>({X_{2d}})</td>
<td>({X_{2e}})</td>
<td>({X_{2f}})</td>
<td>({X_{2g}})</td>
<td>({X_{2h}})</td>
</tr>
<tr>
<td>((q - 1)(q^4 + q^2 + 1))</td>
<td>({X_{2a}})</td>
<td>({X_{2b}})</td>
<td>({X_{2c}})</td>
<td>({X_{2d}})</td>
<td>({X_{2e}})</td>
<td>({X_{2f}})</td>
<td>({X_{2g}})</td>
<td>({X_{2h}})</td>
</tr>
</tbody>
</table>
$q \equiv -1 \pmod{3}$ and $q > 2$. If $q = 2$ we have a faithful 3-modular character of degree 6.

The degrees of the irreducible Brauer characters are:

<table>
<thead>
<tr>
<th>Char.</th>
<th>Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_{11}$</td>
<td>$\frac{1}{2}q(q-1)^2(q^2 - q + 1)$</td>
</tr>
<tr>
<td>$\phi_{12}$</td>
<td>$\frac{1}{2}(q^2 - 1)(q^3 + 3q^2 - q + 6)$</td>
</tr>
<tr>
<td>$\phi_{17}$</td>
<td>$\frac{1}{2}q(q-1)^2(q^2 + q + 1)$</td>
</tr>
<tr>
<td>$\phi_{16}$</td>
<td>$q^2 - 1$</td>
</tr>
<tr>
<td>$\phi_{12}$</td>
<td>$\frac{1}{2}(q-1)^2[6q^4 + (11 - \alpha - 2\beta - 3\gamma)q^3 + (13 + \alpha - 4\beta - 3\gamma)q^2$ $+ (11 - \alpha - 2\beta - 3\gamma)q + 6]$</td>
</tr>
</tbody>
</table>

4. Determination of Decomposition Matrices

4.1. Some Scalar Products

The following tables list some scalar products between characters of $G$. The subgroups $SL_3(q)$ and $SU_3(q)$ are denoted by $L$ and $M$, respectively. The character tables of $L$ and $M$ are given in [11]. It is routine to determine the fusion of the conjugacy classes of $L$ and $M$ into the classes of $G$. We only give those scalar products we shall need in our proofs and which have not already been given in Appendix A of [5]. As always, missing entries are 0.

(a) $q \equiv -1 \pmod{3}$

<table>
<thead>
<tr>
<th>Char.</th>
<th>$X_{11}$</th>
<th>$X_{12}$</th>
<th>$X_{13}$</th>
<th>$X_{14}$</th>
<th>$X_{15}$</th>
<th>$X_{16}$</th>
<th>$X_{17}$</th>
<th>$X_{18}$</th>
<th>$X_{19}$</th>
<th>$X_{19}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{15} \otimes X_{32}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$X_{11}^{G}$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$X_{q(q-1)}^{G}$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$X_{q(q-1)+1}^{G}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

(b) $q \equiv 1 \pmod{3}$

<table>
<thead>
<tr>
<th>Char.</th>
<th>$X_{11}$</th>
<th>$X_{12}$</th>
<th>$X_{13}$</th>
<th>$X_{14}$</th>
<th>$X_{15}$</th>
<th>$X_{16}$</th>
<th>$X_{17}$</th>
<th>$X_{18}$</th>
<th>$X_{19}$</th>
<th>$X_{19}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{(q+1)q^{-1}}^{G}$</td>
<td>$q + 1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$X_{11}^{G}\otimes X_{18}$</td>
<td>$\frac{1}{2}q(q-1)(q-2)$</td>
<td>$\frac{1}{2}q(q-1)$</td>
<td>$\frac{1}{2}q(q-1)$</td>
<td>$\frac{1}{2}q(q-1)$</td>
<td>$\frac{1}{2}q(q-1)$</td>
<td>$\frac{1}{2}q(q-1)$</td>
<td>$\frac{1}{2}q(q-1)$</td>
<td>$\frac{1}{2}q(q-1)$</td>
<td>$\frac{1}{2}q(q-1)$</td>
<td>$\frac{1}{2}q(q-1)$</td>
</tr>
</tbody>
</table>
(c) $q$ odd, $q \equiv -1 \pmod{3}$

<table>
<thead>
<tr>
<th>Char.</th>
<th>$X_{21}$</th>
<th>$X_{22}$</th>
<th>$X_{23}$</th>
<th>$X_{24}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{15} \otimes X_{13}$</td>
<td>$\frac{1}{q}(q^2 + q + 6)$</td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>$X_{15} \otimes X_{14}$</td>
<td>$\frac{1}{q}(q^2 + q + 6)$</td>
<td></td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

4.2. Proof in the Case $3 \mid q - 1$

We have the following relations on 3-regular classes:

$$X_{13} = -X_{14} + X_{16} + X_{18} + X_{19}$$
$$\overline{X}_{19} = X_{19}$$
$$X_{31} = X_{12} + X_{14} - X_{19}$$
$$X_{32} = X_{11} - X_{14} + X_{16} + X_{18}$$
$$X_{33} = X_{15} + X_{16} + X_{19}$$
$$X_{1a} = X_{12} + X_{14} + X_{15} + X_{16}$$
$$X_{1a}' = X_{11} - X_{14} + X_{15} + 2X_{16} + X_{18} + X_{19}$$
$$X_{1b} = X_{12} - X_{14} + X_{15} + 2X_{16} + X_{18} + X_{19}$$
$$X_{1b}' = X_{11} + X_{14} + X_{15} + X_{16}$$
$$X_{1} = X_{11} + X_{12} + 2X_{15} + 3X_{16} + X_{18} + X_{19}$$
$$= X_{1a} + X_{1a}'$$
$$= X_{1b} + X_{1b}'$$

Since $X_{11}, X_{12}, X_{14}, X_{15}, X_{16}, X_{18},$ and $X_{19}$ are linearly independent on 3-regular classes, they form a basic set by Lemma 4 of [5].

Let $(K, R, k)$ denote a splitting 3-modular system for $G$ such that $k$ is the modular field of characteristic 3. Let $W$ denote the Weyl group of $G$. We have $W \cong D_{12}$, the dihedral group of order 12. The character table of $W$ can be found, e.g., in Carter's book [1, p. 412]. We shall use Carter's notation for the characters of $W$. By the theory of Howlett–Lehrer there is a bijection between the irreducible characters of $W$ and the constituents of $K^\Lambda G$, originating from the fact that the endomorphism ring of this module is isomorphic to the group algebra $KW$. In our example this correspondence is as follows (see [1, pp. 449, 450]):

$$\phi_{1,0} \quad X_{1,6} \quad \phi_{1,3} \quad \phi_{1,3}'' \quad \phi_{2,1} \quad \phi_{2,2}$$
$$X_{11} \quad X_{12} \quad X_{13} \quad X_{14} \quad X_{16} \quad X_{15}$$
We shall use the 3-modular decomposition matrix of $W$, which is as follows:

\[
\begin{array}{cccc}
\phi_{1,0} & 1 \\
\phi_{1,6} & 1 \\
\phi_{1,8} & 1 \\
\phi_{1,12} & 1 \\
\phi_{2,1} & 1 & 1 \\
\phi_{2,2} & 1 & 1 \\
\end{array}
\]

The reason for considering this decomposition matrix is the following. Let $U$ denote a maximal unipotent subgroup of $G$. Since 3 does not divide the order of $U$, $P = R^U_G$ is a projective module. By a result of Dipper (see [3, Corollary 4.10]), we can find the decomposition of $P$ into indecomposables by only considering $R^U_B$, the permutation module on the cosets of the Borel subgroup. For $S \in \{K, R, k\}$, let $H_S = \text{End}_{SG}(S^U_B)$ denote the Hecke algebra. The decomposition of $R^U_B$ into indecomposables corresponds to the decomposition of its endomorphism ring $H_R$, which in turn can be described by the decomposition matrix of $H_R$ by Brauer reciprocity. Thus the decomposition matrix of the Hecke algebra $H_R$ is a submatrix of the decomposition matrix of $G$.

From the well-known multiplication formulae for $H_R$ it follows that in case $q \equiv 1 \pmod{3}$ $H_k$ is isomorphic to the group algebra $kW$. This gives us the following Cartan matrix of $H_k$:

\[
\begin{pmatrix}
2 & 1 & 0 & 0 \\
1 & 2 & 0 & 0 \\
0 & 0 & 2 & 1 \\
0 & 0 & 1 & 2
\end{pmatrix}
\]

Since we do not know whether $H_R$ is isomorphic to $RW$, we have to be careful in determining the decomposition matrix of $H_R$. From the Cartan matrix and the fact that $\phi_{2,1}$ and $\phi_{2,2}$ are 2-dimensional $H_k$-representations, we obtain the following decomposition matrix:

\[
D = \begin{pmatrix}
A & 1 \\
X_{16} & \phi_{2,1} & 1 & 1 \\
B & 1 \\
C & 1 \\
X_{15} & \phi_{2,2} & 1 & 1 \\
D & 1
\end{pmatrix}
\]

where $\{A, B, C, D\} = \{X_{11}, X_{12}, X_{13}, X_{14}\}$. It remains to determine the missing row labels of $D$. If these are completed, $D$ describes the decomposition of $R^U_B$ and hence of $R^U_G$ into indecomposable summands. Now $X_{17} \otimes X_{17}$ is a projective character which out of the basic set characters
contains only $X_{11}$, $X_{12}$, and $X_{15}$ as constituents (see [5, Appendix B]). This together with the relation for $X_{13}$ shows that we may take $A = X_{13}$, $B = X_{14}$.

Using this, we obtain four indecomposable direct summands of $R^\uparrow_G$. They are denoted by $\Phi_{11}$, $\Phi_{14}$, $\Phi_{15}$, and $\Phi_{16}$ in the following table of scalar products:

<table>
<thead>
<tr>
<th>Char.</th>
<th>$\Phi_{11}$</th>
<th>$\Phi_{14}$</th>
<th>$\Phi_{15}$</th>
<th>$\Phi_{16}$</th>
<th>$\Phi_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_{11}$</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_{18}$</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_{14}$</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_{15}$</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_{16}$</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$X_{12}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$q-2$</td>
</tr>
</tbody>
</table>

Here, $\Phi_{12}$ denotes the Gelfand-Graev character restricted to the principal block. Let $M = SU_3(q)$. Then $\Phi_{19}$ equals the restriction to the principal block of $\chi_{(q+1, q-1)}^G$. It is easy to see that the restriction of $X_{18}$ to $M$ is just the sum of all characters $\chi_{(u,v,w)}^{(q-1)}(q^2-q+1)$ and so is projective. Thus $\Phi = (\chi_{1}^G \otimes X_{13} = (X_{18} \otimes \overline{M}) \otimes \overline{M}$ is the character of a projective module, with all multiplicities divisible by $\frac{1}{\delta} q(q-1)$. Dividing by this common factor gives $\Phi_{19}$ of the above table.

Now $\Phi_{14}$ is contained at most once in $\Phi_{18}$, whereas $\Phi_{12}$ is contained at most $q-2$ times in $\Phi_{18}$, at most $q+1$ times and at least once in $\Phi_{19}$ (by the relation for $X_{21}$).

If $q$ is even, there is a projective character of the parabolic subgroup $Q$ of degree $\frac{1}{2}q(q-1)^2$, which is called $\mathcal{Y}_4(1)$ in [4]. The induced character $\mathcal{Y}_4(1)\uparrow_Q^G$ contains $X_{18}$, but neither $X_{12}$ nor $X_{16}$ (see [4, p. 344]). This shows that in this case $\alpha = \beta = 0$. The character $\mathcal{Y}_4(1)$ of $Q$ is obtained by inducing a linear character of $U$ to $Q$ which does not have $U_1$ (cf. [4, pp. 331-339]) in its kernel. Such a character is not available for odd $q$, since in this case $U_1 = U'$. This completes the proof for the principal block.

The decomposition matrices for the blocks of non-maximal non-cyclic defect can be determined as in [5]. The crucial step is to show that in such blocks all irreducible Brauer characters are liftable. This can be done by appealing to a result of Puig [6]. All irreducible modules in such a situation have trivial source.

4.3. **Proof in the Case $3 | q + 1$**

We have the following relations on 3-regular classes:

\[ X_{13} = -X_{14} + X_{16} + X_{18} + X_{19} \]

\[ \overline{X}_{19} = X_{19} \]
\[X_{31} = X_{12} - X_{14} + X_{19}\]
\[X_{32} = -X_{11} - X_{14} + X_{16} + X_{18}\]
\[X_{33} = X_{17} + X_{18} + X_{19}\]
\[X_{2a} = X_{12} + X_{14} - X_{16} - X_{17} - 2X_{18} - X_{19}\]
\[X_{2b} = X_{12} - X_{14} - X_{17} - X_{18}\]
\[X_{2c} = -X_{11} - X_{14} + X_{16} + X_{17} + 2X_{18} + X_{19}\]
\[X_2 = X_{11} + X_{12} - X_{16} - 2X_{17} - 3X_{18} - X_{19}\]  
\[= X_{2a} - X_{2b}'\]
\[= X_{2b} - X_{2b}'.\]

Since \(X_{11}, X_{12}, X_{14}, X_{16}, X_{17}, X_{18}\), and \(X_{19}\) are linearly independent on 3-regular classes, they form a basic set by Lemma 4 of [5]. We have the following table of scalar products:

<table>
<thead>
<tr>
<th>Char.</th>
<th>(\Phi_{11})</th>
<th>(\Phi_{18})</th>
<th>(\Phi_{19})</th>
<th>(\Phi_{14})</th>
<th>(\Phi_{17})</th>
<th>(\Phi_{16})</th>
<th>(\Phi_{12})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(X_{11})</td>
<td>1</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(X_{18})</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(X_{19})</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(X_{14})</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(X_{17})</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(X_{16})</td>
<td>2</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(X_{12})</td>
<td>1</td>
<td>(q+1)</td>
<td>(q-1)</td>
<td>1</td>
<td>(z)</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

where \(z \in \{q/2, (q-1)/2\}\). The projectives originate from:

<table>
<thead>
<tr>
<th>Char.</th>
<th>Origin</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Phi_{11})</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
\(\frac{\Theta}{q}\) restricted to \(B_1\) | \(X_1 = \hat{X}_{q+1}\) projective of \(L = SL_3(q)\) |
| \(\Phi_{18}\) | \((X_1 + X_\ell)^{\frac{\Theta}{q}} - \Phi_{11}\) | \(X_1^{(q-1)/2} X_{q+1}\) defect 0 of \(L\) |
| \(\Phi_{19}\) | \(X_{(q-1)/2}^{(q+1)} \hat{L}^{\ell}\) | \(X_{q+1}\) defect 0 of \(L\) |
| \(\Phi_{14}\) | \(\frac{1}{2} X_{q+1}\) | \(X_{q+1}\) defect 0 of \(L\) |
| \(\Phi_{17}\) | \(X_{15} \otimes X_{17}\) | \(X_{15}\) defect 0, see [5, Appendix A] |
| \(\Phi_{16}\) | \(X_{15} \otimes X_{32}\) | \(X_{15}\) defect 0 |
| \(\Phi_{12}\) | Gelfand–Graev Char. | | |
The relations show that $\Phi_{11}$, $\Phi_{14}$, $\Phi_{16}$, and $\Phi_{12}$ are indecomposable. The bounds for $\alpha$, $\beta$, and $\gamma$ follow from the projectives given above.

The proofs for the remaining blocks are exactly the same as in [5], except that in the proof for $B_2$ the projectives $X_{13} \otimes X_{19}$ and $X_{14} \otimes X_{19}$ have to be replaced by $X_{14} \otimes X_{15}$ and $X_{13} \otimes X_{15}$, respectively. This completes the proof in the case $q \equiv 1 \mod 3$.

**APPENDIX**

<table>
<thead>
<tr>
<th>$q \equiv 1 \mod 3$</th>
<th>Central chars. (mod 3) on 3'-elements</th>
<th>$2 = d(X_{18}) = d(X_{19}) = d(X_{19})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_2(q)$</td>
<td>$\psi_5$ $\psi_6$ $\psi_{13}$ $\psi_{14}$ $\psi_{15}$ $\psi_{29}$ $\psi_{30}$ $\psi_{31}$ $\psi_{32}$ $\psi_{33}$</td>
<td>$\psi_5$ $\psi_6$ $\psi_{13}$ $\psi_{14}$ $\psi_{15}$ $\psi_{29}$ $\psi_{30}$ $\psi_{31}$ $\psi_{32}$ $\psi_{33}$</td>
</tr>
<tr>
<td>$\psi_{21}$</td>
<td>$\psi_{22}$ $\psi_{23}$ $\psi_{24}$ $\psi_{25}$ $\psi_{26}$ $\psi_{27}$ $\psi_{28}$ $\psi_{29}$ $\psi_{30}$ $\psi_{31}$</td>
<td>$\psi_{22}$ $\psi_{23}$ $\psi_{24}$ $\psi_{25}$ $\psi_{26}$ $\psi_{27}$ $\psi_{28}$ $\psi_{29}$ $\psi_{30}$ $\psi_{31}$</td>
</tr>
<tr>
<td>$\psi_{22}$</td>
<td>$\psi_{23}$ $\psi_{24}$ $\psi_{25}$ $\psi_{26}$ $\psi_{27}$ $\psi_{28}$ $\psi_{29}$ $\psi_{30}$ $\psi_{31}$</td>
<td>$\psi_{23}$ $\psi_{24}$ $\psi_{25}$ $\psi_{26}$ $\psi_{27}$ $\psi_{28}$ $\psi_{29}$ $\psi_{30}$ $\psi_{31}$</td>
</tr>
<tr>
<td>$\psi_{23}$</td>
<td>$\psi_{24}$ $\psi_{25}$ $\psi_{26}$ $\psi_{27}$ $\psi_{28}$ $\psi_{29}$ $\psi_{30}$ $\psi_{31}$</td>
<td>$\psi_{24}$ $\psi_{25}$ $\psi_{26}$ $\psi_{27}$ $\psi_{28}$ $\psi_{29}$ $\psi_{30}$ $\psi_{31}$</td>
</tr>
<tr>
<td>$\psi_{24}$</td>
<td>$\psi_{25}$ $\psi_{26}$ $\psi_{27}$ $\psi_{28}$ $\psi_{29}$ $\psi_{30}$ $\psi_{31}$</td>
<td>$\psi_{25}$ $\psi_{26}$ $\psi_{27}$ $\psi_{28}$ $\psi_{29}$ $\psi_{30}$ $\psi_{31}$</td>
</tr>
<tr>
<td>$\psi_{25}$</td>
<td>$\psi_{26}$ $\psi_{27}$ $\psi_{28}$ $\psi_{29}$ $\psi_{30}$ $\psi_{31}$</td>
<td>$\psi_{26}$ $\psi_{27}$ $\psi_{28}$ $\psi_{29}$ $\psi_{30}$ $\psi_{31}$</td>
</tr>
<tr>
<td>$\psi_{26}$</td>
<td>$\psi_{27}$ $\psi_{28}$ $\psi_{29}$ $\psi_{30}$ $\psi_{31}$</td>
<td>$\psi_{27}$ $\psi_{28}$ $\psi_{29}$ $\psi_{30}$ $\psi_{31}$</td>
</tr>
<tr>
<td>$\psi_{27}$</td>
<td>$\psi_{28}$ $\psi_{29}$ $\psi_{30}$ $\psi_{31}$</td>
<td>$\psi_{28}$ $\psi_{29}$ $\psi_{30}$ $\psi_{31}$</td>
</tr>
<tr>
<td>$\psi_{28}$</td>
<td>$\psi_{29}$ $\psi_{30}$ $\psi_{31}$</td>
<td>$\psi_{29}$ $\psi_{30}$ $\psi_{31}$</td>
</tr>
<tr>
<td>$\psi_{29}$</td>
<td>$\psi_{30}$ $\psi_{31}$</td>
<td>$\psi_{30}$ $\psi_{31}$</td>
</tr>
<tr>
<td>$\psi_{30}$</td>
<td>$\psi_{31}$</td>
<td>$\psi_{31}$</td>
</tr>
</tbody>
</table>

$q \equiv 1 \mod 3$ Central chars. (mod 3) on 3'-elements $2 = d(X_{18}) = d(X_{19}) = d(X_{19})$
$q \equiv -1 \pmod{3}$

Central chars. (mod 3) on 3'-elements

$2 = d(X_{18}) = d(X_{19}) = d(Y_{19})$

<table>
<thead>
<tr>
<th>defect</th>
<th>1</th>
<th>d</th>
<th>d</th>
<th>d</th>
<th>d</th>
<th>2d</th>
<th>2d</th>
<th>2d</th>
<th>2d+1</th>
<th>2d+1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_2(q)$</td>
<td>$u_0$</td>
<td>$u_a$</td>
<td>$w_{2a}$</td>
<td>$w_{1a}$</td>
<td>$w_{2a}$</td>
<td>$w_{1a}$</td>
<td>$w_{1a}$</td>
<td>$w_{1a}$</td>
<td>$w_{1a}$</td>
<td>$w_{1a}$</td>
</tr>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$u_1$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$u_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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$q \equiv 1 \pmod{3}$

Central chars. (mod 3) on unipotent elements

$2k \equiv 1 \pmod{3} \iff k$ even so this is $G_2(q)$

$d(X_{18}) = d(X_{19}) = d(Y_{19}) = 2$

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<th>d</th>
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<td>$u_a$</td>
<td>$w_{2a}$</td>
<td>$w_{1a}$</td>
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3-BLOCKS AND 3-MODULAR CHARACTERS OF $G_2(q)$

\[
\begin{array}{cccccccccc}
q \equiv -1 \pmod{3} & \text{Central char. (mod 3) on unipotent elements} & (\ast \text{ odd}) & a(X_{16}) = a(X_{19}) = a(X_{19}) = 2 \\
\hline
defect: & 1 & 2 & 2d & 2d & 2d+1 & 2d+1 \\
G_2(2^k) & \psi_6 & \psi_9 & \psi_{10} & \psi_9 & \psi_{10} & \psi_{10} & \psi_{10} & \psi_9 & \psi_9 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\psi_1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\psi_2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\psi_{31} & 1 & \frac{1}{2}(q+1) & \frac{1}{2}(q+1) & 0 & 0 & 0 & 0 & 0 & 0 \\
\psi_{32} & 2 & \frac{1}{2}(q+1) & \frac{1}{2}(q+1) & 0 & 0 & 0 & 0 & 0 & 0 \\
\psi_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\psi_{51} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\psi_{52} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

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