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Multiple eigenvalues

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Abstract

The dimensions of sets of matrices of various types, with specified eigenvalue multiplicities, are determined. The dimensions of the sets of matrices with given Jordan form and with given singular value multiplicities are also found. Each corresponding codimension is the number of conditions which a matrix of the given type must satisfy in order to have the specified multiplicities.

Keywords: Multiple eigenvalues; Dimensions of sets of matrices; Jordan form; Singular value decomposition; Conditions on matrices

1. Introduction

We shall determine the dimensions of the sets of diagonalizable, normal, Hermitian, unitary, and real symmetric matrices having one eigenvalue of multiplicity k_1, another of multiplicity k_2, etc. We shall also find the dimensions of the set of square matrices having a specified Jordan form, and of the set of real rectangular matrices having largest singular value of multiplicity k_1, next largest of multiplicity k_2, etc. In each case the codimension of the specified set gives the number of conditions which the elements of the matrix must satisfy in order to be in that set. For example, the elements of a real symmetric matrix must satisfy two conditions in order to have a double eigenvalue, which is a result of von Neuman and Wigner [1].

To determine these dimensions, we first find the set of all matrices which transform a given matrix into its specified diagonal form, Jordan canonical form, or singular value form. Then we

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calculate the dimensions of these sets of transforming matrices, and use them to obtain the desired results.

Our main results are listed in Table 2. The dimensions of some common sets of matrices are given in Table 1.

This work was stimulated by a lecture of Beresford Parlett, based upon [2], which is related to the work of Lax [3]. Overton and Womersley [4] obtained results like some of those in the present paper.

2. Diagonalizable matrices in C^{nn}

Let C^{nn} be the set of complex square matrices of order n. In C^{nn} a matrix A is similar to J if and only if there is an invertible T such that $A = TJT^{-1}$. If in addition $A = T_1JT_1^{-1}$ then $TJT^{-1} = T_1JT_1^{-1}$, so $T_1^{-1}TJ = JT_1^{-1}T$. Thus $S = T_1^{-1}T$ commutes with J:

Table 1
Complex and real dimensions of various sets of matrices

	Set of matrices in C^{nn}	Complex dimension	Real dimension
1.	All	n^2	$2n^2$
2.	Invertible	n^2	$2n^2$
3.	Singular	n^2-1	$2(n^2-1)$
4.	Diagonalizable	n^2	$2n^2$
5.	Normal	$n(n+1)/2$	$n(n+1)$
6.	Hermitian	–	n^2
7.	Unitary	–	n^2
8.	Symmetric	$n(n+1)/2$	$n(n+1)$
9.	Real symmetric	–	$n(n+1)/2$
10.	Antisymmetric	$n(n-1)/2$	$n(n-1)$
11.	Real antisymmetric	–	$n(n-1)/2$
12.	Orthogonal	–	$n(n-1)/2$
13.	Matrices in C^{nm}		
	With rank $r \leqslant \min(n,m)$	$(m+n-r)r$	$2(m+n-r)r$

Table 2
Dimensions of various sets of matrices with specified eigenvalue multiplicities

Set of matrices in C^{nn} mult $k_1,\ldots,k_I$		Complex dimension	Real dimension
1.	Diagonalizable	$n^2-\sum_{i=1}^{I}(k_i^2-1)$	$2\left[n^2-\sum_{i=1}^{I}(k_i^2-1)\right]$
2.	Normal	–	$n^2+I-\sum_{i=1}^{I}(k_i^2-1)$
3.	Hermitian	–	$n^2-\sum_{i=1}^{I}(k_i^2-1)$
4.	Unitary	–	$n^2-\sum_{i=1}^{I}(k_i^2-1)$
5.	Real symmetric	–	$\frac{n(n-1)}{2}+I-\frac{1}{2}\sum_{i=1}^{I}k_i(k_i-1)$
6.	Matrices in R^{nm} with multiplicities $k_1,\ldots,k_I$ and $\sum_{i=1}^{I}k_i=r$	–	$(n+m-r)r-r+I-\frac{1}{2}\sum_{i=1}^{I}k_i(k_i-1)$
7.	With normal form J (Theorem 7)	–	$n^2-\sum_{j=1}^{N^*}(2j-1)\sum_{a=1}^{p}k_{aj}+p$
8.	$A\in R^{nm}$ with singular value multiplicities $k_1,\ldots,k_J$ (Corollary 9.1)	–	$(n+m-r)r-r-\frac{1}{2}\sum_{j=1}^{J}k_j(k_j-1)+J$

$$SJ = JS. \tag{2.1}$$

Since $T_1 = TS^{-1}$, we have the following result:

Lemma 1. *If $A = TJT^{-1}$ then $A = T_1JT_1^{-1}$ if and only if $T_1 = TS^{-1}$, where S is invertible and satisfies* (2.1).

By using Lemma 1, we can calculate the dimension of the set of A which are similar to J. This dimension is just the dimension of the set of invertible T minus the dimension of the set of invertible S satisfying (2.1). Thus we have

Lemma 2

$$\dim \{A|A = TJT^{-1}\} = \dim\{T|TT^{-1} = I\} - \dim \{S|SS^{-1} = I,\ SJ = JS\}. \tag{2.2}$$

The complex dimension of the set of invertible T in C^{nn} is n^2, and its real dimension is $2n^2$.

We shall now find the set of S which commute with J when $J = \Lambda$ is diagonal. We assume that Λ in C^{nn} has I distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_I$ with multiplicities $k_1, k_2, \ldots, k_I$, where $k_1 + k_2 + \cdots + k_I = n$. We write Λ in block diagonal form

$$\Lambda = (\lambda_1 I_1, \lambda_2 I_2, \ldots, \lambda_I I_I). \tag{2.3}$$

Here I_i is the identity matrix of order k_i. We partition S into blocks S_{ij}, of order k_i by k_j, so that S is conformable with Λ. By setting $J = \Lambda$, and using (2.3) and this partition of S in (2.1), we get

$$S_{ij}\lambda_j = \lambda_i S_{ij} \quad \text{(no summation)}. \tag{2.4}$$

Since the λ_i are distinct, it follows that $S_{ij} = 0$ for $i \neq j$. Thus S is block diagonal with the block S_{ii} square of order k_i, and it is unrestricted by (2.4). Therefore, the complex dimension of the set of S_{ii} is k_i^2, and the complex dimension of the set of S is the sum of the k_i^2. We summarize this result as follows:

Lemma 3

$$\text{complex dim } \{S|SS^{-1} = I, \quad S\Lambda = \Lambda S, \quad \text{mult } \Lambda = (k_1, k_2, \ldots, k_I)\} = \sum_{i=1}^{I} k_i^2. \tag{2.5}$$

Now we use (2.5) *in* (2.2) *to get*

Theorem 1

$$\text{complex dim } \{A|A = T\Lambda T^{-1}, \quad \text{mult } \Lambda = (k_1, \ldots, k_I)\} = n^2 - \sum_{i=1}^{I} k_i^2. \tag{2.6}$$

This theorem gives the dimension of the set of diagonalizable A with specified eigenvalues and eigenvalue multiplicities. To obtain the dimension of the set of diagonalizable A with specified multiplicities, we add to the right side of (2.6) the dimension I of the set of eigenvalues:

Corollary 1.1

$$\begin{aligned}\text{complex dim } \{A|A \text{ diagonalizable}, \quad \text{mult}(k_1, \ldots, k_I)\} &= n^2 - \sum_{i=1}^{I} k_i^2 + I \\ &= n^2 - \sum_{i=1}^{I} (k_i^2 - 1).\end{aligned} \tag{2.7}$$

The complex codimension of the set of diagonalizable A with the specified eigenvalue multiplicities is just n^2, the dimension of C^{nn}, minus the dimension given in (2.7):

Corollary 1.2

$$\text{complex codim}\,\{A|A \text{ diagonalizable},\quad \text{mult}(k_1,\ldots,k_I)\} = \sum_{i=1}^{I}(k_i^2-1). \tag{2.8}$$

As an application of Corollary 1.2, we consider diagonalizable A with one k-fold eigenvalue and all others simple. Then $I = n-k+1$, $k_1 = k$, $k_2 = \cdots = k_{n-k+1} = 1$ and (2.8) yields

$$\text{complex codim}\,\{A|A \text{ diagonalizable},\quad \text{mult} = (k,1,\ldots,1)\} = k^2-1. \tag{2.9}$$

3. Normal matrices in C^{nn}

A matrix A in C^{nn} is normal if it commutes with its adjoint $A^* : AA^* = A^*A$. Every normal matrix is similar to a diagonal matrix Λ, $A = U\Lambda U^*$, where U is unitary ($U^{-1} = U^*$). Therefore A is diagonalizable, so Lemmas 1 and 2 apply with the additional condition that S is unitary. This condition follows from the definition $S = T_1^{-1}T = U_1^{-1}U$, since both $T = U$ and $T_1 = U_1$ are unitary. Eq. (2.4) and the consequence that S is block diagonal still apply, but now the block S_{ii} must be unitary. The real dimension of the set of unitary matrices of order k_i is k_i^2 so the real dimension of the set of unitary S that commute with Λ is the sum of the k_i^2. Thus instead of Lemma 3 we have

Lemma 4

$$\text{real dim}\{S|SS^* = I,\; S\Lambda = \Lambda S,\quad \text{mult}\,\Lambda = (k_1,k_2,\ldots,k_I)\} = \sum_{i=1}^{I}k_i^2. \tag{3.1}$$

We now use in (2.2) both (3.1) and the fact that $T = U$ is unitary. Since the real dimension of the set of U is n^2, we obtain

Theorem 2

$$\text{real dim}\,\{A|A = U\Lambda U^*,\quad \text{mult}\,\Lambda = (k_1,\ldots,k_I)\} = n^2 - \sum_{i=1}^{I}k_i^2. \tag{3.2}$$

Upon adding to (3.2) the real dimension of the set of Λ, which is $2I$, we get the dimension of the set of normal A in C^{nn} with specified multiplicities:

Corollary 2.1

$$\text{real dim}\,\{A|A \text{ normal},\quad \text{mult}\,(k_1,\ldots,k_I)\} = n^2 - \sum_{i=1}^{I}k_i^2 + 2I. \tag{3.3}$$

The dimension of the set of normal matrices is the maximum value of the right side of (3.3). This is achieved when $I = n$ and each $k_i = 1$, which gives

$$\text{real dim}\,\{A|A \text{ normal}\} = n^2 + n. \tag{3.4}$$

Upon subtracting the dimension in (3.3) from the real dimension (3.4) of the set of normal matrices, we get

Corollary 2.2

$$\text{real codim}\ \{A|A \text{ normal},\quad \text{mult } (k_1,\ldots,k_I)\} = n + \sum_{i=1}^{I} k_i^2 - 2I$$
$$= \sum_{i=1}^{I} (k_i - 1)(k_i + 2). \tag{3.5}$$

When A is normal, with one k-fold eigenvalue and all the others simple, (3.5) yields

$$\text{real codim}\ \{A|A \text{ normal},\quad \text{mult } (k, 1, \ldots, 1)\} = (k-1)(k+2). \tag{3.6}$$

4. Hermitian matrices

A matrix A in C^{nn} is Hermitian if it equals its adjoint, $A = A^*$, so A is also normal. Therefore the results of section 3 up to and including Theorem 2 apply to Hermitian matrices. The eigenvalues of an Hermitian matrix are real, so the dimension of the set of λ_i is I. Upon adding I to (3.2) we get the dimension of the set of Hermitian A with multiplicities $k_1, \ldots, k_I$:

Corollary 2.3

$$\text{real dim}\ \{A|A \text{ Hermitian},\quad \text{mult } (k_1, k_2, \ldots, k_I)\} = n^2 - \sum_{i=1}^{I} k_i^2 + I$$
$$= n^2 - \sum_{i=1}^{I} (k_i^2 - 1). \tag{4.1}$$

This result was given by von Neuman and Wigner [1].

Upon subtracting the dimension (4.1) from n^2, the dimension of the set of Hermitian matrices of order n, we get

Theorem 3. *In the set of Hermitian matrices of order n, the real codimension of the subset of those having I distinct eigenvalues with multiplicities $k_1, \ldots, k_I$, where $k_1 + \cdots + k_I = n$, is*

$$\text{real codim}\ \{A|A = A^*,\quad \text{mult}(k_1, \ldots, k_I)\} = \sum_{i=1}^{I} (k_i^2 - 1). \tag{4.2}$$

When $k_1 = k$ and the right side of (4.2) is minimized over the other k_i, the result is $k^2 - 1$. This is the number of real conditions on the elements of an Hermitian matrix for it to have a k-fold eigenvalue:

Corollary 3.1

$$\text{real codim}\ \{A|A \text{ Hermitian},\quad \text{mult } (k, 1, \ldots, 1)\} = k^2 - 1. \tag{4.3}$$

A matrix A in C^{nn} is skew-Hermitian if $A = -A^*$. Then iA is Hermitian, so the results of this section yield corresponding results for skew-Hermitian matrices.

5. Unitary matrices

A matrix A in C^{nn} is unitary if $AA^* = I$, so A is also normal. Therefore, the results of Section 3 through Theorem 2 apply to unitary matrices. Since the eigenvalues of a unitary matrix have absolute value one, the dimension of the set of λ_i is I. This is the same as the dimension of the set of λ_i for Hermitian matrices, so the dimension of the set of unitary A with multiplicities $k_1, \ldots, k_I$ is also given by the right side of (4.1):

Corollary 2.4

$$\text{real dim}\,\{A|A \text{ unitary}, \quad \text{mult}\,(k_1, \ldots, k_I)\} = n^2 - \sum_{i=1}^{I}(k_i^2 - 1). \tag{5.1}$$

When all $k_i = 1$, (5.1) yields n^2, which is the dimension of the set of unitary matrices. Therefore, the results for unitary matrices, analogous to (4.2) and (4.3), are

Theorem 4. *In the set of unitary matrices of order n, the real codimension of the subset of those having I distinct eigenvalues with multiplicities $k_1, \ldots, k_I$, where $k_1 + \cdots + k_I = n$, is*

$$\text{real codim}\,\{A|A \text{ unitary}, \quad \text{mult}\,(k_1, \ldots, k_I)\} = \sum_{i=1}^{I}(k_i^2 - 1). \tag{5.2}$$

The number of conditions on the elements of a unitary matrix for it to have a k-fold eigenvalue, with all other eigenvalues simple, is given by

Corollary 4.1

$$\text{real codim}\,\{A|A \text{ unitary}, \quad \text{mult}\,(k, 1, \ldots, 1)\} = k^2 - 1. \tag{5.3}$$

6. Real symmetric matrices

A matrix A is real symmetric if A is real and $A = A^{\mathrm{T}}$, where A^{T} is the transpose of A. Therefore, it is Hermitian, normal, and diagonalizable. Every real symmetric A is similar to a real diagonal matrix Λ, $A = O\Lambda O^{\mathrm{T}}$, where O is orthogonal, i.e., $O^{-1} = O^{\mathrm{T}}$. Consequently, Lemmas 1 and 2 apply with $T = O$ and $S = O_1^{-1}O$ both orthogonal. Eq. (2.4) holds so S is block diagonal with blocks S_{ii} which are orthogonal and of order k_i.

The dimension of the set of orthogonal matrices of order k_i is $k_i(k_i - 1)/2$. Thus the dimension of the set of orthogonal S that commute with Λ, when the multiplicity of Λ is $(k_1, k_2, \ldots, k_I)$, is

Lemma 5

$$\text{real dim}\,\{S|SS^{\mathrm{T}} = I,\ S\Lambda = \Lambda S, \quad \text{mult}\,\Lambda = (k_1, k_2, \ldots, k_I)\} = \sum_{i=1}^{I} k_i(k_i - 1)/2. \tag{6.1}$$

By using (6.1) in the result (2.2), we have

Theorem 5

$$\text{real dim}\,\{A|A = O\Lambda O^{\mathrm{T}},\quad \text{mult}\,\Lambda = (k_1, k_2, \ldots, k_I)\}$$
$$= n(n-1)/2 - \sum_{i=1}^{I} k_i(k_i - 1)/2. \tag{6.2}$$

Furthermore, the number of distinct eigenvalues in Λ is I. Therefore by adding I to the dimension in (6.2), we get the real dimension of the set of real symmetric A with multiplicities $k_1, k_2, \ldots, k_I$:

Corollary 5.1

$$\text{real dim}\,\{A|A \text{ real symm},\quad \text{mult}\,(k_1, k_2, \ldots, k_I)\}$$
$$= n(n-1)/2 + I - \sum_{i=1}^{I} k_i(k_i - 1)/2. \tag{6.3}$$

Next we subtract the dimension given in (6.3) from the dimension of the space of real symmetric matrices of order n, which is $n(n+1)/2$, and we state the result as

Corollary 5.2. *In the space of real symmetric matrices A of order n, the real codimension of the set of matrices having eigenvalue multiplicities $k_1, k_2, \ldots, k_I$ with $k_1 + k_2 + \cdots + k_I = n$, is*

$$\text{real codim}\,\{A|A \text{ real symm},\quad \text{mult}\,(k_1, k_2, \ldots, k_I)\}$$
$$= \frac{n(n+1)}{2} - \left[\frac{n(n-1)}{2} + I - \sum_{i=1}^{I}\frac{k_i(k_i-1)}{2}\right] = \frac{1}{2}\sum_{i=1}^{I}(k_i+2)(k_i-1). \tag{6.4}$$

This is the number of conditions which must be satisfied by the elements of a real symmetric nth order matrix in order for it to have eigenvalues with the specified multiplicities.

From (6.4) we can calculate the number of conditions which A must satisfy to have one eigenvalue of multiplicity k. We set $k_1 = k$ and minimize (6.4) with respect to the other k_i, which requires $k_2 = \cdots = k_I = 1$. Thus we get

Corollary 5.3

$$\text{real codim}\,\{A|A \text{ real symm},\quad \text{mult}\,(k, 1, \ldots, 1)\} = (k+2)(k-1)/2. \tag{6.5}$$

This codimension is independent of n. When $k = 2$ it yields the value 2, obtained by von Neuman and Wigner [1].

7. Jordan canonical form

We shall now use Lemma 2 to determine the dimension of the set of A in C^{nn} which have the Jordan canonical form J. To find the set of S which satisfy (2.1), we let J have the block diagonal form

$$J = (J^1, J^2, \ldots, J^K), \quad J^i = \lambda_i I^i + H^i,$$
$$I^i = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \quad H^i = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}. \tag{7.1}$$

Here J^i, I^i and H^i are square matrices of order k_i. We write $S=(S^{ij})$ where the block S^{ij} is k_i by k_j. Then $(SJ)^{ij}=S^{ij}J^j$ and $(JS)^{ij}=J^iS^{ij}$, so (2.1) becomes

$$S^{ij}J^j=J^iS^{ij}. \tag{7.2}$$

Upon using the definition (7.1) of J^i in (7.2) we get

$$S^{ij}\lambda_j+S^{ij}H^j=\lambda_iS^{ij}+H^iS^{ij}. \tag{7.3}$$

Next we use the definition of H^i in (7.3), and we write the st element of S^{ij} as S^{ij}_{st}. In this way we obtain

$$\begin{aligned} S^{ij}_{st}(\lambda_j-\lambda_i) &= -\sum_q S^{ij}_{sq}H^j_{qt}+\sum_r H^i_{sr}S^{ij}_{rt} = -\sum_q S^{ij}_{sq}\delta_{q,t-1}+\sum_r \delta_{s,r-1}S^{ij}_{rt} \\ &= -S^{ij}_{s,t-1}+S^{ij}_{s+1,t}. \end{aligned} \tag{7.4}$$

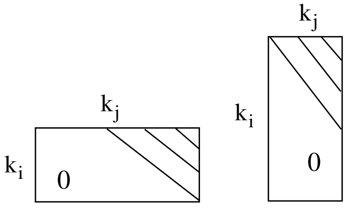
For $\lambda_i\neq\lambda_j$, (7.4) determines S^{ij}_{st} in terms of $S^{ij}_{s,t-1}$ and $S^{ij}_{s+1,t}$. Repeated use of (7.4) leads either to $t-1=0$ or to $s+1=k_i+1$. But $S^{ij}_{s0}=0$ and $S^{ij}_{k_i+1,t}=0$ so

$$S^{ij}_{st}=0 \quad \text{for } \lambda_i\neq\lambda_j. \tag{7.5}$$

If $\lambda_i=\lambda_j$ then (7.4) yields $S^{ij}_{s,t-1}=S^{ij}_{s+1,t}$, so S^{ij} is a Toeplitz matrix, i.e., it is constant on lines parallel to the main diagonal. The entries are zero where $t<s+[k_j-k_i]_+$:

$$S^{ij}_{st}=0 \quad \text{if } t<s+[k_j-k_i]_+ \quad \text{when } \lambda^i=\lambda^j. \tag{7.6}$$

Thus for $\lambda_i=\lambda_j$, S^{ij} has the first form in (7.7) for $k_i\leqslant k_j$ and the second form in (7.7) for $k_i\geqslant k_j$:



(7.7)

This form of the S^{ij} was found by Gantmacher [5, p. 221].

The complex dimension of the set of S^{ij} satisfying (7.6) is

$$\text{complex dim}\{S^{ij} \text{ satisfying (7.6)}\} = \begin{Bmatrix} k_j-[k_j-k_i]_+=k_i & \text{if } k_i\leqslant k_j \\ k_j-0=k_j & \text{if } k_i>k_j \end{Bmatrix} = \min(k_i,k_j). \tag{7.8}$$

Thus the dimension of the set of all S satisfying (2.1) is

$$\text{complex dim}\,\{S|SJ=JS\}=\sum_{i,j}\dim\,\{S^{ij}\}=\sum_{\substack{i,j\\ \lambda_i\neq\lambda_j}}0+\sum_{\substack{i,j\\ \lambda_i=\lambda_j}}\min(k_i,k_j). \tag{7.9}$$

We can write the sum in (7.9) more explicitly by specifying that there are p distinct eigenvalues λ_a, and that λ_a occurs in N_a blocks. We write their orders $k_{a1} \geqslant k_{a2} \geqslant \cdots \geqslant k_{aN_a}$, and set $k_{aj} = 0$ for $j > N_a$. Then $k_{a1} + \cdots + k_{aN_a} = n_a$, where n_a is the multiplicity of λ_a. Now we set $\lambda_1 = \lambda_j = \lambda_a$ in (7.9), replace k_i and k_j by k_{ai} and k_{aj}, and sum over a. Thus we get

$$\begin{aligned}\text{complex dim}\,\{S|SJ = JS\} &= \sum_{\substack{i,j\\ \lambda_i=\lambda_j}} \min(k_i, k_j)\\ &= \sum_{a=1}^{p}\sum_{i=1}^{Na}\sum_{j=1}^{Na} \min(k_{ai}, k_{aj}) = \sum_{a=1}^{p}\sum_{j=1}^{Na}(2j-1)k_{aj}\\ &= \sum_{j=1}^{N^*}(2j-1)\sum_{a=1}^{p} k_{aj}. \end{aligned} \tag{7.10}$$

Here $N^* = \max_a N_a$ is the maximum number of blocks with a given eigenvalue. The final interchange of the order of summation is valid because $k_{aj} = 0$ for $j > N_a$.

The sum of the k_{aj} in (7.10) is just the degree m_j of the jth invariant polynomial of J, and therefore also of any A which is similar to J. Therefore, (7.10) can be rewritten in the following form [5, p. 222, Theorem 2]:

$$\text{complex dim}\,\{S|SJ = JS\} = \sum_{j=1}^{N^*}(2j-1)m_j. \tag{7.11}$$

Upon using (7.10) and (7.11) in (2.2), and recalling that complexdim$\{T\} = n^2$, we have

Theorem 6. *The complex dimension of the set of $A \in C^{nn}$ with the Jordan form J is*

$$\text{complex dim}\,\{A|A=TJT^{-1}\} = n^2 - \sum_{j=1}^{N^*}(2j-1)\sum_{a=1}^{p} k_{aj} = n^2 - \sum_{j=1}^{N^*}(2j-1)m_j. \tag{7.12}$$

The complex dimension of the set of p distinct eigenvalues is p. Upon adding p to (7.12) we get the complex dimension of the set of A with the structure of J, but with any p distinct eigenvalues. We can state this result as follows:

Theorem 7. *Let $A \in C^{nn}$ have p distinct eigenvalues λ_a with multiplicities $n_a, a = 1, \ldots, p$ such that $n_1 + n_2 + \cdots + n_p = n$. Suppose that λ_a occurs in N_a blocks of the Jordan canonical form of A, with the blocks of orders $k_{a1} \geqslant k_{a2} \geqslant \cdots \geqslant k_{aN_a}$, with $k_{a1} + k_{a2} + \cdots + k_{aN_a} = n_a$. The complex dimension of the set of A with these properties, and any values of the λ_a, is*

$$\begin{aligned}\text{complex dim}\,\{A|n_a, N_a, k_{aj}\} &= n^2 - \sum_{j=1}^{N^*}(2j-1)\sum_{a=1}^{p} k_{aj} + p\\ &= n^2 - \sum_{j=1}^{N^*}(2j-1)m_j + p. \end{aligned} \tag{7.13}$$

When all the eigenvalues of A are simple, then $p = n$, $n_a = N_a = N^* = k_{aj} = 1$ and (7.13) yields complex $\dim\{A\} = n^2$. When A has one n-fold eigenvalue and just one block in its Jordan form, then $p = 1$, $n_1 = n$, $N^* = N_1 = 1$, $k_{11} = n$ and (7.13) yields

$$\text{complex dim}\ \{A|p = 1, n_1 = n, N_1 = 1, k_{11} = n\} = n^2 - n + 1. \tag{7.14}$$

When A has one n-fold eigenvalue and n blocks in its Jordan form, then $p = 1$, $n_1 = n$, $N^* = N_1 = n$, $k_{11} = 1$ and (7.13) yields

$$\text{complex dim}\ \{A|p = 1, n_1 = n, N_1 = n, k_{11} = 1\} = 1. \tag{7.15}$$

8. Singular value decomposition

Every $A \in R^{nm}$, the space of real n by m matrices, has a singular value decomposition

$$A = U\Sigma V^{\mathrm{T}}. \tag{8.1}$$

Here $U \in R^{nn}$ and $V \in R^{mm}$ are orthogonal. $\Sigma \in R^{nm}$ is diagonal with $r \leqslant \min(n, m)$ non-zero elements, on the main diagonal, starting at the upper left corner. They are all positive, and are called singular values of A. We seek the real dimension of the set of A with $J \leqslant r$ distinct singular values $\sigma_1 > \sigma_2 > \cdots > \sigma_J > 0$, with respective multiplicities $k_1, k_2, \ldots, k_J$. Then $r = k_1 + k_2 + \cdots + k_J$.

Suppose that in addition to (8.1), A can be written as $A = U_1\Sigma V_1^{\mathrm{T}}$ where U_1 and V_1 are also orthogonal. Then we equate these two expressions for A to get $U\Sigma V^{\mathrm{T}} = U_1\Sigma V_1^{\mathrm{T}}$. From this we conclude that $U_1^{\mathrm{T}}U\Sigma = \Sigma V_1^{\mathrm{T}}V$, which we rewrite as

$$Q\Sigma = \Sigma P. \tag{8.2}$$

Here $Q = U_1^{\mathrm{T}}U \in R^{nn}$ and $P = V_1^{\mathrm{T}}V \in R^{mm}$ are both orthogonal, being products of orthogonal factors. Then $U_1 = UQ^{\mathrm{T}}$ and $V_1 = VP^{\mathrm{T}}$ give the same A as do U and V when used in (8.1).

From (8.1) and this conclusion we obtain the following lemma:

Lemma 6. *The real dimension of the set of $A \in R^{nm}$ with the singular value matrix Σ is*

$$\begin{aligned}\dim\{A|A \in R^{nm},\quad A = U\Sigma V^{\mathrm{T}}\}\\ &= \dim\{U|U \in R^{nn},\quad UU^{\mathrm{T}} = I\} + \dim\{V|V \in R^{mm},\quad VV^{\mathrm{T}} = I\}\\ &-\dim\{Q, P|Q \in R^{nn},\quad QQ^{\mathrm{T}} = I,\quad P \in R^{mm},\quad PP^{\mathrm{T}} = I,\quad Q\Sigma = \Sigma P\}.\end{aligned} \tag{8.3}$$

The real dimensions of the sets of orthogonal U and V are $n(n-1)/2$ and $m(m-1)/2$ respectively. Therefore to use the lemma we shall determine the set of pairs Q, P and its dimension.

We begin by writing the elements Σ_{ij} of Σ in the form

$$\Sigma_{ii} = \sigma_i',\quad i = 1, \ldots, r;\quad \Sigma_{ij} = 0\quad \text{if } i \neq j\quad \text{or}\quad \text{if } i = j > r.$$

Here the first k_1 of the σ_i' are equal to σ_1, the next k_2 of the σ_i' equal σ_2, etc. Then we find that

$$(Q\Sigma)_{ik} = \begin{cases} Q_{ik}\sigma_k', & k \leqslant r\\ 0, & k > r\end{cases} \tag{8.4}$$

$$(\Sigma P)_{ik} = \begin{cases} \sigma_i' P_{ik}, & i \leqslant r\\ 0, & i > r.\end{cases} \tag{8.5}$$

Now we use (8.4) and (8.5) in (8.2) to find

$$Q_{ik}\sigma'_k = \sigma'_i P_{ik}, \quad i \leqslant r, \ \ k \leqslant r \tag{8.6a}$$

$$Q_{ik}\sigma'_k = 0, \quad i > r, \ \ k \leqslant r \tag{8.6b}$$

$$0 = \sigma'_i P_{ik}, \quad i \leqslant r, \ \ k > r. \tag{8.6c}$$

Since the $\sigma'_i \neq 0$, we obtain from (8.6)

$$Q_{ik} = \frac{\sigma'_i}{\sigma'_k} P_{ik}, \quad i, k \leqslant r \tag{8.7a}$$

$$Q_{ik} = 0, \quad i > r, \quad k \leqslant r \tag{8.7b}$$

$$P_{ik} = 0, \quad i \leqslant r, \quad k > r. \tag{8.7c}$$

From (8.7) and the orthogonality of Q and P, one can prove

Theorem 8. *The orthogonal matrices Q and P satisfying (8.2) are block diagonal, each with $J + 1$ square blocks $Q_1, \ldots, Q_{J+1}$ and $P_1, \ldots, P_{J+1}$. Each block is an orthogonal matrix. For $j = 1, \cdots, J$ the blocks Q_j and P_j are of order k_j and they are equal: $Q_j = P_j$, $j = 1, \ldots, J$. Q_{J+1} is of order $n - r$ and P_{J+1} is of order $m - r$.*

This theorem is given by Horn and Johnson [6, p. 147, (3.1.1′)].

From Theorem 8 we obtain

Corollary 8.1

$$\begin{aligned}&\dim\{Q, P | Q \in R^{nn}, \quad QQ^{\mathrm{T}} = I, \quad P \in R^{mm}, \quad PP^{\mathrm{T}} = I, \quad Q\Sigma = \Sigma P\}\\ &\quad = \sum_{j=1}^{J} \frac{k_j(k_j - 1)}{2} + \frac{(n-r)(n-r-1)}{2} + \frac{(m-r)(m-r-1)}{2}.\end{aligned} \tag{8.8}$$

We now use (8.8) in (8.3), and simplify the result to get

Theorem 9. *The real dimension of the set of $A \in R^{nm}$ with the singular values $\sigma_1, \ldots, \sigma_J$ having multiplicities $k_1, \ldots, k_J$ is*

$$\begin{aligned}&\dim\{A | A \in R^{nm}, \quad A = U\Sigma V^{\mathrm{T}}\}\\ &\quad = \frac{1}{2}n(n-1) + \frac{1}{2}m(m-1) - \frac{1}{2}(n-r)(n-r-1)\\ &\qquad -\frac{1}{2}(m-r)(m-r-1) - \frac{1}{2}\sum_{j=1}^{J} k_j(k_j-1)\\ &\quad = (n+m-r)r - r - \frac{1}{2}\sum_{j=1}^{J} k_j(k_j - 1).\end{aligned} \tag{8.9}$$

By adding to (8.9) the dimension J of the set of singular values, we get the dimension of the set of A having singular values with the specified multiplicities $k_1, \ldots, k_J$:

Corollary 9.1

$$\dim\{A|A \in R^{nm},\ \text{mult}(k_1, \ldots, k_J)\} = (n + m - r)r - r - \frac{1}{2}\sum_{j=1}^{J} k_j(k_j - 1) + J. \tag{8.10}$$

The codimension in R^{nm} is just nm minus the right side of (8.10).

As an example, if all the σ_j are simple then $J = r$ and each $k_j = 1$, so (8.10) yields

$$\dim\{A|A \in R^{nm}, \quad \text{mult}(1, \ldots, 1)\} = (n + m - r)r. \tag{8.11}$$

This is just the dimension of the set of $A \in R^{nm}$ of rank r. The codimension of the set of A in Corollary 9.1 with respect to this set of matrices is just the right side of (8.11) minus the right side of (8.10):

$$\text{codim (with respect to rank } r \text{ matrices) } \{A|A \in R^{nm},\ \text{mult}(k_1, \ldots, k_J)\}$$
$$= \frac{1}{2}\sum_{j=1}^{J} k_j(k_j - 1) + r - J. \tag{8.12}$$

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