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Note

An Elementary Proof of Moon's Theorem on Generalized Tournaments

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Let $X = \{1, 2, \ldots, n\}$ and let $T = (t_{ij}: i, j \in X)$ denote a *generalized tournament* on X; i.e., $0 \leqslant t_{ij} \leqslant 1$, and $t_{ij} + t_{ji} = 1$ for each $i \neq j$, and $t_{ii} = 0$ for each i. A real vector $S = (s_1, s_2, \ldots, s_n)$, $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n$, is a score vector if there exists a generalized tournament T on X such that $\sum_{j=1}^{n} t_{ij} = s_i$ for each i.

Extending Landau's well-known theorem, Moon [4] proved the following theorem.

THEOREM. *A real vector* $S = (s_1, s_2, \ldots, s_n)$, $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n$, *is a score vector if, and only if,*

$$\sum_{k=1}^{m} s_k \begin{cases} \geqslant \binom{m}{2} & \text{if } m < n, \\ = \binom{n}{2} & \text{if } m = n. \end{cases} \qquad (*)$$

Moon's proof is based upon Gale's feasibility theorem, which in turn rests on two major results: the minimum cut theorem of Ford and Fulkerson [2], and Hall's theorem on systems of distinct representatives. In this note Moon's theorem is proved by a direct application of Hall's theorem coupled with the following elementary convergence argument. If S is approximated by a componentwise convergent sequence $Q^{(1)}, Q^{(2)}, \ldots$ of *rational* vectors, each $Q^{(k)}$ being the score vector of a generalized tournament T_k, then because tournament entries are bounded there exists a subsequence converging to a tournament T having the desired score vector S. To complete the proof, therefore, we need only bridge two gaps: (1) construct an appropriate sequence $Q^{(1)}, Q^{(2)}, \ldots$ of rational vectors converging to S componentwise, and (2) prove that each $Q^{(k)}$ is a score vector.

Suppose first that $S = (s_1, s_2, \ldots, s_n)$ is a real nondecreasing vector satisfying (*) and having at least one irrational component. There exists a largest integer $m < n$ such that $s_m < s_{m+1}$. For each positive integer l, choose rational numbers $q_1, q_2, \ldots, q_m$ such that $q_1 \leqslant q_2 \leqslant \cdots \leqslant q_m$, and for $1 \leqslant i \leqslant m$

$$s_i < q_i \leqslant s_i + (s_{m+1} - s_m)/2ml.$$

If we put $q_{m+1} = \cdots = q_n = (\binom{n}{2} - \sum_{k=1}^{m} q_k)/(n-m)$, then a routine calculation confirms that $Q^{(l)} = (q_1, q_2, \ldots, q_n)$ is a rational nondecreasing vector satisfying (*). Clearly, $Q^{(l)}$ converges to S.

To build the second bridge, we assume that S is a nondecreasing rational vector satisfying (*). Let p be a positive integer so large that pS is a vector of nonnegative integers. Since pS may fail to satisfy (*), we cannot resort to Landau's theorem for (ordinary) tournaments. Instead, we use a technique [1] based upon the simple extension of Hall's theorem [3, Theorem 2.2], stated next.

LEMMA. *If $H = \{H_i : i \in B\}$ is a finite set of nonempty subsets of G, and if $P = \{1, 2, \ldots, p\}$, then there exists an injection $f : B \times P \to G$ satisfying $f(i, \cdot) \in H_i$ for each $i \in B$, if, ond only if,*

$$\left| \bigcup_{i \in A} H_i \right| \geqslant p \mid A \mid, \qquad \textit{for each } A \subseteq B.$$

For each $i \in X$, let G_i be a set for which $| G_i | = ps_i$. Assume that these sets are pairwise disjoint and that G is their union. Let $B = \{(i,j) : 1 \leqslant i < j \leqslant n\}$ and set $H = \{G_i \cup G_j : (i,j) \in B\}$. If $A \subseteq B$ and $Y = \{i : (i,k)$ or $(k,i) \in A$ for some $k\}$, then

$$\left| \bigcup \{G_i \cup G_j : (i,j) \in A\} \right| = p \sum_{i \in Y} s_i$$

$$\geqslant p \binom{|Y|}{2} \geqslant p \mid A \mid.$$

By the Lemma there exists an injection $f : B \times P \to G$ satisfying

$$f((i,j), \cdot) \in G_i \cup G_j \qquad \text{for each} \quad i < j.$$

Now define a generalized tournament T on X by $t_{ii} = 0$ for $i \in X$, and for each $i < j$

$$t_{ij} = (1/p) \left|\{k : f((i,j), k) \in G_i\}\right| \qquad \text{and} \qquad t_{ji} = 1 - t_{ij}.$$

By the hypothesis $|B \times P| = |G|$, hence f is a bijection and it follows that

$$\sum_j t_{ij} = (1/p) \sum_j |\{k : f((i,j), k) \in G_i\}| = (1/p) \, | G_i | = s_i \, .$$

Thus the necessary generalized tournaments exist, and the argument is complete.

References

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4. J. W. Moon, An extension of Landau's theorem on tournaments, *Pacific J. Math.* **13** (1963), 1343–1345.