NOTE

On a Crossing Number Result of Richter and Thomassen

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We show that if $G$ is a graph minimal with respect to having crossing number at least $k$, and $G$ has no vertices of degree 3, then $G$ has crossing number at most $2k + 35$.

Richter and Thomassen proved that if $G$ is minimal with respect to having crossing number at least $k$, then the crossing number $cr(G)$ of $G$ is at most $2.5k + 16$ [1]. Our aim in this note is to observe that if $G$ has no vertices of degree 3, then the proportionality constant in this bound can be improved to 2.

**Theorem 1.** Let $G$ be a graph minimal with respect to having crossing number at least $k$. Suppose that $G$ has no vertices of degree 3. Then $cr(G) \leq 2k + 35$.

**Proof.** Our proof is largely based on the proof of Theorem 3 in [1].

As proved in Theorem 3 in [1], we can assume $G$ is simple and has no vertices of degree 2. Since $G$ clearly cannot have vertices of degree 1, it follows that we can assume $G$ has minimum degree at least 4.

Let $t_0$ be least possible such that there is a set $E$ of $t_0$ edges such that $G - E$ is planar. Let $C$ be a cycle and $v$ a vertex as in Theorem 2 in [1]. Thus \( \sum_{u \in P(C) \setminus \{v\}} (d(u) - 2) \leq t_0 + 36 \). Let $e = vw$ be an edge of $C$ incident with $v$, and let $P$ be the path $C - e$. Since the minimum degree of $G$ is at least 4, it follows that there are at most $(t_0 + 36)/2$ edges in $P$.

By minimality of $G$, there is a drawing of $G - e$ with at most $k - 1$ crossings. In this drawing some edges in $P$ may cross other edges in $P$. We regard the drawing of $P$ as a planar graph $H$ with vertices of degrees 2 and 4. Let $P'$ be a path in $H$ joining $v$ and $w$.

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There are two ways of drawing $e$ close to $P$, one for each side of $P$. It is readily checked that the total number of crossings in these two drawings of $e$ is at most $(t_0 + 36) + 2\text{cr}(P)$, where $\text{cr}(P)$ is the number of crossings of edges of $P$ in the drawing of $G - e$.

Removing from $G - e$ the (at most $(t_0 + 36)/2$) edges of $P$ leaves a drawing with at most $k - 1 - \text{cr}(P)$ crossings. Therefore, there is a set of at most $1 + (t_0 + 36)/2 + (k - 1 - \text{cr}(P))$ edges whose removal from $G$ leaves a planar graph. By the definition of $t_0$, this implies that $\text{cr}(P) \leq k - t_0/2 + 18$.

Thus, the number of crossings in the two drawings of $e$ is at most $(t_0 + 36) + 2\text{cr}(P) \leq 2k + 72$, and so $e$ can be drawn with at most $k + 36$ crossings. Since $G - e$ is drawn with at most $k - 1$ crossings, it follows that $G$ can be drawn with at most $2k + 35$ crossings.

Theorem 1 implies the following versions of Corollaries 1 and 2 in [1].

**Corollary 1.** Let $G$ be a graph with crossing number $k$. Suppose that $G$ has no vertices of degree 3. Then there is an edge $e$ of $G$ such that $\text{cr}(G - e) \geq (k - 37)/2$.

**Corollary 2.** Let $G$ be a simple graph minimal with respect to having crossing number at least $k$. If all vertices of $G$ have degree at least 7, then $G$ has at most $4k + 58$ vertices.

**REFERENCE**