Asymptotic Behavior of a Nonhomogeneous Linear Recurrence System

Mihály Pituk

Department of Mathematics and Computing, University of Veszprém, P.O. Box 158, 8201 Veszprém, Hungary

Submitted by B. G. Pachpatte

Received March 28, 2000

Consider the nonhomogeneous linear recurrence system

\[ x_{n+1} = (A + B_n)x_n + g_n, \]

where \( A \) and \( B_n \) (\( n = 0, 1, \ldots \)) are square matrices and \( g_n \) (\( n = 0, 1, \ldots \)) are column vectors. In this paper, we describe, in terms of the initial condition, the asymptotic behavior of the solutions of this equation in the case when \( A \) has a simple dominant eigenvalue \( \lambda_0 \) and \( \sum_{n=0}^\infty \| B_n \| < \infty \) and \( \sum_{n=0}^\infty |\lambda_0|^{-n} \| g_n \| < \infty \). The proof is based on the duality between the solutions of the above equation and the solutions of the associated adjoint equation. As a consequence, we obtain a similar result for higher order scalar equations.

1. INTRODUCTION

Consider the linear autonomous recurrence system

\[ x_{n+1} = Ax_n \]  

and the perturbed equation

\[ x_{n+1} = (A + B_n)x_n + g_n, \]

where the coefficients \( A \) and \( B_n \) (\( n = 0, 1, \ldots \)) are \( k \times k \) matrices with complex entries and \( g_n \) (\( n = 0, 1, \ldots \)) are \( k \)-dimensional complex column vectors.

1 Research supported in part by Hungarian National Foundation for Scientific Research Grants F 023772 and T 31935.
Let \( \| \cdot \| \) denote any norm of a vector or the associated induced norm of a square matrix.

Throughout the paper, we shall assume the following hypothesis.

(H) The constant matrix \( A \) has a simple dominant eigenvalue \( \lambda_0 \).

(An eigenvalue \( \lambda_0 \) of \( A \) is said to be dominant if any other eigenvalue \( \lambda \) of \( A \) satisfies \( |\lambda| < |\lambda_0| \).)

It follows from the theory of linear autonomous difference equations [1, 9, 17] that under the above hypothesis for every solution \( \{x_n\}_{n=n_0}^\infty \) of the unperturbed equation (1.1) there exists a constant \( \alpha \) such that
\[
\lim_{n \to \infty} (\lambda_0^{-n} x_n) = \alpha \xi, \tag{1.3}
\]
where \( \xi \) is an eigenvector of \( A \) corresponding to the eigenvalue \( \lambda_0 \). Note that the power method in numerical analysis (an iterative procedure for approximating the dominant eigenvalue and eigenvector of a constant matrix) is based on this observation; see, e.g., [26, Sect. 10.2]. (For a similar qualitative result concerning nonlinear equations, see [19, 20] and the references therein.)

In this paper, among others, we shall show that if
\[
\sum_{n=0}^\infty \|B_n\| < \infty \tag{1.4}
\]
and
\[
\sum_{n=0}^\infty |\lambda_0|^{-n} \|g_n\| < \infty, \tag{1.5}
\]
then the same conclusion is true for the solutions of the perturbed equation (1.2). Since Eq. (1.3) gives a genuine asymptotic characterization of \( \{x_n\}_{n=n_0}^\infty \) provided \( \alpha \neq 0 \), it is important to be able to compute the value of the constant \( \alpha \) explicitly. In our main theorem (see Theorem 3 in Section 2) we prove that the constant \( \alpha \) in the asymptotic relation (1.3) can be characterized by a certain special solution of the associated adjoint equation and the initial value of a given solution \( \{x_n\}_{n=n_0}^\infty \) of Eq. (1.2). The corresponding formula for \( \alpha \) shows that if, in addition to the above hypotheses, we assume that \( n_0 \) is sufficiently large, then for “practically all” initial values Eq. (1.3) gives a genuine asymptotic representation of \( \{x_n\}_{n=n_0}^\infty \). The proof of our main theorem will be based on the duality between the solutions of (1.2) and the solutions of the adjoint equation. In Section 3, we prove a similar asymptotic result for higher order scalar equations.

2. MAIN RESULTS

**Theorem 1.** Suppose conditions (H), (1.4), and (1.5) hold. Then for every solution \( \{x_n\}_{n=n_0}^\infty \) of Eq. (1.2) there exists a constant \( \alpha \) such that (1.3) holds.
Proof. We shall prove Theorem 1 in two steps. First we give a proof in the case when \( \lambda_0 = 1 \) and then we show that the general case can be reduced to the previous one.

Step 1: By a well-known result from linear algebra, \( \mathbb{C}^k \) has a basis consisting of generalized eigenvectors of \( A \). Consequently, hypothesis (H) implies that \( \mathbb{C}^k \) can be decomposed into a direct sum, \( \mathbb{C}^k = E \oplus S \), where \( E = \text{span}\{\xi\} \) (the one-dimensional subspace spanned by vector \( \xi \)) and \( S \) is the stable subspace of \( A \) (the linear subspace of \( \mathbb{C}^k \) spanned by the generalized eigenvectors of \( A \) corresponding to the eigenvalues having moduli less than one). The projections of \( \mathbb{C}^k \) onto \( E \) and \( S \) are represented by \( k \times k \) matrices \( P \) and \( Q \), respectively, which satisfy the relations

\[
P^2 = P, \quad Q^2 = Q, \quad P + Q = I, \quad PQ = 0.
\]

Since \( E \) and \( S \) are invariant subspaces of \( A \), the projections \( P \) and \( Q \) commute with \( A \) (see [16, Sect. 43, Theorem 2]),

\[
AP = PA, \quad AQ = QA.
\]

From the fact that, for every \( x \in \mathbb{C}^k \), \( Px \) is a constant multiple of \( \xi \) and \( A\xi = \xi \), it follows that \( AP = P \) and hence

\[
A^n P = P \quad \text{for } n = 0, 1, \ldots
\]

By the stable subspace theorem (see [17, Theorem 4.7 and its proof]), the solutions starting from \( S \) tend to zero exponentially. More precisely, there exist constants \( K > 0 \) and \( 0 < \rho < 1 \) such that

\[
\|A^n x\| \leq K \rho^n \|x\| \quad \text{for } x \in S \text{ and } n = 0, 1, \ldots
\]

The last relation can be written equivalently as

\[
\|A^n Qx\| \leq K \rho^n \|Qx\| \quad \text{for every } x \in \mathbb{C}^k \text{ and } n = 0, 1, \ldots
\]

From this, taking into account that \( \|Qx\| \leq \|Q\| \|x\| \) for \( x \in \mathbb{C}^k \), we obtain

\[
\|A^n Qx\| \leq L \rho^n \|x\| \quad \text{for } x \in \mathbb{C}^k \text{ and } n = 0, 1, \ldots
\]

where \( L = K \|Q\| \). Hence

\[
\|A^n Q\| = \sup_{\|x\|=1} \|A^n Qx\| \leq L \rho^n \quad \text{for } n = 0, 1, \ldots
\]
Let \( \{x_n\}_{n=n_0}^\infty \) be a solution of the perturbed equation (1.2). By the variation-of-constants formula (see [1, p. 68] or [17, Theorem 4.3]), we have for \( n \geq n_0 \),

\[
x_n = A^{n-n_0}x_{n_0} + \sum_{i=n_0}^{n-1} A^{n-i-1}(B_i x_i + g_i).
\]  

(2.6)

By virtue of (2.1), (2.3), and (2.5), we have for \( n \geq 0 \),

\[
\|A^n\| = \|A^n P + A^n Q\| \leq \|A^n P\| + \|A^n Q\| \leq \|P\| + L \varrho^n.
\]

Since \( \varrho < 1 \), this implies

\[
\|A^n\| \leq M \quad \text{for } n = 0, 1, \ldots, \quad (2.7)
\]

where \( M = \|P\| + L \). From (2.6), using (1.5) (with \( \lambda_0 = 1 \)) and (2.7), we find for \( n \geq n_0 \),

\[
\|x_n\| \leq N + M \sum_{i=n_0}^{n-1} \|B_i\| \|x_i\|,
\]

where \( N = M(\|x_{n_0}\| + \sum_{i=n_0}^\infty \|g_i\|) \). From this, by the discrete Gronwall inequality [1, Corollary 4.1.2], we obtain for \( n \geq n_0 \),

\[
\|x_n\| \leq N \prod_{i=n_0}^{n-1} (1 + M \|B_i\|) \leq N \exp \left( M \sum_{i=n_0}^{n-1} \|B_i\| \right).
\]

This, together with (1.4), implies that the solution \( \{x_n\}_{n=n_0}^\infty \) is bounded,

\[
\|x_n\| \leq C \quad \text{for } n \geq n_0, \quad (2.8)
\]

where \( C = N \exp(M \sum_{i=n_0}^\infty \|B_i\|) \).

Multiplying Eq. (2.6) by \( P \) from the left and using (2.2) and (2.3), we find for \( n > n_0 \),

\[
Px_n = P x_{n_0} + P \sum_{i=n_0}^{n-1} z_i, \quad (2.9)
\]

where

\[
z_n = B_n x_n + g_n \quad \text{for } n \geq n_0. \quad (2.10)
\]

By virtue of (1.4), (1.5), and (2.8), we have

\[
\sum_{n=n_0}^{\infty} \|z_n\| \leq C \sum_{n=n_0}^{\infty} \|B_n\| + \sum_{n=n_0}^{\infty} \|g_n\| < \infty. \quad (2.11)
\]
This, together with (2.9), implies
\[
\lim_{n \to \infty} Px_n = P u, \tag{2.12}
\]
where \( u = x_{n_0} + \sum_{i=n_0}^{\infty} z_i. \)

Applying the projection \( Q \) in the variation-of-constants formula (2.6) and using (2.2), we find for \( n \geq n_0, \)
\[
Qx_n = A^{n-n_0} Qx_{n_0} + \sum_{i=n_0}^{n-1} A^{n-1-i} Qz_i,
\]
where \( z_n \) is given by (2.10). From this, using estimate (2.4), we have for \( n \geq n_0, \)
\[
\|Qx_n\| \leq L q^{n-n_0} \|x_{n_0}\| + L \varphi_n, \tag{2.13}
\]
where
\[
\varphi_n = \sum_{i=n_0}^{n-1} q^{n-1-i} \|z_i\| \quad \text{for} \quad n \geq n_0.
\]
Since \( 0 < q < 1 \) and \( \|z_n\| \to 0 \) as \( n \to \infty \) (cf. (2.11)), [11, Lemma 3.1] implies that \( \varphi_n \to 0 \) as \( n \to \infty. \) (For a different argument, see [23, Lemma 3.1].) Consequently, (2.13) yields
\[
\lim_{n \to \infty} Qx_n = 0. \tag{2.14}
\]

By virtue of (2.1), the limit relations (2.12) and (2.14) imply that
\[
\lim_{n \to \infty} x_n = \lim_{n \to \infty} (Px_n + Qx_n) = Pu.
\]
Thus, \( \lim_{n \to \infty} x_n \) is a constant multiple of \( \xi, \) which completes the proof in the case when \( \lambda_0 = 1. \)

Step 2: Assume that \( \lambda_0 \) is an arbitrary simple dominant eigenvalue of \( A. \) Let \( \{x_n\}_{n=n_0}^{\infty} \) be a solution of (1.2). Define
\[
u_n = \lambda_0^{-n} x_n \quad \text{for} \quad n \geq n_0.
\]
Then \( \{u_n\}_{n=n_0}^{\infty} \) is a solution of the equation
\[
u_{n+1} = (C + D_n)u_n + h_n,
\]
where \( C = \lambda_0^{-1} A, \ D_n = \lambda_0^{-1} B_n, \) and \( h_n = \lambda_0^{-n} g_n \) for \( n \geq 0. \) Clearly, if \( \lambda_0 \) is a simple dominant eigenvalue of \( A \) with a corresponding eigenvector \( \xi, \) then \( \mu_0 = 1 \) is a simple dominant eigenvalue of \( C \) with the same eigenvector \( \xi. \) Furthermore, assumptions (1.4) and (1.5) imply that \( \sum_{n=0}^{\infty} \|D_n\| < \infty \) and \( \sum_{n=0}^{\infty} \|h_n\| < \infty. \) According to the previous part of the proof (Step 1),
\[
\lim_{n \to \infty} u_n = \lim_{n \to \infty} (\lambda_0^{-n} x_n) = \alpha \xi
\]
for some constant \( \alpha. \) \( \Box \)
To compute the constant $\alpha$ in the asymptotic relation (1.3) explicitly, we need to construct a special solution of the adjoint equation associated with the homogeneous part of Eq. (1.2).

**Theorem 2.** Suppose conditions (H) and (1.4) hold. Let $\mathbf{y}$ be an eigenvector of the adjoint matrix $A^*$ corresponding to the eigenvalue $\lambda_0$. Then the adjoint equation

$$y_n = (A^* + B^*_n)y_{n+1}$$

has a solution $\{y_n\}_{n=0}^{\infty}$ such that

$$\lim_{n \to \infty} (\lambda_0^{-n}y_n) = \mathbf{y}.$$  

**Proof.** Since all norms on $\mathbb{C}^k$ are equivalent, we may consider the Euclidean norm.

Let $n_1$ be a nonnegative integer. Denote by $Z$ the linear space of those vector sequences $y = \{y_n\}_{n=n_1}^{\infty}$, $y_n \in \mathbb{C}^k$ for $n \geq n_1$, for which

$$\|y\|_Z \overset{\text{def}}{=} \sup_{n \geq n_1} |\lambda_0|^{-n}\|y_n\| < \infty.$$  

($Z, \|\cdot\|_Z$) is a Banach space.

For $y = \{y_n\}_{n=n_1}^{\infty} \in Z$, define a vector sequence $Ky = \{Ky_n\}_{n=n_1}^{\infty}$ by

$$Ky_n = \sum_{i=n}^{\infty} (A^*)^{-i-n}B^*_iy_{i+1}$$

for $n \geq n_1$. (2.18)

Since the eigenvalues of $A^*$ are the conjugate pairs of the eigenvalues of $A$, hypothesis (H) implies that $\mu_0 = 1$ is a simple dominant eigenvalue of the matrix $\sum_{i=0}^{\infty} (A^*)^{-i-n}B^*_iy_{i+1}$. The same argument as in the proof of Theorem 1 (see also [17, Theorem 4.6 and its proof]) shows that there exists a constant $M > 0$ such that for each $n \geq 0$,

$$\|\sum_{i=n}^{\infty} (A^*)^{-i-n}B^*_iy_{i+1}\| \leq M.$$  

From (2.18), using the last estimate, we find for $n \geq n_1$,

$$\|Ky_n\| \leq |\lambda_0|^{-n}\|\mathbf{y}\| + \sum_{i=n}^{\infty} M|\lambda_0|^{-i-n}\|B^*_iy_{i+1}\|$$

$$\leq |\lambda_0|^{-n}\|\mathbf{y}\| + |\lambda_0|^{-n-1}M\|y\|_Z \sum_{i=n}^{\infty} \|B_i\|,$$

$^2$According to the standard notation, $A^*$ and $\sum_{i=0}^{\infty} (A^*)^{-i-n}B^*_iy_{i+1}$ denote the conjugate transpose of $A$ and the conjugate of $\lambda_0$, respectively.
the last inequality being a consequence of (2.17) and the fact that $B_i$ and $B_i^*$ have the same norm. Consequently, $K_y$ is well defined and

$$
\|K_y\|_Z = \sup_{n \geq n_1} |\lambda_0|^n \|K_n y\| \leq \|\eta\| + |\lambda_0|^{-1} M \|y\|_Z \sum_{i=n_1}^{\infty} \|B_i\|. 
$$

(2.19)

Thus, $K$ maps $Z$ into itself.

Let $y, z \in Z$. By similar estimates as in the proof of (2.19), we obtain

$$
\|K_y - K_z\|_Z \leq |\lambda_0|^{-1} M \|y - z\|_Z \sum_{i=n_1}^{\infty} \|B_i\|. 
$$

(2.20)

Choose $n_1$ such that

$$
q = |\lambda_0|^{-1} M \sum_{i=n_1}^{\infty} \|B_i\| < 1. 
$$

(2.21)

(Such an integer certainly exists). Then $K: Z \rightarrow Z$ is a contraction mapping.

It is easily seen that the unique fixed point $y = \{y_n\}_{n=n_1}^{\infty} \in Z$ is a solution of (2.15) satisfying (2.16). The solution $\{y_n\}_{n=n_1}^{\infty}$ can be extended to all nonnegative $n < n_1$ by Eq. (2.15).

We are now in a position to state our main theorem, which provides an explicit representation of the constant $\alpha$ in Theorem 1.

**Theorem 3.** Suppose conditions (H), (1.4), and (1.5) hold. Let $\{y_n\}_{n=0}^{\infty}$ be a solution of the adjoint equation (2.15) satisfying (2.16). If $\{x_n\}_{n=n_0}^{\infty}$ is the solution of Eq. (1.2) with initial vector

$$
x_{n_0} = v, 
$$

(2.22)

then (1.3) holds with

$$
\alpha = \left( y_{n_0}^* v + \sum_{n=n_0}^{\infty} y_{n+1}^* g_n \right) / (\eta^* \xi), 
$$

(2.23)

the last series being absolutely convergent.

Several remarks are appropriate at this point.

**Remark 1.** As shown in [24, Remark 1], the constant $\eta^* \xi$ in (2.23) is nonzero.

**Remark 2.** Hypothesis (H) is satisfied if, for example, $A$ is a nonnegative irreducible matrix. By the Perron–Frobenius theorem (see [5]) in that case $A$ has a simple positive dominant eigenvalue $\lambda_0$ and there is a corresponding eigenvector $\xi > 0$.
**Remark 3.** Let us mention some previous results which are relevant to our study. Several authors have considered the homogeneous equation

\[ x_{n+1} = (A + B_n)x_n, \]  

(2.24)

where \( \|B_n\| \to 0 \) as \( n \to \infty \) (see [2, 4, 6, 11, 12, 14, 21, 22, 24, 25, 28]). In most of these papers, the authors, assuming some additional assumptions on \( A \) and \( B_n \), established the existence of certain special solutions of Eq. (2.24) associated with the characteristic values of the unperturbed equation (1.1). (The recent papers by Elaydi [10, 11] contain an interesting survey and the latest results along these lines.) For example, Li [21] has proved that if the eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_k \) of \( A \) are nonzero and distinct, then assumption (1.4) implies that, for each \( i = 1, 2, \ldots, k \), Eq. (2.24) has a solution \( \{x_n\}_{n=n_0}^\infty \) such that

\[ \lim_{n \to \infty} (\lambda_i^{-n}x_n) = \xi_i, \]  

(2.25)

where \( \xi_i \) is an eigenvector of \( A \) corresponding to \( \lambda_i \) (see also [9, Theorem 8.25]). Note that the conclusion of Theorem 2 is the same (compare (2.16) and (2.25)) in the case of the dominant eigenvalue of the adjoint matrix \( A^* \). (Of course, Li's result does not apply to Eq. (2.15), since the latter equation is of advanced type.) Theorem 3 is different from the above results in the sense that it gives an asymptotic formula for all solutions of (2.24) and of the more general nonhomogeneous equation (1.2).

**Remark 4.** Applying Theorem 3 to the homogeneous equation (2.24), we conclude that assuming (H) and (1.4), Eq. (1.3) gives a genuine asymptotic characterization \( \alpha \neq 0 \) of the solution \( \{x_n\}_{n=n_0}^\infty \) of (2.24) with initial value (2.22) except the case when

\[ y_{n_0}^y v = 0, \]  

(2.26)

i.e., when \( v \) is orthogonal to \( y_{n_0} \). Since, for \( n_0 \) sufficiently large, \( y_{n_0} \neq 0 \) (cf. (2.16)), a randomly chosen initial vector \( v \) will practically never satisfy (2.26). So, assuming H, (1.4), and \( n_0 \) to be sufficiently large, Eq. (1.3) almost always provides a genuine asymptotic representation of the solution \( \{x_n\}_{n=n_0}^\infty \) of (2.24). A similar remark holds for the nonhomogeneous equation (1.2). The same phenomenon has been observed by Driver et al. [8] in the case of a special system of higher order difference equations with constant coefficients.

**Remark 5.** Assumption (1.4) is important for the validity of Theorem 3. [24, Theorem 2] shows that if in Theorem 3 condition (1.4) is replaced with the weaker assumption \( \sum_{n=0}^\infty \|B_n\|^2 < \infty \), then the asymptotic formula (1.3) is no longer valid for the solutions of the homogeneous equation (2.24).
The proof of Theorem 3 will be based on the following lemma.

**Lemma.** Let \( \{x_n\}_{n=n_0}^{\infty} \) and \( \{y_n\}_{n=n_0}^{\infty} \) be solutions of Eq. (1.2) and the adjoint equation (2.15), respectively. Then

\[
y^*_n x_n = y^*_n x_{n_0} + \sum_{i=n_1}^{n-1} y^*_i + g_i \quad \text{for } n \geq n_0.
\]

(2.27)

**Proof.** Let \( \langle \cdot, \cdot \rangle \) denote the standard inner product in \( \mathbb{C}^k \). Using Eqs. (1.2) and (2.15) and the well-known property of the adjoint operator, we find for \( n \geq n_0 \),

\[
y^*_n x_n = \langle x_n, y_n \rangle = \langle x_n, (A^* + B_n^*) y_{n+1} \rangle = \langle x_n, (A + B_n) y_{n+1} \rangle
\]

\[
= \langle (A + B_n) x_n, y_{n+1} \rangle = \langle x_{n+1} - g_n, y_{n+1} \rangle = y^*_n x_{n+1} - y^*_n x_{n_0}.
\]

Hence

\[
y^*_n x_{n+1} = y^*_n x_n + y^*_n x_{n_0} \quad \text{for } n \geq n_0,
\]

which implies (2.27). \( \square \)

**Proof of Theorem 3.** By Theorem 1, there exists a constant \( \alpha \) such that (1.3) holds. According to the previous lemma, for \( n \geq n_0 \),

\[
y^*_n x_n = y^*_n v + \sum_{i=n_0}^{n-1} y^*_i + g_i.
\]

(2.28)

From (2.16), it follows that the sequence \( \{\lambda_0^{-n} y_n\}_{n=0}^{\infty} \) is bounded; i.e., there exists a constant \( K > 0 \) such that, using the Euclidean norm on \( \mathbb{C}^k \),

\[
\|\lambda_0^{-n} y_n\| = |\lambda_0|^{-n} \|y_n\| \leq K \quad \text{for } n = 0, 1, \ldots.
\]

This, together with (1.5), implies

\[
\sum_{i=n_0}^{\infty} |y^*_i + g_i| \leq \sum_{i=n_0}^{\infty} \|g_i\| \|y_{i+1}\| \leq K |\lambda_0|^{-1} \sum_{i=n_0}^{\infty} |\lambda_0|^{-i} \|g_i\| < \infty,
\]

where the first inequality is a consequence of the Schwarz inequality. Thus, the series \( \sum_{i=n_0}^{\infty} y^*_i + g_i \) is absolutely convergent.

Letting \( n \to \infty \) in (2.28) and taking into account that, in view of (1.3) and (2.16),

\[
y^*_n x_n = (\lambda_0^{-n} y_n) \langle \lambda_0^{-n} y_n \rangle \to \alpha \eta^* \xi \quad \text{as } n \to \infty,
\]

we obtain

\[
\alpha \eta^* \xi = y^*_n v + \sum_{i=n_0}^{\infty} y^*_i + g_i.
\]

Since \( \eta^* \xi \neq 0 \) (see Remark 1), this implies (2.23). \( \square \)

\( ^3 \langle x, y \rangle = y^* x \) for \( x, y \in \mathbb{C}^k \).
If in Eq. (2.15) each \( B_n = 0 \), then the special solution \( \{y_n\}_{n=0}^\infty \) of the adjoint equation (2.15) satisfying the asymptotic relation (2.16) is given by \( y_n = \overline{\lambda}^{-n} \eta \) for \( n = 0, 1, \ldots \). Thus, Theorem 3 yields

**Corollary 1.** For the equation

\[
x_{n+1} = Ax_n + g_n,
\]

suppose conditions (H) and (1.5) hold. If \( \{x_n\}_{n=n_0}^\infty \) is the solution of (2.29) with initial condition (2.22), then (1.3) holds with \( y_n \) defined by

\[
y_n = \overline{\lambda}^{-n} \eta + \sum_{n=n_0}^{\infty} \overline{\lambda}^{-n-1} \eta^* g_n.
\]

For the equation

\[
y_{n+1} = \overline{\lambda}^{-n} \eta + \sum_{n=n_0}^{\infty} \overline{\lambda}^{-n-1} \eta^* g_n
\]

of operator \( K \) from the proof of Theorem 2, combined with formula (2.23), yields a possible approximation of the constant \( \alpha \) corresponding to a given solution \( \{x_n\}_{n=n_0}^\infty \) of Eq. (1.2).

Choose \( n_1 \geq n_0 \) so large that condition (2.21) is satisfied and consider the successive approximations of the unique fixed point \( y = \{y_n\}_{n=n_0}^\infty \) of operator \( K \) from the proof of Theorem 2.

Define \( y_0^{[0]} = 0 \) and \( y_{\nu+1}^{[\nu]} = K y^{[\nu]} \) for \( \nu = 0, 1, \ldots \). That is, using the notation \( y^{[\nu]} = \{y_n^{[\nu]}\}_{n=n_0}^\infty \) for \( \nu = 0, 1, \ldots \),

\[
y_0^{[0]} = 0,
\]

\[
y_n^{[\nu+1]} = \overline{\lambda}^{-n} \eta + \sum_{i=n}^{\infty} (A^*)^{i-n} B_i y_{i+1}^{[\nu]}
\]

for \( n \geq n_1 \) and \( \nu = 0, 1, \ldots \). From (2.20) and (2.21), it follows by easy induction on \( \nu \) that

\[
\|y^{[\nu+1]} - y^{[\nu]}\|_Z \leq \|\eta\| q^{\nu} \quad \text{for} \quad \nu = 0, 1, \ldots
\]

By simple computations,

\[
\|y - y^{[\nu]}\|_Z \leq \sum_{\mu=0}^{\infty} \|y^{[\mu+1]} - y^{[\mu]}\|_Z \leq \|\eta\| (1 - q^{-1}) q^{\nu} \quad \text{for} \quad \nu = 0, 1, \ldots
\]

Consequently,

\[
\|y_n - y_n^{[\nu]}\| \leq \|\eta\| (1 - q^{-1}) q^n |\lambda|^{-n} \quad \text{for} \quad n \geq n_1 \text{ and } \nu = 0, 1, \ldots
\]

By Theorem 3,

\[
\alpha = \left( \frac{y_{n_1}^* x_{n_1} + \sum_{n=n_1}^{\infty} y_{n+1}^* g_n}{(\eta^* \xi)} \right).
\]
For $\nu \geq 0$, define

$$\alpha_\nu = \left( (y^{[\nu]}_{n_1})^* x_{n_1} + \sum_{n=n_1}^{\infty} (y^{[\nu]}_{n+1})^* g_n \right) \left( \eta^* \xi \right).$$

Then, using estimate (2.30) and considering the Euclidean norm on $\mathbb{C}^k$, we obtain

$$|\alpha - \alpha_\nu| \leq C q^\nu \quad \text{for} \quad \nu = 0, 1, \ldots,$$

where the constant $C$ is given by

$$C = \|\eta\| (1 - q)^{-1} \|\eta^* \xi\|^{-1} \left( |\lambda_0|^{-n_1} \|x_{n_1}\| + \sum_{n=n_1}^{\infty} |\lambda_0|^{-n_1} \|g_n\| \right).$$

Since $q < 1$, $\alpha_\nu$ can be used for the approximation of $\alpha$.

### 3. HIGHER ORDER SCALAR EQUATIONS

In this section, we shall apply our results to the scalar equation of order $k + 1$

$$u(n + 1) = \sum_{j=0}^{k} (c_j + d_j(n)) u(n - j) + h(n), \quad (3.1)$$

where $c_j \ (0 \leq j \leq k)$ are complex constants $\{d_j(n)\}_{n=0}^{\infty} \ (0 \leq j \leq k)$ and $\{h(n)\}_{n=0}^{\infty}$ are complex sequences. We shall consider this equation as a perturbation of the equation with constant coefficients

$$u(n + 1) = \sum_{j=0}^{k} c_j u(n - j). \quad (3.2)$$

Rewriting Eq. (3.1) in the form as given in (1.2) and applying Theorems 2 and 3, we shall prove the following theorem.

**Theorem 4.** Suppose that the characteristic equation for the unperturbed equation (3.2),

$$f(\lambda) = 0, \quad f(\lambda) = \lambda - \sum_{j=0}^{k} c_j \lambda^{-j} \quad (3.3)$$

has a simple dominant root $\lambda_0$ and for each $j = 0, 1, \ldots, k$,

$$\sum_{n=0}^{\infty} |d_j(n)| < \infty. \quad (3.4)$$
Then the adjoint equation
\[ w(n) = \sum_{j=0}^{k} (c_j + d_j(n+j))w(n+j+1) \]  \hspace{1cm} (3.5)
has a solution \( \{w(n)\}_{n=n_0}^{\infty} \) such that
\[ \lim_{n \to \infty} [\lambda_0^n w(n)] = 1. \] \hspace{1cm} (3.6)

In addition to the above hypotheses, suppose that
\[ \sum_{n=0}^{\infty} |\lambda_0|^{-n}|h(n)| < \infty. \] \hspace{1cm} (3.7)
If \( \{u(n)\}_{n=n_0-k}^{\infty} \) is the solution of Eq. (3.1) with initial conditions
\[ u(n) = \phi(n) \quad \text{for} \quad n = n_0 - k, n_0 - k + 1, \ldots, n_0, \] \hspace{1cm} (3.8)
then \( \lim_{n \to \infty} [\lambda_0^{-n}u(n)] \) is finite and its value is given by
\[ \lim_{n \to \infty} [\lambda_0^{-n}u(n)] = \frac{1}{f'(\lambda_0)} \left( \phi(n_0)w(n_0) + \sum_{i=n_0-k}^{n_0-1} \phi(i) \sum_{j=n_0-i}^{k} (c_j + d_j(i+j)) \right) \times w(i+j+1) + \sum_{n=n_0}^{\infty} w(n+1)h(n). \] \hspace{1cm} (3.9)

Proof. For \( n \geq 0 \), define the column vector \( x_n = (x_{0n}, x_{1n}, \ldots, x_{kn})^T \) by
\[ x_{in} = u(n - i), \quad 0 \leq i \leq k. \] \hspace{1cm} (3.10)

Clearly, the system of recurrence equations
\[ x_{0,n+1} = \sum_{j=0}^{k} (c_j + d_j(n))x_{jn} + h(n), \]
\[ x_{i,n+1} = x_{i-1,n} \quad (1 \leq i \leq k) \]
is equivalent to Eq. (3.1). These recurrence equations can be written in the form as given in (1.2). The \((k+1) \times (k+1)\) matrix \( A \) has the form \( A = [a_{ij}]_{0 \leq i, j \leq k} \), where \( a_{ij} = c_j \) \((0 \leq j \leq k)\), \( a_{i-1,i} = 1 \) \((1 \leq i \leq k)\), and all other \( a_{ij} \) are 0; for \( n \geq 0 \) the matrix \( B_n \) has the form \( B_n = [b_{ij}(n)]_{0 \leq i, j \leq k} \), where \( b_{0j}(n) = d_j(n) \) \((0 \leq j \leq k)\) and all other \( b_{ij}(n) \) are 0; for \( n \geq 0 \), the
column vector $\mathbf{g}_n$ has the form $\mathbf{g}_n = (g_{0n}, g_{1n}, \ldots, g_{kn})^T$, where $g_{0n} = h(n)$ and all other $g_{in}$ are 0. That is,

$$
\mathbf{g}_n = \begin{pmatrix}
c_0 & c_1 & \ldots & c_{k-1} & c_k \\
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
& & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & 0
\end{pmatrix},
$$

$$
A = \begin{pmatrix}
d_0(n) & d_1(n) & \ldots & d_k(n) \\
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
& & \ddots & \ddots \\
0 & 0 & \ldots & 0
\end{pmatrix},
$$

$$
\mathbf{g}_n = \begin{pmatrix}
h(n) \\
0 \\
0 \\
0 \\
\ldots
\end{pmatrix}.
$$

Since

$$
\det(A - \lambda I) = (-1)^k \left( \sum_{j=0}^{k} c_j \lambda^{k-j} - \lambda^{k+1} \right) = (-1)^{k+1} \lambda^k f(\lambda),
$$

$\lambda_0$ is a simple dominant eigenvalue of $A$. It is easily seen that $\mathbf{\xi} = (\xi_0, \xi_1, \ldots, \xi_k)^T$ and $\mathbf{\eta} = (\eta_0, \eta_1, \ldots, \eta_k)^T$ defined by

$$
\xi_i = \lambda_0^{-i}, \quad 0 \leq i \leq k
$$

(3.11)

and

$$
\eta_i = \sum_{j=i}^{k} c_j \lambda_0^{i-j-1}, \quad 0 \leq i \leq k; \quad \eta_0 = 1 \text{ (cf. (3.3))},
$$

(3.12)

are eigenvectors of $A$ and $A^*$ corresponding to the eigenvalues $\lambda_0$ and $\lambda_0$, respectively. Evidently, assumptions (3.4) and (3.7) imply (1.4) and (1.5).

By Theorem 2, the adjoint equation (2.15) has a solution $\{\mathbf{y}_n\}_{n=0}^{\infty}$, $\mathbf{y}_n = (y_{0n}, y_{1n}, \ldots, y_{kn})^T$ for $n \geq 0$, satisfying (2.16); i.e.,

$$
\lim_{n \to \infty} [\lambda_0^n \mathbf{y}_n] = \mathbf{\eta}_i = \sum_{j=i}^{k} c_j \lambda_0^{i-j-1}, \quad 0 \leq i \leq k.
$$

(3.13)

The adjoint equation (2.15) is equivalent to the equations

$$
y_{in} = (c_i + d_i(n)) y_{0n+1} + y_{i+1, n+1}, \quad 0 \leq i \leq k - 1,
$$

$$
y_{kn} = (c_k + d_k(n)) y_{0n+1}.
$$
From this, we find for \( n \geq 0 \),
\[
y_{in} = \sum_{j=i}^{k} (c_j + d_j(n + j - i))y_{0, n+j-i+1}, \quad 0 \leq i \leq k.
\] (3.14)

Define
\[
w(n) = \overline{y_0n} \quad \text{for} \quad n \geq 0.
\] (3.15)

By virtue of (3.14), we have for \( n \geq 0 \),
\[
\overline{y_i} = \sum_{j=i}^{k} (c_j + d_j(n + j - i))w(n + j - i + 1), \quad 0 \leq i \leq k.
\] (3.16)

From (3.13) and (3.16) (taking \( i = 0 \)) and the fact that \( \eta_0 = 1 \), it follows that \( \{w(n)\}_{n=0}^{\infty} \) is a solution of Eq. (3.5) satisfying (3.6).

Let \( \{u(n)\}_{n=0}^{\infty} \) be the solution of Eq. (3.1) with initial conditions (3.8). Then \( \{x_n\}_{n=0}^{\infty} \), where \( x_n = (x_{0n}, x_{1n}, \ldots, x_{kn})^T \) is defined by (3.10), is a solution of Eq. (1.2) with initial value \( x_{n0} = v = (v_0, v_1, \ldots, v_k)^T \), where
\[
u_i = \phi(n_0 - i), \quad 0 \leq i \leq k.
\] (3.17)

By Theorem 3, using relations (3.10) and (3.11), we have
\[
\lim_{n \to \infty} \left[ \lambda_0^n u(n - i) \right] = \lim_{n \to \infty} \left[ \lambda_0^n x_{in} \right] = \alpha \xi_i = \alpha \lambda_0^{n-i}, \quad 0 \leq i \leq k,
\] (3.18)

where the constant \( \alpha \) is given by (2.23). We shall compute \( \alpha \) explicitly.

By virtue of (3.11) and (3.12),
\[
\eta^* \xi = \sum_{i=0}^{k} \sum_{j=i}^{k} c_j \lambda_0^{j-i-1}.
\]

From this, changing the order of summation and using Eq. (3.3), we obtain
\[
\eta^* \xi = \sum_{j=0}^{k} (j + 1)c_j \lambda_0^{j-1} = 1 + \sum_{j=0}^{k} j c_j \lambda_0^{j-1}.
\]

Hence
\[
\eta^* \xi = f^*(\lambda_0).
\] (3.19)

Equations (3.16) and (3.17) imply that
\[
y_{n0}^* v = \sum_{i=n_0-k}^{n_0} \phi(i) \sum_{j=n_0-i}^{k} (c_j + d_j(i + j))w(i + j + 1).
\]
From this, using Eq. (3.5), we find
\[ y_{n_0}^* = \frac{\phi}{\lambda_{n_0}} - \sum_{i=n_0-k}^{n_0-1} \sum_{j=n_0-i}^{k} (c_j + d_j(i+j))w(i+j+1). \] (3.20)

Finally, in view of (3.15) and the definition of \( g_n \), we have
\[ \sum_{n=n_0}^{\infty} y_{n+1}^*g_n = \sum_{n=n_0}^{\infty} w(n+1)h(n). \] (3.21)

Substituting (3.19)–(3.21) into (2.23), we obtain that \( \alpha \) is equal to the constant on the right-hand side of (3.9). This together with (3.18) \((i = 0)\) implies (3.9).

**Remark 6.** For sufficient conditions for the existence of a simple dominant root of Eq. (3.3) (with real coefficients), see [8].

If in Eq. (3.5) each \( d_j(n) \equiv 0 \), then the special solution \( \{w(n)\}_{n=0}^{\infty} \) of Eq. (3.5) satisfying (3.6) is given by \( w(n) = \lambda_{n_0}^{-n} \) for \( n \geq 0 \). Thus, Theorem 4 yields

**Corollary 2.** Suppose that Eq. (3.3) has a simple dominant root \( \lambda_0 \) and condition (3.7) holds. If \( \{u(n)\}_{n=n_0-k}^{\infty} \) is the solution of the equation
\[ u(n+1) = \sum_{j=0}^{k} c_ju(n-j) + h(n) \] (3.22)
with initial conditions (3.8), then \( \lim_{n \to \infty} [\lambda_0^{-n}u(n)] \) is finite and its value is given by
\[ \lim_{n \to \infty} [\lambda_0^{-n}u(n)] = \frac{1}{f'(\lambda_0)} \left( \phi(n_0)\lambda_0^{-n_0} + \sum_{i=n_0-k}^{n_0-1} \phi(i) \sum_{j=n_0-i}^{k} c_j\lambda_0^{-i-j-1} \right) + \sum_{n=n_0}^{\infty} \lambda_0^{-n-1}h(n). \] (3.23)

**Remark 7.** In the homogeneous case \((h(n) \equiv 0)\) the limit (3.23) has been computed by Driver et al. [8]. For similar results concerning linear homogeneous scalar equations, see [15, 18].

**REFERENCES**


