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On the location of critical point for the Poisson equation in plane ☆

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Abstract

The location of the unique critical point of $\Delta u = -1$ is investigated by conformal mapping method in complex variables. It is found that if the domain is given by $r = 1 + \epsilon p(\theta)$, the critical point coincides with the center of mass up to the order of ϵ. However, the two do not exactly match in general as shown by simple examples.

Keywords: Critical point; Location; Vortex; Level curve; Stagnation point

1. Introduction

Solutions of partial differential equations generally possess many interesting and useful characteristics. Among them, the critical points where the gradient vanishes would be one of the most important. In fact, enough information (e.g., number, location, nature, etc.) on these points combined with a proper inspection of level sets provides a complete geometry and topology of the solution in two or three dimensions [8].

However, one finds just a few research results on critical points of partial differential equations and most of them are concerned with the number or the nature of critical points for certain specific types of partial differential equations. (See, e.g., [2,12].) Among them, we briefly refer works for elliptic partial differential equations [1,11]. These are interesting results but none of them except

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[4] discusses where the critical points are located. Thus, research on this issue is desired for complete and accurate description of the solution.

In this paper, we concentrate on the location of the (unique) critical point of Poisson equation in two dimensions which is one of the most fundamental equations. By the method of conformal map in complex plane, we successfully calculate the location of the critical point and the asymptotic shape of level curves for the perturbed domain from a unit disk. We then verify that the critical point agrees with the center of mass of the domain up to the small perturbation parameter. Moreover, we suggest an example to show that the critical point does not exactly coincide with the center of mass. Additionally, another illustration is provided to show the emergence of new critical points during domain perturbation.

2. Location of the critical point

We first describe the problem:

Determine the location of the critical point of the Poisson equation for $u = u(x, y)$,

$$\Delta u = -1 \quad \text{on } D, \tag{1}$$

$$u = 0 \quad \text{on } \partial D, \tag{2}$$

where D is any convex domain in R^2.

The solution $u(x, y)$ is known to possess unique critical point [11]. Moreover, the level curves of u (i.e., $u =$ constant) are all convex as in [7]. At this point, we like to pose the following questions. First, where is the unique critical point located? In particular, if the domain is a disk, the critical point is exactly the center. Then, where does it go if we perturb the domain (i.e., disk)? Is there any mathematical or physical meaning of the new location of critical point? Secondly, in relation to the first question, how do the level curves near the new moved critical point look asymptotically? Thirdly, if we deform the disk more or in other way, we can obtain a nonconvex domain. How and where does the new critical point appear?

In the following, we try to answer the questions above. We start by analyzing how the critical point moves as we deform the unit disk by the boundary perturbation $r = 1 + \epsilon p(\theta)$ of order ϵ. We state the main result in the following:

Theorem 1. *Let $u(x, y)$ be the solution of* (1), (2) *and (x_c, y_c) be the unique critical point of u in D. If $D = U_\epsilon$ is a slightly perturbed domain from the unit disc $U = \{(x, y)\colon x^2 + y^2 < 1\}$ and has the form $r < 1 + \epsilon p(\theta)$ in polar coordinates where ϵ is a sufficiently small parameter, then (x_c, y_c) agrees with the center of mass of D up to order of ϵ.*

We need to be not too optimistic from the theorem. In fact, the exact location of the critical point does not, in general, coincide with the center of mass as illustrated by an example in Section 5.

Now let us prove the theorem.

3. Conformal mapping and proof of Theorem 1

We introduce the complex variable $z = x + iy$ and consider a conformal map which sends U to D as in the above theorem. (See Fig. 1.) Forgetting the higher order terms, the conformal map $z = f(\zeta)$ for $\zeta \in U$ is given by

$$f(\zeta) = \zeta + \epsilon \frac{\zeta}{2\pi} \int_0^{2\pi} \frac{e^{it} + \zeta}{e^{it} - \zeta} p(t)\, dt + O(\epsilon^2) \tag{3}$$

which, in essence, comes from the Hadamard's variational formula for the Green function for the Laplacian. (See Nehari [10, pp. 263–265].) This map, in particular sends the unit circle onto $r = 1 + \epsilon p(\theta)$.

Let $v(x, y) = u(x, y) + \frac{x^2+y^2}{4}$, then (1), (2) are interpreted as

$$\begin{cases} \Delta v = 0 \quad \text{on } D, \\ v|_{\partial D} = \frac{x^2+y^2}{4}. \end{cases} \tag{4}$$

We then consider $v^* = v^*(\zeta) = v(f(\zeta))$ which is defined on $\overline{U}$ and satisfies

$$\begin{cases} \Delta_\zeta v^* = 0 \quad \text{on } U, \\ v^*|_{\partial U} = \frac{|f(e^{i\phi})|^2}{4}, \end{cases} \tag{5}$$

where $\Delta_\zeta = \frac{1}{4}\partial_\zeta \partial_{\bar{\zeta}}$ is the Laplacian on ζ. Under the given perturbation we compute

$$v^*|_{\partial U} = v^*(e^{i\phi}) = \frac{1}{4} + \frac{\epsilon p(\phi)}{2} + O(\epsilon^2) \tag{6}$$

and thus the harmonic function $v^* = v^*(\zeta) = v^*(\rho e^{i\phi})$ pertaining to the boundary values on ∂U is

$$v^*(\rho e^{i\phi}) = \frac{1}{4} + \frac{1}{2} H(\rho, \phi)\epsilon + O(\epsilon^2), \tag{7}$$

where

$$H(\rho, \phi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \rho^n (a_n \cos n\phi + b_n \sin n\phi) \tag{8}$$

for the following given

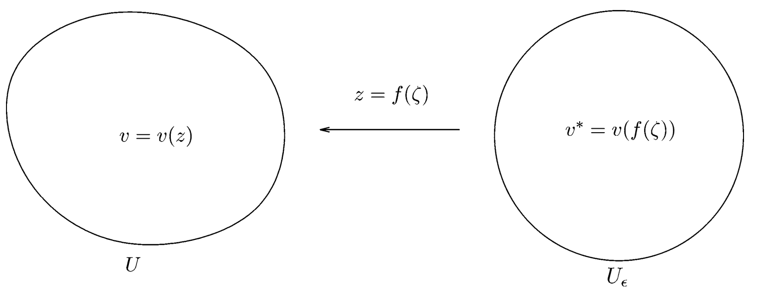

Fig. 1. Conformal mapping of the domain from the unit disk.

$$p(\phi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi). \tag{9}$$

Note that $H(\rho, \phi)$ is the harmonic function with boundary value $p(\phi)$ for $0 \leqslant \phi \leqslant 2\pi$.

To find a critical point of $u(x, y)$, we first seek points ζ satisfying

$$\nabla_\zeta v^* = \nabla_\zeta \frac{r^2}{4}, \tag{10}$$

where $r^2 = \rho^2(1 + 2\epsilon H(\rho, \phi)) + O(\epsilon^2)$. In polar coordinates $\zeta = \rho e^{i\phi}$, this reduces to, up to $O(\epsilon)$ terms,

$$\epsilon\left(1 - \rho^2\right)\frac{\partial v}{\partial \rho} - 2\epsilon\rho v - \rho = 0, \tag{11}$$

$$\rho = 1 \quad \text{or} \quad \frac{\partial v}{\partial \phi} = 0 \quad (\text{if } \rho > 0). \tag{12}$$

Incorporating the obvious expansion $\rho = 0 + \rho_1\epsilon + O(\epsilon^2)$ into (11), (12) and collecting terms up to $O(\epsilon)$,

$$\frac{\rho_1}{2} = \frac{1}{2}(a_1 \cos\phi + b_1 \sin\phi), \tag{13}$$

$$\rho_1(b_1 \cos\phi - a_1 \sin\phi) = 0. \tag{14}$$

Solving this system, we obtain the unique solution:

$$\rho_1 = a_1 \cos\phi + b_1 \sin\phi = \sqrt{a_1^2 + b_1^2}, \tag{15}$$

$$\phi = \tan^{-1}\frac{b_1}{a_1}. \tag{16}$$

Hence the unique critical point is placed at

$$(x_c, y_c) = (a_1, b_1)\epsilon + O\left(\epsilon^2\right). \tag{17}$$

This is a reasonable result in the sense that higher order frequency terms ($n > 1$) in $p(\phi)$ themselves are rotationally symmetric perturbations on $0 \leqslant \phi \leqslant 2\pi$. Thus, from the well-known symmetry property of the Laplace operator, the critical point does not change its position under these symmetric transformations of domain. We conclude simply that only the first order terms of the perturbation contributes the movement of critical point.

Next we easily compute the center of gravity of U_ϵ as follows:

$$\iint_{U_\epsilon} 1\, dA = \iint_{U_\epsilon} r\, dr\, d\theta = \pi + a_0\pi\epsilon + O\left(\epsilon^2\right), \tag{18}$$

$$\iint_{U_\epsilon} x\, dA = \iint_{U_\epsilon} r^2 \cos\theta\, dr\, d\theta = a_1\pi\epsilon + O\left(\epsilon^2\right), \tag{19}$$

$$\iint_{U_\epsilon} y\, dA = b_1\pi\epsilon + O\left(\epsilon^2\right). \tag{20}$$

Thus, from the definition, the center of gravity (x_m, y_m) resides at

$$(x_m, y_m) = (a_1, b_1)\epsilon + O\left(\epsilon^2\right), \tag{21}$$

which completes the proof.

4. Asymptotic configuration of the level curves

Let us turn to the second question and study the asymptotic behavior of solution near the perturbed critical point P. More precisely, we derive the equation of level curves asymptotically at P by computing the 2nd order terms of Taylor expansion of v^* there. Again in polar coordinates, putting

$$u^* = u^*(\zeta) = v^*(\zeta) - \frac{r^2}{4}, \tag{22}$$

we observe, for $\vec{h} = h_r\vec{e}_r + h_\theta\vec{e}_\theta$,

$$(\vec{h}\cdot\nabla)^2 u^*(P) = \left(h_r\partial_r + \frac{h_\theta}{r}\partial_\theta\right)^2 u^*(P) \tag{23}$$

$$= \left(h_r^2\partial_{rr} - \frac{h_r h_\theta}{r^2}\partial_\theta + \frac{2h_r h_\theta}{r}\partial_{r\theta} + \frac{h_\theta^2}{r^2}\partial_{\theta\theta}\right)u^*(P). \tag{24}$$

We then compute and collect terms in the order of ϵ as

$$O(1):\quad -\frac{1}{2}h_r^2, \tag{25}$$

$$O(\epsilon):\quad Ah_r^2 + Bh_r h_\theta + Ch_\theta^2, \tag{26}$$

where

$$A = \frac{1}{2}\left[H_{\rho\rho}\left(1-\rho^2\right) - 4\rho H_\rho - 2H\right], \tag{27}$$

$$B = -\frac{1}{2}\left(3 + \frac{1}{\rho^2}\right)H_\phi + \frac{1}{\rho}H_{\rho\phi}\left(1-\rho^2\right), \tag{28}$$

$$C = \frac{1-\rho^2}{2\rho^2}H_{\phi\phi}. \tag{29}$$

We proceed to calculate the beginning terms of these coefficients,

$$A = -\frac{a_0}{2} + O(\epsilon), \qquad B = 0 + O(\epsilon), \qquad C = -\frac{1}{2} + O(\epsilon), \tag{30}$$

and thus the 2nd order terms have an expansion

$$-\frac{1}{2}h_r^2 - \frac{1}{2}\left(a_0h_r^2 + h_\theta^2\right)\epsilon + O\left(\epsilon^2\right). \tag{31}$$

If $\epsilon = 0$, the level curves are just $h_r =$ constant which are concentric circles around the critical point. While, for $\epsilon \neq 0$, they are deformed in such a way that only the zeroth order of perturbation a_0 contributes. Specifically, the level curves are asymptotically ellipses with two major axis in the direction of $\phi = \tan^{-1}\frac{b_1}{a_1}$ and $\phi + \frac{\pi}{2}$. If ϵ is small enough then the major axis is in ϕ direction.

5. Examples

Example 1 *(Critical point and the center of mass do not coincide).* From the theorem above we wonder if the two points are coinciding. This is not the case in general as shown in the following example. Take $z = f(\zeta) = \zeta + a\zeta^2$ where a is a properly small real number. (We need to assume $|a| < \frac{1}{2}$ for conformality.) Then we compute

$$v^*(\zeta) = v^*\big(\rho e^{i\phi}\big) = \frac{1}{2}\left(\frac{1+a^2}{2} - a\rho\cos\phi\right). \tag{32}$$

Since the conformal mapping $z = f(\zeta)$ inherits a symmetry with respect to x-axis, the critical point is placed somewhere on the x-axis. Then the equation to solve becomes

$$\frac{d}{dx}\left(\frac{|f(x)|^2}{4} - v^*(x)\right) = 0 \tag{33}$$

that is,

$$x(ax-1)(2ax-1) + a = 0. \tag{34}$$

For $|a| \ll 1$, this has three distinct real roots of which two are sitting outside $[-1, 1]$. The unique meaningful solution is approximately calculated by the perturbation method. Assuming the solution has an expansion in a we obtain the asymptotic series

$$x_c = 0 - a + 0a^2 + 3a^3 + \cdots \tag{35}$$

while the center of gravity is placed at

$$x_m = -\frac{a}{1+2a^2} \tag{36}$$

and thus asymptotically

$$x_m = 0 - a + 0a^2 + 2a^3 + \cdots. \tag{37}$$

We observe the agreement up to $O(\epsilon^2)$ and the difference is in the third order of a which is comparatively small for $|a| \approx 0$. In fact, we obtain sufficiently accurate numerical solutions of (34) by Maple and draw a graph to see how the difference behaves as a assumes various values in $(-\frac{1}{2}, \frac{1}{2})$. (See Fig. 2.)

Example 2 *(Multiple critical points case).* We find another interesting example where three critical points exist in the domain D and one of them is the center of mass. To construct such an example, by the theorem of Pagani-Masciadri [11], the domain is necessarily concave. After some trial and error, we choose $z = f(\zeta) = \zeta + a\zeta^3 + b\zeta^5 + c\zeta^7$ where we specifically take $a = 1/2$, $b = 1/4$, $c = 1/14$, for instance. (Look at the domain D in Fig. 3.) Computations below are performed by Maple.

First, we need to check if this is a conformal mapping from U onto a dumbbell-like domain $D = f(U)$ by computing the roots of $f'(z)$. In the case of $a = 1/2$, $b = b$, $c = 1/14$, $f'(z)$ is factorized into the form

$$f'(z) = \frac{1}{2}\big(z^2 + dz + e\big)\big(z^2 + f\big)\big(z^2 - dz + e\big),$$

where d, e, f should satisfy the equations

$$1 = \frac{1}{2}e^2 f, \tag{38}$$

$$\frac{3}{2} = -\frac{1}{2}d^2 f + \frac{1}{2}e^2 + ef, \tag{39}$$

$$5b = -\frac{1}{2}d^2 + e + \frac{1}{2}f. \tag{40}$$

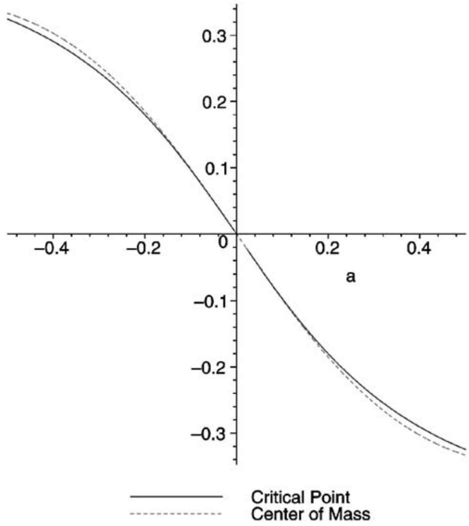

Fig. 2. The difference for various a.

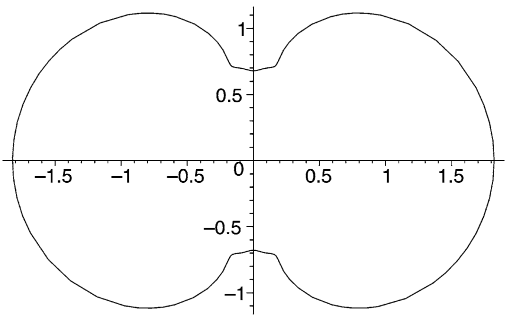

Fig. 3. The mapped domain $D = f(U)$ for $a = 1/2$, $b = 1/4$, $c = 1/14$.

Solving this system of equations, we obtain a set of real solutions d, e, f given by

$$d = \pm\sqrt{\frac{\alpha^4 - 3\alpha^2 + 4\alpha}{2}}, \qquad e = \alpha, \qquad f = 10b + \frac{1}{2}\alpha^4 - \frac{3}{2}\alpha^2 \tag{41}$$

and where α is a positive real root of

$$Z^6 - 3Z^4 + 20Z^2 b - 4 = 0. \tag{42}$$

Equation (42) always has a positive real solution since by putting $X = Z^2$, it becomes a cubic equation with the highest order coefficients 1 and with y intercepts -4. Under the setting, we select proper b so that the factors in (3) have all roots outside the unit circle. After some experiments, we choose $b = 1/4$ and find $1.1 < \alpha < 1.3$. This gives the roots of $f'(z)$ approximately, $\pm 1.173066873i$, $\pm 0.5672796379 \pm 0.9400868273i$, all of which have the moduli strictly greater than 1.

We proceed to compute the corresponding harmonic function on U by

$$v^*\left(\rho e^{i\phi}\right) = 1 + a^2 + b^2 + c^2 + 2(a + ab + bc)\rho^2\cos(2\phi) + 2(b + ac)\rho^4\cos(4\phi) + 2c\rho^6\cos(6\phi). \tag{43}$$

Since the domain D is symmetric with respect to x-axis, we consider the function on the x-axis and then it becomes

$$v^*(x) = 2cx^6 + (2ac + 2b)x^4 + (2ab + 2a + 2bc)x^2 + 1 + b^2 + c^2 + a^2. \tag{44}$$

Thus, the possible critical points are obtained by solving the equation for real x,

$$h(x) = \frac{d}{dx}\left(\frac{|f(x)|^2}{4} - v^*(x)\right) = 0, \tag{45}$$

which is

$$-\frac{1}{2}\left(x + ax^3 + bx^5 + cx^7\right)\left(1 + 3ax^2 + 5bx^4 + 7cx^6\right) + 2acx^3 + abx + 2bx^3 + 3cx^5 + ax + bcx = 0. \tag{46}$$

In the case of $a = 1/2$, $b = b$, $c = 1/14$, it becomes

$$h(x) = -\frac{1}{56}x^{13} - \frac{3}{7}bx^{11} + \left(-\frac{5}{28} - \frac{5}{2}b^2\right)x^9 + \left(-2b - \frac{2}{7}\right)x^7 + \left(-\frac{9}{56} - 3b\right)x^5 + \left(2b - \frac{13}{14}\right)x^3 + \frac{4}{7}bx. \tag{47}$$

Let $X = x^2$ and put $H(X) = h(x)/x$, we consider

$$H(X) = -\frac{1}{56}X^6 - \frac{3}{7}bX^5 + \left(-\frac{5}{28} - \frac{5}{2}b^2\right)X^4 + \left(-2b - \frac{2}{7}\right)X^3 + \left(-\frac{9}{56} - 3b\right)X^2 + \left(2b - \frac{13}{14}\right)X + \frac{4}{7}b. \tag{48}$$

Differentiating, we obtain

$$\frac{dH}{dX} = -\frac{3}{28}X^5 - \frac{15}{7}bX^4 + \left(-10b^2 - \frac{5}{7}\right)X^3 + \left(-\frac{6}{7} - 6b\right)X^2 + \left(-\frac{9}{28} - 6b\right)X + 2b - \frac{13}{14} \tag{49}$$

and conclude that

$$\frac{dH}{dX} < 0 \tag{50}$$

for all $X > 0$ if $0 < b < \frac{13}{28}$. From $H(0) = \frac{4}{7}b > 0$, we finally conclude the existence of a single real root for $H(X) = 0$. Again, for instance, if $a = 1/2$, $b = 1/4$, $c = 1/14$, the real roots of $h(x) = 0$ are 0 and approximately -0.24696, 0.24696 which, in fact, are a local minimum, a local maximum and a local maximum, respectively. To make it clear, we draw the graph of the solution $z = u(x, y)$ in Fig. 4.

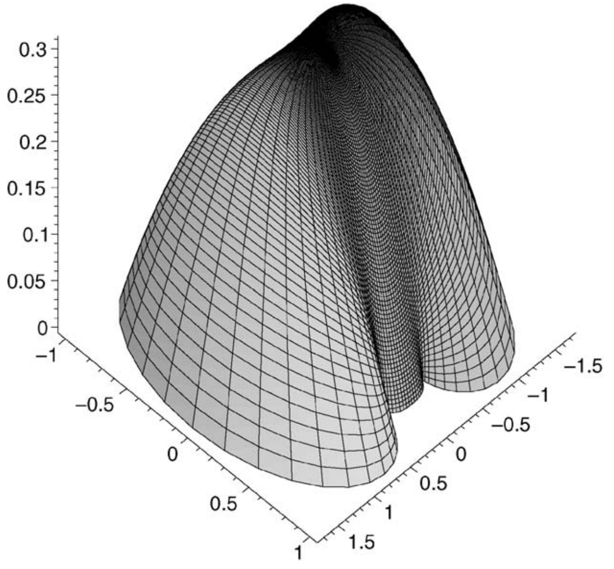

Fig. 4. The solution $u(x, y)$ for $a = \frac{1}{2}, b = \frac{1}{4}, c = \frac{1}{14}$.

6. Concluding remarks and application

The feature of the critical point in the Poisson equation has not been investigated thoroughly yet. An approximate location is obtained, however, we are still unaware of the exact location of critical points as well as their mathematical or physical implications. We also illustrate two examples to show how the critical points are different from the center of mass. It also happens that for some nonconvex domains the center of gravity is located outside the domain while the critical point is always inside.

The generalization of results obtained to higher dimensional setting will be another interesting topic. This seems to be more difficult since there is no proper tool analogous to conformal mapping in three or higher dimensional space. Besides, the method of nodal lines is not directly applicable in three dimensions as mentioned in [11].

Finally, we remark that the current study has its application to incompressible fluid dynamics. Let u represent the stream function of the given fluid flow in two dimensions then $-\Delta u$ corresponds to the vorticity. It is now broadly known that the limit of viscous Navier–Stokes flows in two dimensions as the viscosity approaches to zero is characterized by the constancy of the vorticity in a closed streamline region. (Such region is often called a *vortex patch*. For more details, look at [5,6].) Thus, the problems in this paper are interpreted to configure the inviscid limit flow by locating the stagnation point of vortex patch and by determining the streamlines nearby. To be more precise, from the information above we like to determine the geometry and topology of fluid flow in the region. This is an important subject in the motion of vortex related flows [3,9].

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