Complexity lower bounds for computation trees with elementary transcendental function gates

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Abstract

We consider computation trees which admit as gate functions along with the usual arithmetic operations also algebraic or transcendental functions like exp, log, sin, square root (defined in the relevant domains) or much more general Pfaffian functions. A new method for proving lower bounds on the depth of these trees is developed which allows to prove a lower bound $\Omega(\sqrt{\log N})$ for testing membership to a convex polyhedron with $N$ facets of all dimensions, provided that $N$ is large enough.

1. Pfaffian computation trees

Definition 1. By a Pfaffian computation tree $\mathcal{T}$ we mean a generalization of an algebraic decision tree (see e.g. [1,4,12,28–30]) in which at any node $v$ of $\mathcal{T}$ a Pfaffian function $f_v$ in the variables $X_1, \ldots, X_n$ (see Definition A.2 in the Appendix) is attached, which satisfies the following properties. Let $f_{v_0}, \ldots, f_{v_r}, f_{v_{r+1}} = f_v$ be the functions attached to all the nodes along the branch $\mathcal{T}_v$ of $\mathcal{T}$ leading from the root $v_0$ to $v$. We assume that Pfaffian function $f_v$ satisfies the following differential equation

$$df_v = \sum_{1 \leq j \leq n} g_{v,j}(X_1, \ldots, X_n, f_{v_0}, \ldots, f_{v_r}) \, dX_j,$$

with $g_{v,j} \subset \mathbb{R}[X_1, \ldots, X_n, U_0, \ldots, U_{r+1}]$. The tree $\mathcal{T}$ branches at $v$ to its three sons according to the sign of $f_v$ (cf. [1]). Thereby, to each node $v$ one can naturally assign a semi-Pfaffian set $U_v \subset \mathbb{R}^n$ (see Definition A.3 in the Appendix) consisting of all the points for which the sign conditions for functions along the branch $\mathcal{T}_v$ are valid. Thus, to three sons of $v$ one assigns the semi-Pfaffian sets $U_v \cap \{f_v > 0\}, U_v \cap \{f_v = 0\}, U_v \cap \{f_v < 0\}$, respectively. We assume that $f_v$ is defined on a certain domain (see Definition A.2) containing $U_v$. To any leaf of $\mathcal{T}$ an output either "yes" or "no" is assigned and we say that $\mathcal{T}$ tests the membership problem to the set of all points

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(x₁, ..., xₙ) ∈ ℝⁿ for which the outputs of the corresponding leaves of ℋ are “yes” (see [11]).

Note that a more general notion of a Pfaffian sigmoid was introduced in [10] and a method for obtaining lower bounds on the (parallel) complexity was developed.

If we take only arithmetic operations as the gate Pfaffian functions fᵢ in ℋ then we come to the algebraic computation trees (see e.g. [1]). As examples of gate Pfaffian functions fᵢ one could take \( \exp(fᵢ), \log(fᵢ), \) where log is defined on the positive half-line, \( \sqrt{fᵢ}, \) where square root is defined on the positive half-line, \( \sin(fᵢ), \) where sin is defined on the interval \((-π, π), \) \( \tan(fᵢ), \) where tan is defined on the interval \((-π/2, π/2), \) \( 0 ≤ q ≤ l. \) Other examples can be found in Section A.1 of the Appendix. Trees ℋ restricted to some special classes of Pfaffian functions (for instance, those mentioned above) can be of particular interest, but since we are interested in the complexity lower bounds we shall consider arbitrary Pfaffian functions.

Suppose that the degrees \( \deg gₘᵢ \) of the polynomials occurring in the definition of the gate functions \( fᵢ, \) in ℋ are less than \( d. \)

Now we are able to formulate the main result of the paper. This result was announced in [15].

**Theorem.** Let a Pfaffian computation tree ℋ test a membership problem to a closed convex polyhedron \( P ⊂ \mathbb{R}ⁿ, \) having \( N \) facets of all the dimensions. Then the depth \( K \) of ℋ is greater than \( \Omega(√log N), \) provided that \( N ≥ (dn)^{Q(n^d)}, \)

In [11] a particular case of the theorem for \( n = 2, \) when \( P \) is a polygon, was proved.

Several methods based on topological characteristics are known for obtaining complexity lower bounds for algebraic computation trees testing membership to a semialgebraic set \( S ⊂ \mathbb{R}ⁿ. \) In [1], the bound \( \Omega(\log C) \) was proved, where \( C \) is the number of connected components of \( S \) or its complement, in [3, 4, 28] the bound \( \Omega(\log χ) \) was proved, where \( χ \) is the Euler characteristic. The most general (among the listed) bound \( \Omega(\log B) \) was proved in [3, 29], where \( B \) is the sum of Betti numbers of \( S. \)

Actually one could directly extend these results to Pfaffian computation trees, replacing in the proofs the references to Milnor’s bound [23] on the sum of Betti numbers of a semialgebraic set by the references to Khovanskii’s bound [20] for the sum of Betti numbers of a semi-Pfaffian set. This leads to the following proposition [11]. If a Pfaffian computation tree tests the membership problem to a semi-Pfaffian set \( W \) with the sum of Betti numbers \( B, \) then the depth of the tree is greater than \( \Omega(√log B) \) [11].

There is a conjecture that the bound in [20] could be improved (see Section A.1 in the Appendix). This conjecture implies the lower bounds \( \Omega(\log N) \) in the theorem and \( \Omega(\log \mathcal{B}) \) in the proposition from [11], respectively.

Observe that as the sum of Betti numbers of a convex polyhedron equals 1, the theorem does not follow, apparently, from the proposition. Note that in [12] the complexity lower bound \( \Omega(\log N) \) was proved for testing membership to a polyhedron
with $N$ facets by an algebraic decision tree (for large enough $N$, cf. the theorem). In [30] a similar bound was shown for a weaker model of linear decision trees. The method from [12] cannot be directly generalized to Pfaffian computation trees, since in [12] the efficient quantifier elimination procedure for the first-order theory of reals (see [9, 14, 17, 24]) was essentially used whereas for the theories involving Pfaffian functions (in particular, elementary transcendental) the quantifier elimination does not exist.

We remark that the computations involving other functions, rather than arithmetic, were considered in several papers: in [18], for the computations involving root extractions, a lower bound for computing an algebraic function was obtained, in [13] this result was extended for the computations involving exp and log.

We mention that for testing membership to a polyhedron an upper bound $O(\log N)$ $n^{O(1)}$ was shown in [22] even for linear decision trees.

Now we proceed to the proof of the theorem which will continue up to the end of Section 3.

We start with introducing some necessary concepts and notations. In Section 2 we introduce the notion of $i$-angle points and prove that the set of $i$-angle points has the dimension at most $i$. This notion differs from the concept of sharp points introduced and used in [12], the latter does not work for Pfaffian computation trees. In Section 3 we introduce and study another important technical concept, flat points. All the necessary information about Pfaffian functions and sub-Pfaffian sets is included in the Appendix (in which the numbering of all the statements, definitions and sections begins with A).

For an $m$-plane $Q \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ denote by $Q(x)$ the $m$-plane, collinear to $Q$ and containing $x$. For a facet $\Pi$ of the polyhedron $P$ denote by $\bar{\Pi}$ the $\dim(\Pi)$-plane containing $\Pi$ (we assume a facet to be open, i.e. without its boundary).

Two planes $Q_1$, $Q_2$ or arbitrary dimensions are called transversal if
\[
\dim(Q_1(0) \cap Q_2(0)) = \max\{0, \dim(Q_1(0)) + \dim(Q_2(0)) - n\}.
\]

The proofs of the following two easy lemmas can be found in [12] (Lemma 1 is also proved in [5]).

**Lemma 1.** For each $j$ with $1 \leq j \leq n$ there exists a family $\mathcal{A}_j$ consisting of $j(n - j) + 1$ $j$-subspaces in $\mathbb{R}^n$ such that for any $i$-subspace $Q \subset \mathbb{R}^n$, $1 \leq i \leq n$, there is a $j$-subspace $R \in \mathcal{A}_j$ which is transversal to $Q$.

**Lemma 2.** There exists a rotation of coordinates $X_1, \ldots, X_n$ such that after this rotation for every $j$, every $Q \in \mathcal{A}_j$ and for every facet $\Pi$ of $P$, the subspace $Q$ and the plane $\bar{\Pi}$ become transversal.

In what follows we suppose that the coordinate system meets the requirements of Lemma 2. Now we reduce consideration to the case when the polyhedron $P$ is bounded. The next construction follows the beginning of the proof of Lemma 5 [12].

Let $t$ be the minimal dimension of facets of $P$. Fix a certain $t$-facet $P_i$ of $P$, then $t$-plane $\mathcal{P}_i$ is contained in $P$. On each facet $\Pi$ of $P$ choose a point $x_{\Pi} \in \Pi$. Take
an arbitrary hyperplane $\sigma$ transversal to $P_t$ and such that the points $x_{\Pi}$ for all facets $\Pi$ of $P$ lie in the same of two open half-spaces of $\mathbb{R}^n \setminus \sigma$ (denote this half-space by $\Sigma$). Consider the polyhedron $P \cap (\Sigma \cup \sigma)$, it contains a facet of a dimension less than $t$. Continuing this process while $t > 1$, we come eventually to the case $t = 0$, i.e. polyhedron $P'$ obtained as a result of this process has a vertex.

There exists a linear form $L = \beta_1 x_1 + \cdots + \beta_n x_n$ with $\beta_i \in \mathbb{R}$, $1 \leq i \leq n$, such that for every $\gamma \in \mathbb{R}$ an intersection $P'' = \{L + \gamma \geq 0\} \cap P'$ is compact. Take $\gamma$ such that $x_{\Pi} \in \{L + \gamma \geq 0\} \cap P'$ for all $\Pi$.

In order to reduce consideration to the compact polyhedron $P''$, observe that from a Pfaffian computation tree of depth $K$ for the membership problem to $P$, one can easily produce a Pfaffian computation tree of a depth at most $K + n$ for the membership problem to $P''$. Assuming that the theorem is valid for the compact $P''$, and thus $K + n \geq \Omega(\sqrt{\log N})$, we get a similar bound $K \geq \Omega(\sqrt{\log N})$ under the supposed inequality for $N$ in the hypothesis of the theorem. Therefore, in what follows we assume that $P$ is bounded.

In Section A.2 a sequence of nonstandard extensions of fields is introduced. One can choose in each $\mathbb{R}_{i+1}$ an element infinitesimal relative to $\mathbb{R}_i$. We denote these elements, respectively, by

$$\varepsilon_1 \in \mathbb{R}_1, \{\delta^\ell \in \mathbb{R}_{(-1)(n^2+1)+\ell+1} : 1 \leq \ell \leq n-1, 1 \leq j \leq n^2 + 1\},$$

$$\varepsilon_2 \in \mathbb{R}_{n^2-n^2+n+1}, \varepsilon_3 \in \mathbb{R}_{n^2-n^2+n+2}$$

(the reason for these notations would become clear later on). To match the notations denote the fields $\mathbb{R}_1 = \mathbb{R}_{\varepsilon_1}, \mathbb{R}_{(-1)(n^2+1)+\ell+1} = \mathbb{R}_{\delta^\ell}$, $1 \leq \ell \leq n-1, 1 \leq j \leq n^2 + 1$, $\mathbb{R}_{n^2-n^2+n+1} = \mathbb{R}_{\varepsilon_2}, \mathbb{R}_{n^2-n^2+n+2} = \mathbb{R}_{\varepsilon_3}$, respectively. For brevity set also $\mathbb{R}_5 = \mathbb{R}_{n^2-n^2+n} = \mathbb{R}_{\delta^{n^2+1}}$. The completion (see Section A.2) for any sub-Pfaffian set $U$ (see Definition A.4) we denote by $U(\delta) = U(n^2-n^2+n), U(\varepsilon) = U(n^2-n^2+n+2)$. Analogously we denote the languages (see Section A.2) $\mathcal{L}_5 = \mathcal{L}_{n^2-n^2+n}, \mathcal{L}_{\varepsilon_2} = \mathcal{L}_{n^2-n^2+n+1}, \mathcal{L}_{\varepsilon_3} = \mathcal{L}_{n^2-n^2+n+2}$. In Section A.2, for each $i$ the standard part $s_t$ is described. Actually, throughout the paper we will use in almost all the cases $s_{t_{n^2-n^2+n}}$, which we will for brevity denote by $s_t$ (on occasions we will also use $s_{t_{n^2-n^2+n+1}}$, which we denote by $s_{t_{\varepsilon_2}}$).

Consider a Pfaffian computation tree $T$ testing the membership to $P$ with depth $K$. Fix any of its branches with the output "yes", and let $f_{t_0}, \ldots, f_{t_K}$ be the Pfaffian functions attached to the nodes along this branch. We rename the functions $f_{t_0}, \ldots, f_{t_K}$ by $u_0, \ldots, u_K$ in such a way that $u_0, \ldots, u_K$, for a certain $K_i \leq K$, correspond to the sign zero, and $u_{K_i+1} > 0, \ldots, u_K > 0$ correspond to nonzero signs along the branch. More precisely, consider a semi-Pfaffian set (see Definition A.3)

$$W = \{x \in \mathbb{R}_{t_{\varepsilon_3}}^n : u_0(x) = \cdots = u_{K_1}(x) = 0, u_{K_1+1}(x) > 0, \ldots, u_K(x) > 0\},$$

which is the accepting set corresponding to the branch. Then the set $W \cap \mathbb{R}^n$ is the set of points on which $T$ along the fixed branch outputs "yes", hence $W \cap \mathbb{R}^n \subset P$. 
Since the functions $u_0, \ldots, u_K$ are defined over $\mathbb{R}$, the completion (see Section A.2) $(W \cap \mathbb{R}^n)^{(e_3)} = W$. In the sequel we will estimate the number of $i$-facets $\Pi$ of $P$ such that $\dim(W \cap \Pi \cap \mathbb{R}^n) = i$.

When $K_1 < 0$ the set $(W \cap \mathbb{R}^n)$ lies in the interior of $P$, so this estimate is trivial. Therefore, we assume that $K_1 \geq 0$ and denote $f = u_0^2 + \cdots + u_K^2$.

2. Angle points

**Definition 2.** A point $x \in W$ is called a $0$-quasiangle if $u_{K_1 + 1}(x) \geq \varepsilon_1, \ldots, u_K(x) \geq \varepsilon_1$, and there exist points $y_1, \ldots, y_n \in \{ f - e_3 = 0 \}$ such that the Euclidean distances $\| y_i - x \| \leq \varepsilon_2, 1 \leq i \leq n$, and

$$
\left( \begin{array}{c}
\frac{\partial f}{\partial X_1}(y_1) \\
\vdots \\
\frac{\partial f}{\partial X_n}(y_1) \\
\vdots \\
\frac{\partial f}{\partial X_1}(y_n) \\
\frac{\partial f}{\partial X_n}(y_n)
\end{array} \right)^2
\geq \varepsilon_1^2 A(y_1) \cdots A(y_n),
$$

where $A = \sum_{1 \leq i \leq n} (\partial f/\partial X_i)^2$.

Observe that Corollary A.5 states that for any point $y \in \{ f = e_3 \} \subset \mathbb{R}^{e_3}$, the gradient $\nabla f(y) = (\partial f/\partial X_1, \ldots, \partial f/\partial X_n)(y)$ does not vanish. Notice that the inequality (0) in the definition means that the absolute value of the determinant of the matrix formed by the normalized gradient vectors of $f$ at the points $y_1, \ldots, y_n$ is greater than $\varepsilon_1$.

**Definition 3.** A point $x \in W$ is called an $i$-quasiangle $(0 < i < n)$ if for each $(n - i)$-subspace $\Pi \in \mathcal{A}_{n-1}$ (see Lemma 1) the point $x$ is a 0-quasiangle point in the semi-Pfaffian set $W \cap \Pi(x)$ (here we understand 0-quasiangle with respect to a basis in $\Pi$ whose elements are from $\mathbb{R}^n$, in other words have coordinates from $\mathbb{R}$, the role of $f$ plays the restriction of $f$ on $\Pi(x)$).

The set of $i$-quasiangle points of $W$ we denote by $A_i$. Observe that $A_i$ can be determined by a Pfaffian formula and thus is a sub-Pfaffian set (see Definition A.4).

**Definition 4.** The points of the set $A_i = st(A_i) \subset \mathbb{R}^n_0$ are called $i$-angle.

Lemma A.7 implies that $A_i$ is sub-Pfaffian and definable over $\mathbb{R}_i$. Due to Lemma A.4, $A_i \subset W$.

**Lemma 3.** Let $P_i$ be an $i$-facet of $P$ with dimension (see Definition A.5) $\dim(W \cap P_i) = i$. If for two points $\tilde{x} \in W \cap \mathbb{R}^n \cap P_i$ and $x \in P_i^{(e)}$ the distance $\| x - \tilde{x} \|$ is infinitesimal relative to $\mathbb{R}$, then $x \in A_i$.

**Remark.** Actually, the lemma states that $x \in A_i$ since $x = st(x) \in st(\tilde{A}_i) = A_i$.

**Proof of the Lemma.** Since $\dim(W \cap P_i) = i$, Lemma A.1 implies that $f$ vanishes on $\tilde{P}_i$. Throughout this paper $B_z(s) \subset \mathbb{R}^n_0$ denotes the open ball centred at $z$ with radius
There exists $0 < c \in \mathbb{R}$ such that $u_j(x) > c$, $K_1 + 1 \leq j \leq K$, therefore there exists $0 < c \in \mathbb{R}$ such that $u_j(y) > c/2$, $K_1 + 1 \leq j \leq K$ for any $y \in B_1(2r) \cap \mathbb{R}^n$. Taking into account that the Pfaffian functions $u_j$ are defined over $\mathbb{R}$. According to the transfer principle (see Section A.2), $u_j(y) > c/2$, $K_1 + 1 < j < K$ for any $y \in B_3(2r) \cap \mathbb{R}^n$. In particular, $u_j(x) > c/2 > e_1$, $K_1 + 1 \leq j \leq K$.

Fix an arbitrary subspace $\Pi \in \mathcal{F}_{n-i}$. Our purpose is to show that $x$ is 0-quasiangle in the set $W \cap \Pi(x)$, which will imply the lemma (see Definition 3). Since $\Pi$ is transversal to $P$, the point $x$ is a vertex of the polyhedron $\mathcal{P} = (P \cap \Pi(x))^{(e_3)}$ (see Lemma 2 and the supposition just after it). The vertex $x$ belongs to at least $n - i$ of $(n - i - 1)$-facets (of the maximal dimension) of $\mathcal{P}$. Observe that for each of these facets the normalized orthogonal (in $\Pi(x)$) vector has the coordinates in $\mathbb{R}$. Choose any $T_1, \ldots, T_{n-i}$ among these facets.

Notice that for any point $y \in cl(B_3(r)) \cap \mathbb{R}^n$ (where $cl$ denotes the closure in the topology with a base of all open balls) the inequalities $u_j(y) > e_1$, $K_1 + 1 \leq j \leq K$ hold since $cl(B_3(r)) \subset B_3(2r)$. Hence, $\bigcap_{K_1 + 1 \leq j \leq K} \{f = 0, \ & u_j \geq e_1\}^{(e_3)} \cap cl(B_3(r)) = \{f = 0\}^{(e_3)} \cap cl(B_3(r))$.

Denote by $\mathcal{K} \subset \Pi(x)$ the intersection of the unique closed cone $\mathcal{K}$ with the vertex at $x$ formed in $\Pi(x)$ by $(n - i - 1)$-planes $T_1, \ldots, T_{n-i}$ containing $\mathcal{P}$, with the ball $cl(B_3(r))$. For any point $z \in \{f = e_3\} \cap \Pi(x) \cap cl(B_3(r))$ we have $st_e(z) \in \{f = 0\} \cap \Pi(x) \cap cl(B_3(r))$, due to Lemma A.4. Therefore, $st_e(z) \in W \cap cl(B_3(r)) \cap \Pi(x) \subset \mathcal{P} \cap cl(B_3(r)) \subset \mathcal{D}$, in particular the distance $\rho(z, \mathcal{D})$ from the point $z$ to the set $\mathcal{D}$ is infinitesimal relative to $\mathbb{R}$. Since the set $\{f = e_3\} \cap \Pi(x) \cap cl(B_3(r))$ is closed in the topology with a base of all open balls, and bounded, the maximum value $\rho_0$ of $\rho(z, \mathcal{D})$ over all the points $z \in \{f = e_3\} \cap \Pi(x) \cap cl(B_3(r))$ exists (here we use the transfer principle), and is infinitesimal relative to $\mathbb{R}$.

Shift (in $\Pi(x)$) each of $(n - i - 1)$-planes $T_1, \ldots, T_{n-i}$ parallel to itself outward from $\mathcal{D}$ to the distance $\rho_0$. Denote the resulting shifted $(n - i - 1)$-planes by $T_1', \ldots, T_{n-i}'$, respectively. Denote by $x'$ the (unique) common point of $T_1' \cap \ldots \cap T_{n-i}'$. Denote by $\mathcal{D}'$ the intersection of the closed cone $\mathcal{K}'$ formed by $T_1', \ldots, T_{n-i}'$, having the vertex in $x'$, with the ball $cl(B_3(r))$. Then $\{f = e_3\} \cap \Pi(x) \cap cl(B_3(r)) \subset \mathcal{D}'$. Observe that the distance $\|x - x'\|$ is infinitesimal relative to $\mathbb{R}$. We replace $(n - i - 1)$-planes $T_j'$, $1 \leq j \leq n - i$ (in $\Pi(x)$), by some $(n - i - 1)$-planes $T''_j$, $1 \leq j \leq n - i$, respectively, in the following way. Take any hyperplane $\Omega$ (in $\Pi(x)$), defined over $\mathbb{R}$, such that the intersection $C_1 = \Omega \cap K \subset \mathcal{D}$. Then $C_1$ is an $(n - i - 1)$-dimensional simplex, let its $(n - i - 2)$-facets which are the intersections of $\Omega$ with $T_1, \ldots, T_{n-i}$, respectively, be determined in $\Omega$ by the equations $\{L_j = 0\}$, $1 \leq j \leq n - i$, for some linear polynomials $L_j$ defined over $\mathbb{R}$. Thus $C_1 = \{L_1 \geq 0, \ldots, L_{n-i} \geq 0\} \subset \Omega$. Consider now the $(n - i - 1)$-dimensional simplex $C_2 = \{L_1 + e_2 \geq 0, \ldots, L_{n-i} + e_2 \geq 0\} \cap \Omega \supset C_1$. The facets of $C_2$ are $\{L_j = -e_2\} \cap \Omega$, $1 \leq j \leq n - i$, and, therefore, they are parallel to the corresponding facets of $C_1$. Denote by $\overline{T}_j^{(3)}$, $1 \leq j \leq n - i$, the hyperplane (in $\Pi(x)$) containing $x$ and $\{L_j = -e_2\} \cap \Omega$. Denote by $\mathcal{K}^{(3)} \subset \Pi(x)$ the cone formed by $\overline{T}_j^{(3)}$, $1 \leq j \leq n - i$, containing $C_2$; observe that $\mathcal{K}^{(3)} \supset \mathcal{K}$.
We claim that the sine of the angle $x$ between the hyperplanes $\overline{T}_j$ and $\overline{T}^{(3)}_j$ (i.e. between vectors, orthogonal to $\overline{T}_j$ and $\overline{T}^{(3)}_j$, respectively) is infinitesimal relative to $\mathbb{R}_\delta$. Indeed, consider the unique 2-plane $\omega_j$, $1 \leq j \leq n-i$, passing through $x$ and orthogonal to $\{L_j = 0\} \cap \Omega$. It intersects the $(n-i-2)$-plane $\{L_j = 0\} \cap \Omega$ (respectively, the $(n-i-2)$-plane $\{L_j = -\varepsilon_2\} \cap \Omega$) at the unique point $y_j$ (respectively, $y_j^{(3)}$). Observe that the vector in $\omega_j$ orthogonal to the line $l_j$ passing through $x$ and $y_j$ (respectively, the line $l_j^{(3)}$ passing through $x$ and $y_j^{(3)}$) is orthogonal to $\overline{T}_j$ (respectively, $\overline{T}^{(3)}_j$). The segment $(y_j, y_j^{(3)})$ lies on the line $\omega_j \cap \Omega$ and is orthogonal to $\{L_j = 0\} \cap \Omega$. Hence the distance between the $(n-i-2)$-planes $\{L_j = 0\} \cap \Omega$ and $\{L_j = -\varepsilon_2\} \cap \Omega$, which is equal to the length of the segment $[y_j, y_j^{(3)}]$, is infinitesimal relative to $\mathbb{R}_\delta$. Since the angle $x$ equals the angle between the lines $l_j$ and $l_j^{(3)}$, we conclude that $\sin x$ is infinitesimal relative to $\mathbb{R}_\delta$, taking into account that in the triangle $(x, y_j, y_j^{(3)})$ the vertices $x$ and $y_j$ are defined over $\mathbb{R}_\delta$, therefore the sides $(x, y_j)$ and $(x, y_j^{(3)})$ are not infinitesimal relative to $\mathbb{R}_\delta$ and $(y_j, y_j^{(3)})$ is infinitesimal relative to $\mathbb{R}_\delta$. This proves the claim.

Let us show that there exists an element $0 < \beta \in \mathbb{R}_\varepsilon$ such that for any two points $x_1 \in \partial C_1, z_1 \in \partial C_2$ from the boundaries (see Definition A.8 (here we mean the boundary in the hyperplane $\Omega$)), the sine of the angle between the lines $(x_1, z_1)$ and $(x, z_2)$ is greater than or equal to $\beta$. Since both points $z_1, z_2$ range over bounded closed sets, there exists (due to the transfer principle) the minimum $\beta$ of these sines. Observe that $\beta > 0$ since $\partial C_1 \cap \partial C_2 = \emptyset$. One could define the element $\beta$ by a formula of the language $\mathcal{L}_\varepsilon$. Therefore, $\beta \in \mathbb{R}_\varepsilon$ by the transfer principle, as was to be shown.

Note that the cones $\mathcal{K}$ and $\mathcal{K}'$ are isometric. We define the desired $(n-i-1)$-planes $\overline{T}''_j, 1 \leq j \leq n-i$, as the images of $\overline{T}^{(3)}_j$, respectively, under the shift mapping the cone $\mathcal{K}$ onto $\mathcal{K}'$, then the cone $\mathcal{K}''$ formed by $\overline{T}''_j, 1 \leq j \leq n-i$, is the image of the cone $\mathcal{K}^{(3)}$.

For every $1 \leq j \leq n-i$, pick a point $x_j \in \{f = \varepsilon_3\} \cap \Pi(x) \cap cl B_r(x)$ with the property that $x_j$ is the nearest to $\overline{T}''_j$ on the (bounded and closed) set $\{f = \varepsilon_3\} \cap \Pi(x) \cap cl B_r(x)$. Lemma A.4 entails that there exists a point $y_j \in \{f = \varepsilon_3\} \cap \Pi(x) \cap B_r(x)$ such that $\|x - y_j\|$ is infinitesimal relative to $\mathbb{R}_\varepsilon$, therefore $\|x' - y_j\|$ is infinitesimal relative to $\mathbb{R}_\varepsilon$ as well, hence the distance from $x_j$ to $\overline{T}''_j$ is also infinitesimal relative to $\mathbb{R}_\varepsilon$.

Denote by $x_j'' \in \overline{T}''_j$ the orthogonal projection of $x_j$ on $\overline{T}''_j$. Let us prove that $\|x_j - x_j''\|$ is infinitesimal relative to $\mathbb{R}_\varepsilon$. Since $x_j \in \{f = \varepsilon_3\} \cap \Pi(x) \cap cl B_r(x)$, the segment $(x_j, x_j'')$ intersects $\partial \mathcal{K}$ (here we mean the boundary in $\Pi(x)$) at the unique point $x_j'$. Since the sine of the angle $\gamma$ between the lines $(x', x_j)$ and $(x', x_j'')$ is greater than or equal to that of the angle between the lines $(x', x_j')$ and $(x', x_j'')$, which, in turn, is greater than or equal to $\beta$ (see above), we conclude that $\sin \gamma \geq \beta \in \mathbb{R}_\varepsilon$. Therefore, $\|x_j - x_j''\| = \|x_j - x_j''\| / \sin \gamma$ is infinitesimal relative to $\mathbb{R}_\varepsilon$, $1 \leq j \leq n-i$, which was to be proved. Hence, $\|x - x_j\|$ is infinitesimal relative to $\mathbb{R}_\varepsilon$ as well, in particular $x_j \in B_r(x)$.

Observe that the gradient $\nabla_x (f^\beta)$ (where $f^\beta$ denotes the restriction of $f$ on $\Pi(x)$, cf. Definition 3) does not vanish because $x_j \in \{f = \varepsilon_3\} \cap \Pi(x)$ (see Corollary A.5) and it is orthogonal to the hyperplane $\overline{T}''_j$ (in $\Pi(x)$), as $x$ is the nearest to $\overline{T}''_j$ on
the set \( \{ f = c_3 \} \cap \Pi(x) \cap cl B_{k}(r) \). Since the sines of the angles between any pair of
hyperplanes \( \overline{T_{j_1}} , \overline{T_{j_2}} \) (in \( \Pi(x) \)) are greater than a certain \( c, 0 < c \in \mathbb{R} \), we conclude
that the sines of the angles between any pair of hyperplanes \( \overline{T_{j_1}}'' , \overline{T_{j_2}}'' \) are greater than
\( c/2 \) according to the claim proved above (stating that the sine of the angle between
\( \overline{T_{j_1}} \) and \( \overline{T_{j_2}}'' \) is infinitesimal relative to \( \mathbb{R}_\delta \)). Therefore
\[
\det \left( \frac{\text{grad}_{x_1}(f)}{\| \text{grad}_{x_1}(f) \|}, \ldots, \frac{\text{grad}_{x_n}(f)}{\| \text{grad}_{x_n}(f) \|} \right) > c_1 > 0
\]
for a suitable \( c_1 \in \mathbb{R} \).

Taking the points \( x_1, \ldots, x_{n-i} \) as the points \( y_1, \ldots, y_{n-i} \) in Definition 2 we get that
\( x \) is 0-quasiangle in the semi-Pfaffian set \( W \cap \Pi(x) \), whence \( x \) is \( i \)-quasiangle because
the \( (n-i) \)-plane \( \Pi \in A_{n-i} \) was chosen arbitrarily. \( \square \)

**Corollary.** Let a point \( \tilde{x} \subset W \cap P_i \cap \mathbb{R}^n \) and the dimension in the point \( \tilde{x} \)
\( \dim_{\tilde{x}}(W \cap P_i) = i \), then
(a) \( \dim_{\tilde{x}}(A_i \cap P_i^\delta) = i \);
(b) \( \dim(A_i \cap P_i^\delta) = i \).

**Proof.** Lemma 3 and the remark following this lemma imply that for any \( 0 < \rho \in \mathbb{R}_\delta \)
which is infinitesimal relative to \( \mathbb{R} \), we have the inclusion \( (B_{\tilde{x}}(\rho) \cap P_i^\delta) \subset A_i \), this
provides (a).

Moreover, Lemma 3 and the remark imply that \( (B_{\tilde{x}}(\rho) \cap P_i^\delta) \subset st(A_i \cap P_i^\delta) \). Thus,
(b) follows from Lemma A.8. \( \square \)

**Lemma 4.** \( \dim(A_i) \leq i \).

**Proof.** First let us reduce the proof to the case \( i = 0 \), in which \( \tilde{A}_0, A_0 \) are defined for
a set \( W \) given by Pfaffian functions \( u_1, \ldots, u_K \) defined over \( \mathbb{R}_\delta \) (rather than \( \mathbb{R} \)), see
Section A.2.

Thus, let \( i \geq 1 \) and suppose that \( e = \dim(A_i) \geq i+1 \). Due to Corollary A.1, there exists
a nonsingular point \( y \in A_i \) such that \( \dim_{y}(A_{i}) = e \). Denote by \( T_y \) the tangent plane to
\( A_i \) at the point \( y \). Since \( \dim(T_y) = e \) one can find the \( (n-i) \)-subspace \( \Pi \in A_{n-i} \) such
that \( \dim(T_y \cap \Pi(y)) = e-i \). Take any \( (n-e) \)-subspace \( R \subset \Pi \) defined over \( \mathbb{R}_\delta \) for
which \( (T_y \cap R(y)) = \{ y \} \). Consider the linear orthogonal projection \( \pi : \mathbb{R}_\delta^e \rightarrow \mathbb{R}_\delta^e \)
on to \( e \)-subspace along \( R \). Then dim \( \pi(T_y) = e \). Therefore, \( \pi(A_i) \subset \mathbb{R}_\delta^e \) contains \( e \)-dimensional
ball \( B_{\mathbb{R}_\delta}(r) \) for a certain \( 0 < r \in \mathbb{R}_\delta \) (by the implicit function theorem and the
transfer principle).

For any point \( x \in A_i \) there is a point \( x' \in \tilde{A}_i \) such that \( st(x') = x \), hence \( st(\pi(A_i)) \supset B_{\mathbb{R}_\delta}(r) \).

By assumption the lemma is valid for the case \( i = 0 \). Then for any point \( z \in \mathbb{R}_\delta^n \),
applying this assumption to the set of 0-angle points of the intersection \( \Pi(z) \cap W \) we conclude
that the sub-Pfaffian set \( st(\Pi(z) \cap A_i) \) has the dimension at most 0 (taking
into account Definition 3 of \( i \)-quasiangle points and that \( \Pi(z) \) is defined over \( \mathbb{R}_\delta \).
Let us show that \( \pi(\tilde{A}) \) does not contain a ball \( B_0(r) \) for any \( 0 < r_1 \notin R_i \) and \( w \notin R_i^0 \). Assume the contrary, then there exists a point \( w \in B_0(r_1) \cap R_i^0 \). Let \( z_1 \in R_i^0 \) be a point such that \( \pi(z_1) = w \). Denote \( \Pi_1 = \pi(\Pi) \), then \( \dim \Pi_1 = e - 1, \Pi = \pi^{-1}(\Pi_1) \). Then the following inequalities hold:

\[
\dim st(\Pi_1(w_1) \cap \pi(\tilde{A})) \geq \dim st(\Pi_1(w_1) \cap B_0(r_1)) = e - 1 \geq 0
\]

On the other hand, \( \Pi_1(w_1) \cap \pi(\tilde{A}) = \pi(\tilde{A} \cap \Pi(z_1)) \), and, therefore,

\[
\dim st(\Pi_1(w_1) \cap \pi(\tilde{A})) \leq \dim st(\tilde{A} \cap \Pi(z_1)) \leq 0
\]

(the latter inequality was proved above). The obtained contradiction shows that \( \pi(\tilde{A}) \) does not contain a ball \( B_0(r) \) for any \( 0 < r_1 \notin R_i \).

We claim that for any ball \( B_2(r_2) \subset B_0(r) \) defined over \( R_i^0 \) such that \( 0 < r_2 \notin R_i^0 \), the intersection \( B_2(r_2) \cap \tilde{\pi}(\tilde{A}) \neq \emptyset \). Assume the contrary. Then either \( B_2(r_2) \subset \pi(\tilde{A}) \) or \( B_2(r_2) \cap \pi(\tilde{A}) = \emptyset \). The inclusion \( B_2(r_2) \subset \pi(\tilde{A}) \) is impossible as was shown above. If \( B_2(r_2) \cap \pi(\tilde{A}) = \emptyset \), then \( st(z_2) \notin st(\pi(\tilde{A})) \), the latter contradicts the inclusions \( st(\pi(\tilde{A})) \supset B_2(r_2) \supset B_2(r_2)/2 \) of the sets in the space \( R_i^0 \). This proves the claim.

Because of Lemma A.3, \( \dim(\tilde{\pi}(\pi(\tilde{A})) \leq e - 1 \). Applying Lemma A.8, we get

\[
\dim(\tilde{\pi}(\pi(\tilde{A}))) \leq e - 1
\]

On the other hand, we shall now prove that \( \dim(\tilde{\pi}(\pi(\tilde{A}))) \leq B_0(r) \). This contradiction completes the proof of the reduction of the lemma to the case \( i = 0 \). Indeed, let \( z_3 \in B_0(r) \). Observe that the set \( D = \{ ||z - z_3||^2 : z \in \tilde{\pi}(\tilde{A}) \} \) is sub-Pfaffian. Due to Corollary A.4, \( D \) is a finite union of points and intervals. Let \( \omega \) be the minimal among these points and the endpoints of these intervals. Suppose that \( z_3 \notin \tilde{\pi}(\tilde{A}) \), i.e. there does not exist \( z \in \tilde{\pi}(\tilde{A}) \) such that \( st(z) = z_3 \). Then \( \omega > r_3^2 \) for an element \( 0 < r_3 \notin R_i^0 \). It follows that \( B_2(r_3) \cap \tilde{\pi}(\tilde{A}) = \emptyset \). This contradicts the claim just proved.

Now let \( i = 0 \). Suppose the statement of the lemma is wrong and \( \dim(A) = s \geq 1 \). There is a linear projection \( \pi : R_i^0 \to R_i^s \) onto a certain coordinate \( s \)-subspace, such that \( \pi(A_0) \supset B_0(r) \) for some \( z \in R_i^s, \ 0 < r \in R_i^s \). Choose an open interval \( I \subset R_i^s \) of the length \( 2r \) passing through \( z \).

Our nearest purpose is to prove the existence of a sub-Pfaffian curve (i.e. a one-dimensional sub-Pfaffian set) \( C_0 \subset A_0 \) such that \( \pi(C_0) = I \) and the mapping \( \pi : C_0 \to I \) is bijective. This follows from the next, a more general construction.

Let \( V \subset F^m, U \subset F^m \) be sub-Pfaffian sets where \( F \) is one of the fields \( R_i \) defined in Section A.2 and let \( \varphi : V \to U \) be a sub-Pfaffian mapping (i.e. a mapping with a sub-Pfaffian graph). Let us describe one of the possible ways to construct a sub-Pfaffian set \( V_0 \subset V \) such that the restriction \( \varphi : V_0 \to \varphi(V) \) of \( \varphi \) is bijective.

For every point \( u \in \varphi(V) \) take the (unique) point \( v_u \in V \) such that \( \varphi(v_u) = u \) according to the following rule (actually, this rule is quite flexible).

A projection \( \pi_1(\varphi^{-1}(u)) \) of \( \varphi^{-1}(u) \) onto the axis \( X_1 \) is a union of a finite number of intervals (with or without endpoints) since \( \pi_1(\varphi^{-1}(u)) \) is sub-Pfaffian (see Corollary A.4). Let \( a_1, a_2 \) be the endpoints of the leftmost among these intervals (note that a sub-Pfaffian set is always bounded, see Definition A.4). Then \( (a_1 + a_2)/2 \in \pi_1(\varphi^{-1}(u)) \).
Consider the projection $\pi_2(\varphi^{-1}(u) \cap \{X_1 = (a_1 + a_2)/2\})$ onto the axis $X_2$. Continuing in a similar way, after $n$ steps we obtain a point $v_n = ((a_1 + a_2)/2, \ldots) \in \varphi^{-1}(u)$. We define $V_0$ as a set of all the obtained points $v_n$ for all $u \in \varphi(V)$. One can easily prove that $V_0$ is sub-Pfaffian and the mapping $\varphi : V_0 \rightarrow \varphi(V)$ is bijective.

Applying this construction to the mapping $\pi|_{\varphi^{-1}(L) \cap A_0} : \pi^{-1}(L) \cap A_0 \rightarrow L$ we get a required sub-Pfaffian curve $C_0 \subset A_0$. Since there are only a finite number of connected components of $C_0$ (see Corollary A.3), there exists a connected component $C$ such that $\pi(C)$ is an interval of a length $r_0 > 0$ for a certain $r_0 \in \mathbb{R}_\delta$. Then the completion $C^{(\delta)} \subset \mathbb{R}^n_\delta$ is a connected component of the curve $C_0^{(\delta)} \subset \mathbb{R}^n_\delta$ (see Section A.3).

Fix a nonsingular point $x \in C$ (due to Corollaries A.1 and A.4, C has only a finite number of singular points). Denote by $\tau \subset \mathbb{R}^n_\delta$ the tangent line to $C$ at $x$, then its completion $\tau^{(\delta)} \subset \mathbb{R}^n_\delta$ is tangent to $C^{(\delta)}$. After a suitable linear coordinate transformation (defined over $\mathbb{R}_\delta$) one can assume that $x = 0$ and $\tau$ coincides with the axis $X_n$. Denote by $\gamma$ the projection mapping on the axis $X_n$.

There exists $0 < \mu \in \mathbb{R}_\delta$ satisfying the following properties:

(i) the unique connected component $c$ of the intersection $C \cap \{-\mu < X_n < \mu\} \subset \mathbb{R}^n_\delta$, containing 0, is a nonsingular curve and the mapping $\gamma^{-1} : (-\mu, \mu) \rightarrow c$ is definable and doubly differentiable;

(ii) there exists $0 < \lambda \in \mathbb{R}_\delta$ such that for any $y \in (-\mu, \mu)$ the inequality $\|\gamma^{-1}(0, \ldots, 0, y) - (0, \ldots, 0, y)\| \leq \lambda|y|^2$ holds.

One can prove the existence of $\mu$ for the curves in $\mathbb{R}^n$ using the Taylor formula, and then for $C$ applying the transfer principle.

The transfer principle also implies that (i), (ii) hold for the completions $C^{(\delta)} \subset C^{(\delta)}$ and any $y \in (-\mu, \mu)^{(\delta)}$.

The angle between a line $\ell$ and a hyperplane $\mathcal{P}$ in $\mathbb{R}^n_\delta$ is defined as the difference between $\pi/2$ and the angle between $\ell$ and the vector orthogonal to $\mathcal{P}$. Observe that there exists $0 < \nu \in \mathbb{R}_\delta$ such that if $n$ normalized vectors $v_1, \ldots, v_n \in \mathbb{R}^n_\delta$ satisfy the inequality $|\det(v_1, \ldots, v_n)| > \nu$, then for any hyperplane $\mathcal{P}$ there is $i$, $1 \leq i \leq n$, for which the sine of the angle between $v_i$ and $\mathcal{P}$ is greater than $\nu$ (actually, one could take $v = v_i/2$ but we will not use this particular value).

Introduce the sub-Pfaffian set $V \subset \mathbb{R}^n_\delta$ consisting of all the points $z = (z_1, \ldots, z_n) \in \mathbb{R}^n_\delta$ such that

1. $z \in \{f = \nu_3\}$, $|z_n| < \mu$;
2. the sine of the angle between $\text{grad}_z(f - \nu_3)$ and the hyperplane $\{X_n = 0\}$ is greater than $\nu$;
3. for a given $z_n$ the minimum of the distance to the axis $X_n$ (i.e. of the function $(X_1^2 + \cdots + X_{n-1}^2)^{1/2}$) on the set of all the points satisfying (1), (2) is attained at $z$.

Let us apply the above construction to the projection $\gamma : V \rightarrow (-\mu, \mu)$. The construction supplies us with a sub-Pfaffian subset $V_0 \subset V$ such that each nonempty preimage $\gamma^{-1}(y)$ contains exactly one point from $V_0$. Therefore $\dim(V_0) \leq 1$.

We claim that, actually, $\dim(V_0) = 1$. Suppose the contrary, then $V_0$ would consist of a finite number of points (see Corollary A.4). We show, however, that $V_0$ contains infinitely many points.
Indeed, take an arbitrary point \( y \in \mathbb{R}_3 \cap (-\mu, \mu) \) and the (unique) point \( w \in c \) such that \( y(w) = (0, \ldots, 0, y) \). Since \( c \subset A_0 \) there exists (see Definition 4 of 0-angle points) a point \( w_1 \in A_0 \) such that \( st(w_1) = w \), therefore (see Definition 2 of 0-quasiangle points) there exists a point \( w_2 \in \{ f = e_3 \} \) for which \( ||w_1 - w_2|| \leq \varepsilon_2 \) and the sine of the angle between the vector \( \text{grad}_{w_2}(f - e_3) \) and the hyperplane \( \{ X_n = 0 \} \) is greater than \( \nu \) (see (0)). Because \( st\|w_2 - w\| = 0 \) and for the orthogonal projection \( \|y(w_2) - y(w)\| \leq \|w_2 - w\| \), we deduce that \( st(y(w_2)) = st(y(w)) = y \). Since the point \( w_2 \) satisfies the conditions (1), (2) in the definition of \( V \), there exists a point \( w_3 \in \{ f = e_3 \} \) such that \( y(w_3) = y(w_2) \), the sine of the angle between \( \text{grad}_{w_3}(f - e_3) \) and the hyperplane \( \{ X_n = 0 \} \) is greater than \( \nu \), and \( w_3 \) has the minimal distance to the axis \( X_n \) among the points with these properties. Then \( w_3 \in V \).

Thus, we have shown that for each point \( y \in \mathbb{R}_3 \cap (-\mu, \mu) \) there exists a point \( w_3 \in V \) such that \( st(y(w_3)) = (0, \ldots, 0, y) \). Because of the above construction, there exists the unique point \( w_4 \in V_0 \) for which \( y(w_4) = y(w_3) \). Hence \( V_0 \) contains an infinite number of points, i.e. \( \dim(V_0) = 1 \).

Let \( V_0 = \bigcup \gamma_i \) be the decomposition of \( V_0 \) into the connected components. Since \( V_0 \) is sub-Pfaffian, it has only a finite number of singular points and a finite number of points at which the tangent to the curve \( V_0 \) is orthogonal to the axis \( X_n \) (i.e. of the critical points of the mapping \( y \)), here we invoke Corollaries A.1 and A.4. It follows that each \( \gamma_i \) admits a finite partition \( \gamma_i = \bigcup_{i<j} \gamma_{ij} \bigcup_{i} \gamma_i \), where every \( \gamma_{ij} \) is a nonsingular connected sub-Pfaffian curve (without the endpoints) not containing the critical points of \( y \), and every \( \gamma_i \) is a set consisting of a single point.

We have shown above that \( st(y(V_0)) = [-\mu, \mu] \). Since \( \gamma(\gamma_i) \subset (-\mu, \mu) \) is connected (as an image of a connected curve), it is an interval, hence \( st(y(\gamma_i)) \subset [-\mu, \mu] \) is a closed interval. Therefore, there are \( i_0, j_0 \) for which an interval \( I = st(y(\gamma_{i_0j_0})) \) has a positive length \( |I| \in \mathbb{R}_3 \), besides \( I \) contains 0 and does not lie entirely to the left of 0.

Due to the implicit function theorem, one may represent the curve \( \gamma_{i_0j_0} \) in a parametrical form: \( (X_1(z_n), \ldots, X_n(z_n), X_{n+1}) \) where \( X_1, \ldots, X_{n+1} \) are smooth functions. Observe that for any point \( z = (X_1(z_n), \ldots, X_{n+1}(z_n), z_n) \in \gamma_{i_0j_0} \), the tangent vector \( (\dot{X}_1(z_n), \ldots, \dot{X}_{n+1}(z_n), 1) \) at this point to the curve \( \gamma_{i_0j_0} \) has a sine of the angle with the axis \( X_n \) greater than \( \nu \), since this tangent vector is orthogonal to \( \text{grad}_{e_3}(f - e_3) \), taking into account inclusions \( V_0 \subset V \subset \{ f = e_3 \} \). In other words, \( \sum_{1 \leq i \leq n+1} (\dot{X}_i(z_n))^2 > \nu^2/(1 - \nu^2) \).

For each pair of indices \( 1 \leq i < j \leq n - 1 \) either there are at most a finite number of the tangent vectors \( (\dot{X}_1(z_n), \ldots, \dot{X}_{n+1}(z_n), 1) \) at the points of the curve \( \gamma_{i_0j_0} \) such that \( \dot{X}_i(z_n) = \pm \dot{X}_j(z_n) \) or all these vectors satisfy one of the two conditions: \( \dot{X}_i(z_n) = \dot{X}_j(z_n) \) or \( \dot{X}_i(z_n) = -\dot{X}_j(z_n) \), because \( \gamma_{i_0j_0} \) is sub-Pfaffian. Therefore, there exists a connected sub-Pfaffian curve \( \gamma \subset \gamma_{i_0j_0} \) for which the length of the interval \( st(y(\gamma)) \in \mathbb{R}_3 \) is positive, besides \( st(y(\gamma)) \) contains 0 and does not lie entirely to the left of 0. Apart from that, either \( |\dot{X}_i(z_n)| \neq |\dot{X}_j(z_n)| \), for any pair \( 1 \leq i < j \leq n - 1 \) and any point \( (X_1(z_n), \ldots, X_{n+1}(z_n), z_n) \in \gamma \), or for a certain pair \( 1 \leq i < j \leq n - 1 \), one of the two conditions \( \dot{X}_i(z_n) = \dot{X}_j(z_n) \) or \( \dot{X}_i(z_n) = -\dot{X}_j(z_n) \) holds for any point from \( \gamma \). Let us assume that \( |\dot{X}_i(z_n)| \neq |\dot{X}_j(z_n)| \) for any pair \( 1 \leq i < j \leq n - 1 \) (the case \( |\dot{X}_i(z_n)| = |\dot{X}_j(z_n)| \)
can be treated in a similar way). There exists $s$, $1 \leq s \leq n - 1$, such that $|\hat{X}_s(z_n)| > |\hat{X}_j(z_n)|$, $1 \leq j \leq n - 1$, $s \neq j$, for all the points for $\gamma'$. Moreover, $\hat{X}_s(z_n)$ has a constant nonnegative sign for all the points from $\gamma'$. For definiteness suppose that $\hat{X}_s(z_n) > 0$ for all the points from $\gamma'$ (the case $\hat{X}_s(z_n) < 0$ can be considered in a similar manner).

Then $\hat{X}_s(z_n) > \sqrt{(n - 1)(1 - v^2))^{1/2}} = v_0 \in \mathbb{R}_e$, and $v_0 > 0$.

Let an interval $[0, \mu_3] \subset st(\gamma'(\gamma')) \subset [-\mu, \mu]$, where $0 < \mu_2 \in \mathbb{R}_d$. Then for any $\mu_3$, $\mu_4 \in \mathbb{R}_d$, such that $0 < \mu_3 < \mu_4 < \mu_2$, the completion of the interval $[\mu_3, \mu_4]^{(0)} \subset \gamma'(\gamma')$. Since $\hat{X}_s(z_n) > v_0$ for any $z_n \in [\mu_3, \mu_4]^{(0)}$, for any point $\eta \in [\mu_3, \mu_4]^{(0)}$ the inequality

$$X_\eta(\eta) - X_s(\mu_3) \geq v_0(\eta - \mu_3)$$

holds. Indeed, the latter statement could be written as a formula of the first-order theory of real closed fields, in the case of the field $\mathbb{R}$ it is true because

$$X_\eta(\eta) - X_s(\mu_3) = \int_{\mu_3}^\eta \hat{X}_s \geq v_0(y - \mu_3),$$

then use the transfer principle.

Let $y \in (-\mu, \mu) \cap \mathbb{R}_d$. We have proved above that for the unique point $w = \gamma^{-1}(0, \ldots, 0, y) \in c \subset A_0$ there exists a point $w_1 \in \widetilde{A}_0$ such that $st(w_1) = w$, besides there exists a point $w_2 \in \{f = e_3\}$ such that $||w_1 - w_2|| \leq e_2$ and the sine of the angle between the vector $\text{grad}_w(f - e_3)$ and the hyperplane $\{X_n = 0\}$ is greater than $\nu$. Then the distance from $w_2$ to the axis $X_n$ does not exceed $||w_2 - w|| + ||w_1 - w|| + ||w - (0, \ldots, 0, y)|| \leq e_2 + ||w_1 - w|| + \lambda y^2 \leq \lambda_0 y^2$ for $\lambda \in \mathbb{R}_d$, introduced in (ii) above, and any $\lambda < \lambda_0 \in \mathbb{R}_d$. So the distance to the axis $X_n$ from the unique point $w_4 \in V_0$, for which $\gamma(w_4) = \gamma(w_2)$, also does not exceed $\lambda_0 y^2$. Note that $st(\gamma(w_4)) = (0, \ldots, 0, y)$.

On the other hand if $y \in [\mu_3, \mu_4] \cap \mathbb{R}_d$, then applying the above argument to the point $(y + \mu_3)/2$ instead of $y$ we prove the existence of a point $w_5 \in V_0$ such that $st(\gamma(w_5)) = (0, \ldots, 0, (y + \mu_3)/2)$ and the distance to the axis $X_n$ from the point $w_5$ does not exceed $\lambda_0((y + \mu_3)/2)^2$. Arguing as above, we get

$$X_\eta(w_4) - X_s(w_5) \geq v_0||\gamma(w_4) - \gamma(w_5)|| > v_1(y - \mu_3)/2$$

for arbitrary $v_1 \in \mathbb{R}_e$, $0 < v_1 < v_0$. Then either the distance from the point $w_4$ to the axis $X_n$ or the distance from the point $w_5$ to $X_n$ is greater than $v_1(y - \mu_3)/4$, on the other hand both distances do not exceed $\lambda_0 y^2$. Taking any $y$, $0 < y \in \mathbb{R}_d$, such that $y < v_1/(v_1 + 4\lambda_0)$ and $\mu_3 = y^2$, we get a contradiction because $v_1(y - y^2)/4 > \lambda_0 y^2$. □

3. Flat points

**Definition 4.** Let $0 \leq i \leq n - 1$. A point $x \in A_i$ is $i$-flat if there exists an $i$-plane $\Pi$, passing through $x$ such that $\text{dim}(\Pi \cap A_i) = i$.

Denote by $\phi_i \subset A_i$ the set of $i$-flat points. Note that for $i = 0$ Lemma 4 implies that $\text{dim} A_0 \leq 0$, i.e. $A_0$ consists of at most a finite set of points (see Corollary A.4), therefore $\phi_0 = A_0$. 

Lemma 5. (a) There is at most a finite number of i-planes $\Pi$ such that $\dim (\Pi \cap \phi_i) = i$.

(b) $\phi_i$ is contained in the union of all i-planes described in (a).

Proof. If $\phi_i = \emptyset$, the lemma is trivial, so suppose that $\phi_i \neq \emptyset$. Since $\phi_0 = A_0$ consists of a finite number of points, the lemma for the case $i = 0$ is obvious. So, in what follows we assume that $i \geq 1$.

(b) is evident. Note that if $\Pi$ satisfies (a) then $\dim (\Pi \cap A_i) = i$ since $\phi_i \subset A_i$.

Introduce a set $\hat{\phi}_i \subset \phi_i$ consisting of all the points $y \in \phi_i$ for which there exists an i-plane $\Pi$ passing through $y$, such that for suitable $0 < r \in \mathbb{R}_\delta$ we have $B_y(r) \cap \Pi \subset \phi_i$.

The set $\hat{\phi}_i$ is obviously sub-Pfaffian.

Besides, $\dim \hat{\phi}_i = i$. Indeed, Lemma 4 implies that $\dim \hat{\phi}_i \leq i$. On the other hand as $\phi_i \neq \emptyset$, there exists i-plane $\Pi$ such that $\dim (\Pi \cap A_i) = i$, hence $\Pi \cap A_i \supset \Pi \cap B_y(r_1)$ for some $v_1 \in \Pi$, $0 < r_1 \in \mathbb{R}_\delta$. Then $\Pi \cap B_y(r_1) \subset \hat{\phi}_i$, i.e. $\dim \hat{\phi}_i \geq i$.

If suffices to prove that there exists only a finite number of i-planes $\Pi$ for which $\dim (\Pi \cap \hat{\phi}_i) = i$. This would imply the item (a) of the lemma since for any i-plane $\Pi$ such that $\dim (\Pi \cap \hat{\phi}_i) = i$ we have $\dim (\Pi \cap \phi_i) = i$.

Denote by $\hat{\phi}_i \subset \hat{\phi}_i$ the set of all nonsingular points of $\hat{\phi}_i$. The set $\hat{\phi}_i \subset \hat{\phi}_i$ of all singular points is sub-Pfaffian and $\dim (\hat{\phi}_i \subset \hat{\phi}_i) \leq i - 1$ (see Corollary A.1 and Lemma A.2). For any point $y_2 \in \hat{\phi}_i$, there is the unique i-plane $\Pi'$ such that for an appropriate $0 < r_2 \in \mathbb{R}_\delta$ we have $B_{y_2}(r_2) \cap \Pi' \subset \hat{\phi}_i$. Then for a suitable $0 < r_3 \in \mathbb{R}_\delta$, a certain neighbourhood of $y_3$ in $\hat{\phi}_i$ coincides with $B_{y_3}(r_3) \cap \Pi'$, moreover $B_{y_3}(r_3) \cap \Pi'$ is a neighbourhood of $y_3$ in $\hat{\phi}_i$.

If $\dim (\Pi \cap \hat{\phi}_i) = i$ for i-plane $\Pi$ then $\Pi \cap \hat{\phi}_i$ contains a nonsingular point $y_3 \in \hat{\phi}_i$ (since $\dim (\hat{\phi}_i \subset \hat{\phi}_i) \leq i - 1$); moreover a neighbourhood of $y_3$ in $\hat{\phi}_i$ coincides with $B_{y_3}(r_4) \cap \Pi$ for a suitable $0 < r_4 \in \mathbb{R}_\delta$. Thus, it is sufficient to show that there are only a finite number of i-planes $\Pi$ such that $\dim (\Pi \cap \hat{\phi}_i) = i$.

Each connected component of $\hat{\phi}_i$ is contained in an i-plane $\Pi$, since for any point $y_4 \in \hat{\phi}_i$ its certain neighbourhood in $\hat{\phi}_i$ coincides with $B_{y_4}(r_5) \cap \Pi''$ for some $0 < r_5 \in \mathbb{R}_\delta$ and i-plane $\Pi''$. Because the number of connected components of $\hat{\phi}_i$ is finite (see Corollary A.3), the number of i-planes $\Pi$ such that $\dim (\Pi \cap \hat{\phi}_i) = i$ is also finite. \qed

Lemma 6. If a connected component $\varphi$ of $\phi_i$ has a nonempty intersection $\varphi \cap P_i \neq \emptyset$ with an i-facet $P_i$ of $P$, then $\varphi \subset P_i$.

Proof. First we prove for a connected component $\varphi_0$ of $\phi_i$ the following statement: if $\varphi_0 \cap \text{cl}(P_i) \neq \emptyset$ then $\varphi_0 \subset \text{cl}(P_i)$. Assume the contrary. Then there exists a point $y \in \varphi_0 \cap \overline{P_i}$ such that $y \in \text{cl}(\varphi_0 \cap \overline{P_i}) \subset \text{cl}(\hat{\phi}_i \cap \overline{P_i})$. Due to Lemma 5, there is a finite family $\mathcal{P}$ of i-planes $\Pi$ such that $\dim (\Pi \cap \hat{\phi}_i) = i$ and $\phi_i$ lies in the union of all these i-planes. Let us show that there exists $\Pi' \in \mathcal{P}$, $\Pi' \neq \overline{P_i}$ such that $y \in \Pi'$. Indeed, let
$y_j \to y$, where $y_j \in \phi_0 \setminus \overline{P_i}$. For each $j$ there is $\Pi'' \in \mathcal{P}$ such that $y_j \in \Pi''$ (obviously $\Pi'' \neq \overline{P_i}$). Since $\mathcal{P}$ is finite there exists an infinite subsequence $y_j$, $1 \leq \ell < \infty$, and $\Pi'' \in \mathcal{P}$ for which $y_j \in \Pi''$, $1 \leq \ell < \infty$. Thus $y \in \Pi'' \neq \overline{P_i}$.

Since $\phi_i \subset A_i \subset W \cap \mathbb{R}^n_\delta$ (see the remark following Definition 4) the function $f$ vanishes on the intersection of $\Pi''$ with the domain of $f$ (see Lemma A.1), taking into account that $\dim(\Pi'' \cap \phi_i) = i$. Besides $u_{K_{i+1}}(y) > 0, \ldots, u_K(y) > 0$, therefore $u_{K_{i+1}}, \ldots, u_K$ are positive also in $B_j(\rho)$ for an appropriate $0 < \rho \in \mathbb{R}_\delta$. Hence $\Pi'' \cap B_j(\rho) \subset W \cap \mathbb{R}^n_\delta$. This contradicts the inclusion $W \cap \mathbb{R}^n_\delta \subset P(\delta)$ because $y$ belongs to the closure $cl(P_i)$ of the $i$ facet of the convex polyhedron $P(\delta)$. Thus $\phi_0 \subset cl(P_i)$, and the statement is proved.

To complete the proof of the lemma it suffices to show that $\phi \cap (cl(P_i) \setminus P_i) = \emptyset$. If $z \in \phi \cap (cl(P_i) \setminus P_i)$ then there is another $i$-facet $P_i'$ of $P$ such that $z \in cl(P_i')$. Then, by the proved above, $\phi \subset cl(P_i')$, this contradicts $\phi \cap P_i \neq \emptyset$. □

Our next purpose is to explicitly describe (see Lemma 7 below) the sufficient condition for the $i$-flatness of a point $x \in A_i$ by means of the Pfaffian formula with a purely existential quantifier prefix.

Let $\Pi$ be an $i$-plane containing $x$ and, for some points $v_1, \ldots, v_i \in \Pi \cap A_i$, the vectors $v_1, x, \ldots, v_i$ be linearly independent. Denote by $y_1, \ldots, y_{(i+1)n}$ the coordinates of the vectors $x, v_1, \ldots, v_i$. Due to Lemma A.9(1), the degree of sub-Pfaffian transcedency $[y_1, \ldots, y_{(i+1)n}]_{[\mathbb{R}_\delta]} \leq (i+1)n \leq n^2$. Introduce the points $w^{(j)} = x + \sum_{1 \leq \ell \leq i} \delta^{(j)}(v_\ell - x) \in \Pi$, $1 \leq j \leq n^2 + 1$.

Lemma 7. Let the points $x, v_1, \ldots, v_i \in A_i \cap \Pi$. If $w^{(1)}, \ldots, w^{(n^2 + 1)} \in A_i \cap \Pi$, then $x$ is $i$-flat and moreover $\dim(A_i \cap \Pi) = i$.

Proof. Suppose that, on the contrary, $\dim(A_i \cap \Pi) \leq i - 1$. Consider the sub-Pfaffian set $\mathcal{A} \subset \mathbb{R}^{(i+1)n+i}_\delta$ consisting of all the points

$$(y_1, \ldots, y_n, y_{1,1}, \ldots, y_{1,n}, y_{2,1}, \ldots, y_{2,n}, \ldots, y_{i,1}, \ldots, y_{i,n}, z_1, \ldots, z_i)$$

for which $(y_1, \ldots, y_n) + \sum_{1 \leq \ell \leq i} z_{\ell}(y_{\ell,1}, \ldots, y_{\ell,n}) - (y_1, \ldots, y_n)) \in A_i$ (cf. the expressions for $w^{(j)}$). Then $\mathcal{A}$ is definable over $\mathbb{R}_1$ since $A_i$ is definable over $\mathbb{R}_1$ (see the remark following Definition 4). Besides $\dim(\mathcal{A} \cap \{y_1, \ldots, y_n, y_{1,1}, \ldots, y_{1,n}, \ldots, y_{i,1}, \ldots, y_{i,n}\}) = \dim(A_i \cap \Pi) \leq i - 1$ by the supposition. According to Definition A.10, this means that $[(\delta^{(1)}_1, \ldots, \delta^{(1)}_{i+1}) : (y_1, \ldots, y_{(i+1)n})]_{[\mathbb{R}_1]} \leq i - 1$ for each $1 \leq j \leq n^2 + 1$ since $w^{(j)} \in A_i \cap \Pi$. Applying Lemma A.10 several times, proceeding by induction on $j$, and taking into account that $[(\delta^{(1)}_1, \ldots, \delta^{(1)}_{i+1}) : (y_1, \ldots, y_{(i+1)n})]_{[\mathbb{R}_1]} \leq [(\delta^{(1)}_1, \ldots, \delta^{(1)}_{i+1}) : (y_1, \ldots, y_{(i+1)n})]_{[\mathbb{R}_1]} \leq n^2 + j(i - 1)$ for each $0 \leq j \leq n^2 + 1$.

Putting $j = n^2 + 1$ leads to a contradiction since $[y_1, \ldots, y_{(i+1)n}, \delta^{(1)}_1, \ldots, \delta^{(1)}_{i+1}, \ldots, \delta^{(i+1)}_1, \ldots, \delta^{(i+1)}_{i+1}]_{[\mathbb{R}_1]} \geq [\delta^{(1)}_1, \ldots, \delta^{(1)}_{i+1}, \ldots, \delta^{(i+1)}_1, \ldots, \delta^{(i+1)}_{i+1}]_{[\mathbb{R}_1]} = i(n^2 + 1)$ because of Lemma A.9(2). □
Definition 6. A point \( y \in \tilde{A}_i \) is called \( i \)-pseudoflat if there exist the points \( v_1, \ldots, v_i \in \tilde{A}_i \) such that
\[
1 \det(v_1 - y, \ldots, v_i - y)^T(v_1 - y, \ldots, v_i - y) > e_i \]
(where \( (v_1 - y, \ldots, v_i - y)^T \) denotes the transposition of an \( n \times i \) matrix with the columns \( v_1 - y, \ldots, v_i - y \)) and the points \( y + \sum_{1 \leq r \leq i} \delta_r^{(i)}(v_r - y) \in \tilde{A}_i, 1 \leq j \leq n^2 + 1 \).

The sub-Pfaffian set of all \( i \)-pseudoflat points is denoted by \( \tilde{\phi}_i \).

Lemma 8. If \( \dim(W \cap P_i) = i \) then \( W \cap P_i \cap \mathbb{R}^n \subset \tilde{\phi}_i \).

Proof. Let \( \tilde{x} \in W \cap P_i \cap \mathbb{R}^n \). Take arbitrary points \( v_1, \ldots, v_i \in W \cap P_i \cap \mathbb{R}^n \) such that the vectors \( v_1 - \tilde{x}, \ldots, v_i - \tilde{x} \) are linearly independent, then
\[
\mathbb{R} \ni \det(v_1 - \tilde{x}, \ldots, v_i - \tilde{x})^T(v_1 - \tilde{x}, \ldots, v_i - \tilde{x}) > 0,
\]
obviously
\[
| \det(v_1 - \tilde{x}, \ldots, v_i - \tilde{x})^T(v_1 - \tilde{x}, \ldots, v_i - \tilde{x}) | > e_i.
\]
The distance from a point \( \tilde{w}^{(j)} = \tilde{x} + \sum_{1 \leq r \leq i} \delta_r^{(j)}(v_r - \tilde{x}) \in \mathbb{R}^n \) to \( \tilde{x} \) is infinitesimal relative to \( \mathbb{R}^n \) for each \( 1 \leq j \leq n^2 + 1 \). Lemma 3 implies that \( \tilde{w}^{(j)} \in \tilde{A}_i, 1 \leq j \leq n^2 + 1 \), hence \( \tilde{x} \in \tilde{\phi}_i \) by Definition 6. \( \square \)

Lemma 9. \( st(\tilde{\phi}_i) \subset \phi_i \).

Proof. Let \( \tilde{y} \in \tilde{\phi}_i \) and \( v_1, \ldots, v_i \in \tilde{A}_i \) satisfy Definition 6. Observe that
\[
| \det(st(v_1) - st(\tilde{y}), \ldots, st(v_i) - st(\tilde{y}))^T(st(v_1) - st(\tilde{y}), \ldots, st(v_i) - st(\tilde{y})) | > e_i/2,
\]
taking into account Lemma A.4 and that the points \( \tilde{y}, v_1, \ldots, v_i \in A_i \subset W \subset P \) are \( \mathbb{R} \)-finite (see Section A.2). Furthermore, \( st(\tilde{y}) + \sum_{1 \leq r \leq i} \delta_r^{(i)}(st(v_r) - st(\tilde{y})) \in st(\tilde{A}_i) = A_i, 1 \leq j \leq n^2 + 1 \). Denote by \( I \) the unique \( i \)-plane passing through the points \( st(\tilde{y}), st(v_1), \ldots, st(v_i) \). Lemma 7 entails that \( st(\tilde{y}) \in \phi_i \) and \( \dim(\mathbb{R} \cap A_i) = i \). \( \square \)

Let \( \bar{\phi}_i = \bigcup_j \bar{\phi}_j, \phi_i = \bigcup_j \phi_j \) be the representations of \( \tilde{\phi}_i \) and \( \phi_i \), respectively, as the unions of (necessarily sub-Pfaffian, see Section A.3) connected components. Lemmas A.6, and A.7 imply that \( st(\bar{\phi}_j) \) is a sub-Pfaffian connected set. Hence due to Lemma 9, for each \( j \) there is \( \ell \) such that \( st(\bar{\phi}_j) \subset \phi_{\ell} \). For any \( i \)-facet \( P_i \) of \( P \) such that \( \dim(W \cap P_i \cap \mathbb{R}^n) = i \), Lemma 8 entails that \( W \cap P_i \cap \mathbb{R}^n \subset \bar{\phi}_i \). Take a point \( x \in W \cap P_i \cap \mathbb{R}^n \), then \( x \in \bar{\phi}_j \) for a certain \( j \). It follows that \( st(\bar{\phi}_j) \subset \phi_{\ell} \) for a suitable \( \ell \), thus \( x = st(x) \in st(\bar{\phi}_j) \subset \phi_{\ell} \). Due to Lemma 6, \( \phi_{\ell} \subset P_i \). So, to any facet \( P_i \) such that \( \dim(W \cap P_i \cap \mathbb{R}^n) = i \) corresponds (not necessarily unique) connected component \( \bar{\phi}_j \), and to different such \( i \)-facets \( P_i, P'_i \) correspond different connected components, respectively. Thus, we obtain the following lemma.

Lemma 10. The number of \( i \)-facets \( P_i \) such that \( \dim(W \cap P_i \cap \mathbb{R}^n) = i \) does not exceed the number of connected components of \( \bar{\phi}_i \).

Observe that \( \bar{\phi}_i \) can be defined by a Pfaffian formula \( \psi \) having a prefix with only existential quantifiers. Moreover, the prefix contains \( O(n^4) \) quantifiers, since for each of
O(n^2) points \( v_1, \ldots, v_i, y + \sum_{1 \leq j \leq i} \delta_i^{(j)}(v_j - y), 1 \leq j \leq n^2 + 1 \), the formula \( \psi \) expresses the condition of membership to the set \( A_i \) (see Definition 6), which, in turn, requires \( O(n^2) \) existential quantifiers (see Definitions 2 and 3), namely for the coordinates of the points \( v_1, \ldots, v_n \). The polynomials occurring in \( \psi \), and the polynomials of the type \( g_{i,j} \) occurring in the definition of Pfaffian functions \( u_0, \ldots, u_K \) (see the beginning of Section 1) have degrees less than \( O(dn) \) (cf. (0)). The number of all these polynomials (i.e. the number of atomic subformulas of \( \psi \)) can be bounded by \( n^{O(1)}K \) (see Lemma 1 and Definitions 2 and 3). Therefore, the number of all connected components of the sub-Pfaffian set \( \tilde{\phi}_i \) does not exceed \( 2^{K^2}(dnK)^{O(K+n^4)} \), due to Corollary A.2. Together with Lemma 10 this implies the following lemma.

**Lemma 11.** The number of \( i \)-facets \( P_i \) such that \( \dim(W \cap P_i \cap \mathbb{R}^n) = i \) does not exceed \( 2^{K^2}(dnK)^{O(K+n^4)} \).

In order to complete the proof of the theorem one observes that the Pfaffian computation tree \( \mathcal{T} \) contains at most \( 3^K \) branches and for each \( 0 \leq i \leq n-1 \) for each \( i \)-facet \( P_i \) there is a branch of \( \mathcal{T} \) such that \( \dim(W' \cap P_i \cap \mathbb{R}^n) = i \), where \( W' \) is the accepting set corresponding to this branch. Hence \( N \leq 3^K 2^{K^2}(dnK)^{O(K+n^4)} \). Together with the assumption \( N \geq (dn)^{\Omega(n^4 \log d)} \), this entails the inequality \( K \geq \Omega(\sqrt{\log N}) \).

**Appendix. Sub-Pfaffian sets**

**A.1. Gabrielov's theorem and Khovanskii's bound**

In this section we give definitions and describe some properties of concepts related to Pfaffian functions and to the subsets of \( \mathbb{R}^n \) defined by these functions. We skip all the proofs which could be found elsewhere.

The concept of Pfaffian function was introduced by Khovanskii [19, 20], who had established their fundamental properties.

**Definition A.1.** A subset \( A \subset \mathbb{C}^n \) is called complex analytic variety if any point of \( \mathbb{C}^n \) has a neighbourhood \( U \) such that the intersection \( A \cap U \) coincides with the set \( \{ g_1 = \cdots = g_k = 0 \} \cap U \), where \( g_1, \ldots, g_k \) are complex analytic (holomorphic) functions on \( U \) (see e.g. [21]).

We say that a real analytic function \( f \) has a domain \( G \subset \mathbb{R}^n \) if there is an open subset \( \mathcal{G} \subset \mathbb{R}^n \) such that \( f \) is defined on \( \mathcal{G} \) and \( G \subset \mathcal{G} \).

**Definition A.2.** (a) A Pfaffian chain of length \( r \) and degree \( d_1 \geq 1 \) is a sequence of real analytic functions \( f_1, \ldots, f_r \) with the following properties:

1. For each \( 1 \leq j \leq r \) there exists a complex analytic function \( \tilde{f}_j \) defined in a subset \( \tilde{G}_j \subset \mathbb{C}^n \), such that \( \mathbb{C}^n \setminus \tilde{G}_j \) is a complex analytic variety, and \( f_j \) is the restriction of \( \tilde{f}_j \) on \( \mathbb{R}^n \).
Observe that as real analytic function $f_j$ has a domain $G_j \subset \tilde{G} \cap \mathbb{R}^n$.

Let $\tilde{G} = \bigcap_{1 \leq j \leq r} \tilde{G}_j$ and $G = \bigcap_{1 \leq j \leq r} G_j$.

2. Every $f_j$, $1 \leq j \leq r$, satisfies a Pfaffian equation

$$df_j(X) = \sum_{1 \leq j \leq r} g_{ij}(X, f_1(X), \ldots, f_j(X)) \, dX_j$$

for $1 \leq j \leq r$. Here $X = (X_1, \ldots, X_n)$, $g_{ij} \in \mathbb{R}[X, Y_1, \ldots, Y_r]$, $\deg_{X, Y_1, \ldots, Y_r}(g_{ij}) \leq d_1$.

(b) A function $f(X) = P(X, f_1(X), \ldots, f_r(X))$, where $P \in \mathbb{R}[X, Y_1, \ldots, Y_r]$, $\deg_{X, Y_1, \ldots, Y_r}(P) \leq d_2$, is called a Pfaffian function (with a Pfaffian chain $f_1, \ldots, f_r$) of length $r$ and degree $d = d_1 + d_2$.

Note that our definition of a Pfaffian function is more restrictive than a usual one (see [19, 20]) due to the requirement of existence of $f_j$.

**Examples** *(the exposition follows [8]).*

1. Pfaffian functions of length 0 and degree $d+1$ are polynomials of degree not exceeding $d$.

2. The exponential function $f(X) = e^{aX}$ is Pfaffian of length 1 and degree 2, with $\tilde{G} = \mathbb{C}$, $G = \mathbb{R}$, due to the equation

$$df(X) = af(X) \, dX.$$

3. The function $f(X) = 1/X$ is Pfaffian of length 1 and degree 3 with $\tilde{G} = \{X \neq 0\} \subset \mathbb{C}$, $G = \{X \neq 0\} \subset \mathbb{R}$, due to the equation $df(X) = -f^2(X) \, dX$.

4. Logarithm $f(X) = \ln(X)$ is Pfaffian of length 2 and degree 3 with $\tilde{G} = \{X \neq 0\} \subset \mathbb{C}$, $G = \{X > 0\} \subset \mathbb{R}$,

$$df(X) = g(X)^2 \, dX, \quad dg(X) = -g^2(X) \, dX,$$

where $g(X) = 1/X$.

5. Tangent $f(X) = \tan(X)$ is Pfaffian of length 1 and degree 3 with

$$\tilde{G} = \bigcap_{k \in \mathbb{Z}} \{X \neq \frac{\pi}{2} + k\pi\} \subset \mathbb{C}, \quad G = \tilde{G} \cap \mathbb{R},$$

due to the equation $df(X) = (1 + f^2(X)) \, dX$.

6. Cosine $\cos(X)$ is Pfaffian of length 2 and degree 3 with

$$\tilde{G} = \bigcap_{k \in \mathbb{Z}} \{X \neq \pi + 2k\pi\} \subset \mathbb{C}, \quad G = \tilde{G} \cap \mathbb{R},$$

due to the equations

$$\cos(X) = 2f(X) - 1, \quad df(X) = -f(X)g(X) \, dX, \quad dg(X) = 1/2(1 + g^2(X)) \, dX,$$

where $f(X) = \cos^2(X/2)$ and $g(X) = \tan(X/2)$. 
(7) Sine $f(X) = \sin(X)$ is Pfaffian of length 3 and degree 3 in $G = \bigcap_{k \in \mathbb{Z}} \{X \neq \pi + 2k\pi\} \subset \mathbb{C}$, $G = \tilde{G} \cap \mathbb{R}$, due to the equations $df = g(X) \, dx$ where $g(X) = \cos(X)$.

Let us now list some elementary properties of Pfaffian functions, describing the behaviour of their parameters under the basic operations (the proofs are simple, see e.g. [8]).

1. The sum and the product of two Pfaffian functions $f_1$ and $f_2$ of lengths $r_1$ and $r_2$, degrees $d_1$ and $d_2$, with $G = \tilde{H}_1$, $G = \tilde{H}_2$, $G = H_1$, $G = H_2$, respectively, are Pfaffian functions of length $r_1 + r_2$, degree $d_1 + d_2$ and with $G = \tilde{H}_1 \cap \tilde{H}_2$, $G = H_1 \cap H_2$ for both the sum and the product. If two Pfaffian functions are defined by the same Pfaffian chain of length $r$, the length of the sum and the product is also $r$.

2. A partial derivative of a Pfaffian function of length $r$ and degree $d$ is a Pfaffian function of length $r$ and degree $2d$.

3. Let $X = (X_1, \ldots, X_n)$, $Z = (Z_1, \ldots, Z_r)$ be tuples of variables and $f$ be a Pfaffian function in $X, Z$ of length $r_1$, degree $d_1$ and with $G = \tilde{H}_1 \subset \mathbb{C}^{n + r}$, $G = H_1 \subset \mathbb{R}^{n + r}$.

Let $h = (h_1, \ldots, h_r)$ be an $r$-tuple of Pfaffian functions in $X$ of length $r_2$, degree $d_2$, with a common Pfaffian chain, with $G = \tilde{H}_2 \subset \mathbb{C}^n$, $G = H_2 \subset \mathbb{R}^n$, such that $(x, h(x)) \in H_1$ for all $x \in H_2$. Then the complex analytic function $\hat{g} \equiv \hat{f}(X, \hat{h}(X))$ (see (a)(1) of Definition A.2) is defined in a subset $\tilde{H}_3 \subset \mathbb{C}^n$ such that $\mathbb{C}^n \setminus \tilde{H}_3$ is a complex analytic variety of a dimension smaller than $n$. Indeed, the preimage of the complex analytic variety $\mathbb{C}^n \setminus \tilde{H}_1$ in $\mathbb{C}^n \setminus \tilde{H}_2$, under the map $\hat{h}$, is also a complex analytic variety different from $\mathbb{C}^n$ since $\hat{g}$ is a composition of analytic functions. Therefore, the dimension of this preimage is less than $n$ (see [21]). An easy computation (see [8]) shows that $g \equiv f(X, h(X))$ is a Pfaffian function in $G_2$ of length $r_1 + r_2$ and degree $d_1d_2$.

**Lemma A.1.** Let $f$ be a Pfaffian function with $G \subset \mathbb{R}^n$ and $L \subset \mathbb{R}^n$ a $p$-plane. If there exist $x \in G \cap L$ and $r$, $0 < r \in \mathbb{R}$, such that $f$ vanishes in the intersection $L \cap B_x(r)$ then $f$ vanishes in $G \cap L$ (here $B_x(r)$ denotes an open $n$-dimensional ball centred at $x$ with radius $r$).

**Proof.** Consider the complex analytic function $\hat{f}$ corresponding to $f$ as in Definition A.2, and the complex $p$-plane $\tilde{L}$, defined in $\mathbb{C}^n$ by the same system of linear equations as $L$. Since $\tilde{L}$ is an irreducible complex analytic variety, either it is contained in the variety $\mathbb{C}^n \setminus \tilde{G}$ or in the complex dimension $\dim_C(\tilde{L} \cap (\mathbb{C}^n \setminus \tilde{G})) < \dim_C(\tilde{L})$ (by the dimension of the intersection theorem, see [21]). The first alternative is impossible because $x \in L \subset \tilde{L}$. Since $\dim(L \cap B_x(r)) = p$, the second alternative implies that the complex analytic function $\hat{f}$ is defined on $p$-plane $\tilde{L}$ everywhere except a subset $\tilde{L} \setminus \tilde{G}$ of a dimension less than $p$, and vanishes on a subset of complex dimension $p$. Since $\tilde{G} \cap \tilde{L}$ is connected in the topology with the base of all open balls of $\tilde{L}$, treated as $2p$-
dimensional real space, we conclude that \( \tilde{f} \) vanishes on \( \tilde{L} \cap \tilde{G} \). Hence \( \tilde{f} \) is identically zero on \( \tilde{L} \). It follows that the restriction \( f \) of \( \tilde{f} \) vanishes on \( G \cap L \subset \mathbb{R}^n \). \( \square \)

Next we define by induction two closely linked notions: quantifier-free Pfaffian formula and semi-Pfaffian set. Again, our definitions will be more restrictive than the original ones (see \([19,20,7]\)).

**Definition A.3.** Let \( h_0 \) be a Pfaffian chain of length 1, with \( h_0 \) defined in \( \mathbb{R}^n \). A quantifier-free formula of rank 0 is an expression of the form

\[
\phi^{(0)} = \bigvee_{1 \leq i \leq s_0} (f_{i1}^{(0)} = \cdots = f_{in}^{(0)} = 0 \& g_{i1}^{(0)} > 0 \& \cdots \& g_{in}^{(0)} > 0),
\]

where \( f_{ij}^{(0)}, g_{ij}^{(0)} \) are Pfaffian functions (called atomic functions), with \( h_0 \) as a common Pfaffian chain (see Definition A.2(b)), thus, in particular, \( f_{ij}^{(0)}, g_{ij}^{(0)} \) are defined in \( \mathbb{R}^n \). Suppose that we had already defined a concept of a quantifier-free Pfaffian formula \( \chi^{(\ell)} \) of rank \( \ell, 0 \leq \ell \in \mathbb{Z} \). A semi-Pfaffian set \( W \subset \mathbb{R}^n \), determined by \( \chi^{(\ell)} \), is the set of all points \( x \in \mathbb{R}^n \), satisfying \( \chi^{(\ell)} \). We write \( W = \{ \chi^{(\ell)} \} \). A quantifier-free Pfaffian formula of rank \( \ell + 1 \) is of the form

\[
\phi^{(\ell+1)} = \bigvee_{1 \leq i \leq s_{\ell+1}} (f_{i1}^{(\ell+1)} = \cdots = f_{in}^{(\ell+1)} = 0 \& g_{i1}^{(\ell+1)} > 0 \& \cdots \& g_{in}^{(\ell+1)} > 0),
\]

where \( f_{ij}^{(\ell+1)}, g_{ij}^{(\ell+1)} \) are Pfaffian functions with the common Pfaffian chain \( h_0, \ldots, h_{\ell+1} \). Here the function \( h_{\ell+1} \) is defined in a domain \( G \) which is a closure of a semi-Pfaffian set of the kind \( \{ \chi^{(\ell)} \} \), where \( \chi^{(\ell)} \) is a quantifier-free Pfaffian formula of rank \( \ell \). Functions \( f_{ij}^{(\ell+1)}, g_{ij}^{(\ell+1)} \) together with all atomic functions occurring in the description of \( \chi^{(\ell)} \) are called atomic functions of \( \phi^{(\ell+1)} \).

**Example.** The set \( \{ \tan(X) = 0 \& a < X < b \} \subset \mathbb{R} \), where \( -\pi/2 < a < b < \pi/2 \), is semi-Pfaffian, defined by a quantifier-free Pfaffian formula. On the other hand, the set \( \{ \tan(X) = 0 \} \cap \bigcup_{k \in \mathbb{Z}} \{ a + k\pi < X < b + k\pi \} \subset \mathbb{R} \) for \( -\pi/2 < a < b < \pi/2 \) (cf. Example (5) above) is not semi-Pfaffian.

**Definition A.4.** Fix a certain \( R, 0 < R \in \mathbb{R} \), and let \( \mathcal{X}^n \subset \mathbb{R}^n \) be the \( n \)-dimensional cube centred at the origin and having an edge with length \( 2R \). A Pfaffian formula is an expression of the form \( \psi = Q_1 Y_1 Q_2 Y_2 \cdots Q_t Y_t(\Phi) \) where \( \Phi \) is a quantifier-free Pfaffian formula of arbitrary rank (called quantifier-free part of \( \psi \)) with atomic functions in \( n+t \) variables \( Y_1, \ldots, Y_t, X_1, \ldots, X_n \) and \( Q_i, 1 \leq i \leq t \), are quantifiers \( \exists \) or \( \forall \), each restricted on the interval \( (-R,R) \subset \mathbb{R} \). A sub-Pfaffian set \( V \subset \mathbb{R}^n \), determined by \( \psi \), is the set of all points \( x \in \mathcal{X}^n \), satisfying \( \psi \). We write \( V = \{ \psi \} \).

We say that two Pfaffian formulas \( \psi, \psi' \) are equivalent if \( \{ \psi \} = \{ \psi' \} \).

**Definition A.5.** The local dimension \( \dim_p(V) \) of a set \( V \) at a point \( x \in V \) is the maximal \( p, 0 \leq p \in \mathbb{Z} \), such that the linear projection of a neighbourhood of \( x \) in

onto a coordinate $p$-subspace (along all the rest of the coordinates) contains a $p$-dimensional ball. The dimension $\dim(V)$ of $V$ is the maximal value $\dim_x(V)$ for all $x \in V$.

**Definition A.6.** A point $x$ of a set $V \subset \mathbb{R}^n$, with $\dim(V) = p$, is called analytically nonsingular (or nonsingular) if a neighbourhood of $x$ in $V$ is analytically diffeomorphic (respectively, $C^1$-diffeomorphic) to an open $p$-dimensional ball. Denote by $V_a^0$ (or by $V^0$) the set of all analytically nonsingular (respectively, nonsingular) points of $V$. The points of the set $V_a^* = V \setminus V_a^0$ (respectively $V^* = V \setminus V^0$) are called analytically singular (respectively, singular).

For a set $V \subset \mathbb{R}^n$ denote by $cl(V)$ its closure in the topology with a base of all open balls in $\mathbb{R}^n$.

**Definition A.7.** For a set $V \subset \mathbb{R}^n$ the disjoint family $\{V_i\}$ of subsets $V_i \subset V$ is called a smooth stratification of $V$ if

1. $V = \bigcup_i V_i$,
2. each $V_i$, called a stratum, is an analytic manifold in $\mathbb{R}^n$,
3. if $V_i \cap cl(V_j) \neq \emptyset$, then $V_i \cap cl(V_j)$ and $\dim(V_i) < \dim V_j$.

**Proposition A.1** ([16, 26]). For any sub-Pfaffian set $V \subset \mathbb{R}^n$ there exists a finite smooth stratification.

**Corollary A.1.** $\dim(V^*) < \dim(V)$.

**Proof.** The inequality $\dim(V^*) < \dim(V)$ directly follows from Proposition A.1, the inequality $\dim(V^*) \leq \dim(V_a^*)$ follows from the obvious inclusion $V^* \subset V_a^*$.

**Lemma A.2.** For a sub-Pfaffian set $V \subset \mathbb{R}^n$ the subsets $V^0$ and $V^*$ are sub-Pfaffian.

**Proof.** The sets $V^0$ and $V^*$ can be described by appropriate Pfaffian formulas involving a Pfaffian formula defining $V$.

**Definition A.8.** For a set $V \subset \mathbb{R}^n$ the boundary $\partial V$ is a subset of all points $x \in \mathbb{R}^n$ such that for every $r$, $0 < r \in \mathbb{R}$, the intersections $B_x(r) \cap V \neq \emptyset, B_x(r) \cap (\mathbb{R}^n \setminus V) \neq \emptyset$.

**Lemma A.3.** For a sub-Pfaffian set $V \subset \mathbb{R}^n$ the dimension $\dim(\partial V) \leq n - 1$.

**Proof.** Let $\{V_i\}$ be a finite smooth stratification of $V$, see Proposition A.1. Suppose first that $\dim(V) < n$. Then, the closure $cl(V) = \bigcup_i cl(V_i) = \bigcup_i \partial V_i = \partial V$. On the other hand, $\dim(cl(V)) = \dim(V)$ [7], hence the lemma is valid in this case.

Now let $\dim(V) = n$. The set $V$ is representable as $V = V_{\text{max}} \cup V_{\text{min}}$, where $V_{\text{max}}$ is the union of all $n$-dimensional strata of $V$, and $V_{\text{min}}$ is the union of the remaining
strata (of smaller dimensions). Then

\[ \dim(\partial V) \leq \dim(\partial V_{\text{max}} \cup \partial V_{\text{min}}) = \dim((cI(V_{\text{max}}) V_{\text{max}}) \cup \partial V_{\text{min}}) \]

\[ = \max\{\dim(cI(V_{\text{max}}) V_{\text{max}}), \dim(\partial V_{\text{min}})\}. \]

According to [7], \( \dim(cI(V_{\text{max}}) V_{\text{max}}) < \dim(V_{\text{max}}) \). The inequality \( \dim(\partial V_{\text{min}}) \leq n - 1 \) was proved before. \( \square \)

**Definition A.9.** Let \( \psi \) be a Pfaffian formula having \( N \) atomic Pfaffian functions in \( n \) variables with the same Pfaffian chain of length \( r \) and degrees less than \( d \). The 4-tuple \((N, n, r, d)\) is called the format of \( \psi \).

**Proposition A.2** ([7], Theorem 2). For a Pfaffian formula \( \psi \) of a format \((N, n, r, d)\) there exists an equivalent formula \( \psi' \) having only existential quantifiers, and of the format \((N', n', r', d')\), where the values \( N', n', r', d' \) are bounded from above by the value of a suitable function in \( N, n, r, d \).

**Proposition A.3** ([7], Theorem 1). For a sub-Pfaffian set \( \{\psi\} \subset \mathbb{R}^n \) with a Pfaffian formula \( \psi \) of a format \((N, n, r, d)\), any of its connected components can be defined by a Pfaffian formula of a format \((N', n', r', d')\), where the values \( N', n', r', d' \) are bounded from above by the value of an appropriate function in \( N, n, r, d \).

**Proposition A.4** ([19, 20]). The number of connected components of a semi-Pfaffian set \( \{\Phi\} \) defined by a quantifier-free formula \( \Phi \) of the format \((N, n, r, d)\) does not exceed \( 2^r n^{O(r)} (Nd)^{O(r+n)} \).

There is a generally adopted conjecture that, under the hypothesis of Proposition A.4, the bound \( n^{O(r)} (Nd)^{O(r+n)} \) is actually true.

**Corollary A.2.** The number of connected components of a sub-Pfaffian set \( \{\psi\} \), defined by a formula \( \psi \) of the format \((N, n, r, d)\) in which only existential quantifiers can occur, does not exceed \( 2^r n^{O(r)} (Nd)^{O(r+n)} \).

**Proof.** It is sufficient to note that the number of connected components of a projection of a set does not exceed the number of connected components of the set itself. \( \square \)

**Corollary A.3.** The number of the connected components of an arbitrary sub-Pfaffian set \( \{\psi\} \), defined by a formula \( \psi \) of a format \((N, n, r, d)\), is finite, moreover, it is bounded from above by the value of a certain function in \( N, n, r, d \).

**Proof.** Apply to \( \psi \) successively Proposition A.3 and Corollary A.2.

**Corollary A.4.** A zero-dimensional sub-Pfaffian set in \( \mathbb{R}^n \) is finite. A sub-Pfaffian set in \( \mathbb{R}^1 \) is a finite union of points and (open, closed or semiclosed) intervals. In each
case the number of points or intervals is bounded from above by the value of a certain function in the format of a formula representing the sub-Pfaffian set.

Proof. Directly follows from Lemma A.2 and Corollary A.3. □

A.2. Sub-Pfaffian sets over nonstandard extensions of reals

In the main text of the paper we consider the extensions of the field \( \mathbb{R} \) with "nonstandard" (in particular, infinitesimal) elements. The following digest from nonstandard analysis is taken from [27], for a detailed exposition see [6].

There exists a sequence of ordered fields

\[ \mathbb{R}_0 \subset \mathbb{R}_1 \subset \mathbb{R}_2 \subset \cdots \subset \mathbb{R}_k \subset \cdots \]

in which the field \( \mathbb{R}_k \), \( k \geq 1 \), contains an element \( \varepsilon_k > 0 \) infinitesimal relative to the elements of \( \mathbb{R}_{k-1} \) (i.e. for every positive element \( a \in \mathbb{R}_{k-1} \) the inequality \( \varepsilon_k < a \) is true). In addition, for every function \( \varphi : \mathbb{R}_{k-1}^n \rightarrow \mathbb{R}_{k-1} \) there exists a natural extension, being a function \( \varphi \) from \( \mathbb{R}_k^n \) to \( \mathbb{R}_k \). It follows, invoking characteristic functions, that each subset \( S \subset \mathbb{R}_{k-1}^n \) has a natural extension to \( \mathbb{R}_k^n \). We say that \( \mathbb{R}_j \) is a nonstandard extension of \( \mathbb{R}_i \) for \( 0 \leq i < j \).

Consider the language \( \mathcal{L}_k \), \( k \geq 0 \), of the first-order predicate calculus, in which the set of all function symbols is in bijective correspondence with the set of all functions of several arguments from \( \mathbb{R}_k \) taking values in \( \mathbb{R}_k \) and the only predicate is the equality relation. We shall say that the closed (i.e. containing no free variables) formula \( \Phi \) of the language \( \mathcal{L}_k \) is true in \( \mathbb{R}_k \), \( k \geq 0 \), if and only if the statement expressed by this formula with respect to \( \mathbb{R}_k \) is true. The following "transfer principle" is valid: for all integers \( 0 \leq i < j \) the closed formula \( \Phi \) of \( \mathcal{L}_i \) is true in \( \mathbb{R}_i \) if and only if it is true in \( \mathbb{R}_j \).

An element \( z \in \mathbb{R}_k \), \( k \geq 1 \), is called infinitesimal relative to \( \mathbb{R}_j \), \( j < k \), if for every \( 0 < w \in \mathbb{R}_j \) the inequality \( |z| < w \) is valid. An element \( z \in \mathbb{R}_k \) is called infinitely large if \( z = 1/z' \), where \( z' \) is infinitesimal. If \( z \in \mathbb{R}_k \) is not infinitely large relative to \( \mathbb{R}_j \), \( z \) is called \( \mathbb{R}_j \)-finite.

One can prove [6] that if an element \( z \in \mathbb{R}_k \) is \( \mathbb{R}_j \)-finite then there exist unique elements \( z_1 \in \mathbb{R}_j \) and \( z_2 \in \mathbb{R}_k \), where \( z_2 \) is infinitesimal relative to \( \mathbb{R}_j \), such that \( z = z_1 + z_2 \). In this case \( z_1 \) is called the standard part of \( z \) (relative to \( \mathbb{R}_j \)) and is denoted by \( z_1 = \text{st}_j(z) \). One can extend the operation \( \text{st}_j \) (componentwise) to vectors from \( \mathbb{R}_k^n \) and (elementwise) to subsets of \( \mathbb{R}_k^n \).

In what follows, all the functions \( \varphi \) we shall consider in \( \mathbb{R}_k^n \) \( k \geq 0 \), will be Pfaffian. By this we mean that for each \( \varphi \) there exists a Pfaffian function \( \varphi' \) definable over \( \mathbb{R} \) (i.e. in the sense of Definition A.2) such that \( \varphi \) is the result of a replacement of some variables in \( \varphi' \) by some elements of \( \mathbb{R}_k \).

Moreover, we assume that the domain \( G \subset \mathbb{R}_k^n \) of \( \varphi \) is a sub-Pfaffian set, defined by a Pfaffian formula \( \Pi \) with atomic functions definable over \( \mathbb{R} \) and some variables replaced by elements from \( \mathbb{R}_k \). We say that \( \varphi \) is definable over \( \mathbb{R}_k \).
For any $\ell > k$, the same function $\varphi'$, formula $\Pi$ and the replacements determine the function $\varphi^{(\ell)} : G^{(\ell)} \rightarrow \mathbb{R}$, which coincides with $\varphi$ in $\mathbb{R}_k$ and is called the completion of $\varphi$ over $\mathbb{R}_k$, similarly $G^{(\ell)} \subset \mathbb{R}$ (determined by $\Pi$) is called the completion of $G$ over $\mathbb{R}_k$.

Basic notions, introduced in Section A.1, can be naturally extended to a nonstandard field $\mathbb{R}_k$ for $k > 0$. Thus, we shall consider semi-Pfaffian sets, sub-Pfaffian sets, Pfaffian formulas, determined in $\mathbb{R}_k^n$ by Pfaffian functions definable over $\mathbb{R}_k$. In this case we say that the sets and formulas are definable over $\mathbb{R}_k$.

If a sub-Pfaffian set $W \subset \mathbb{R}_k^n$ is determined in $\mathbb{R}_k^n$ by a Pfaffian formula $\Phi$ with atomic subformulas definable over $\mathbb{R}_k$ then the sub-Pfaffian set in $\mathbb{R}_k^n$, $\ell > k$, determined by the same formula in which the atomic functions are replaced by their completions is called the completion of $W$ and is denoted by $W^{(\ell)}$.

Some of the basic statements proved earlier in this Appendix can be extended (using the transfer principle) to the fields $\mathbb{R}_k$ for $k > 0$. This obviously concerns the statements: Lemma A.1, Corollary A.1, Lemma A.2, Lemma A.3, Proposition A.2, Corollary A.4. Propositions A.3, A.4 and Corollaries A.2, A.3 about the estimates of the connected components are also extendable (see below).

The following lemma illustrates a use of the transfer principle and the notion of the standard part.

**Lemma A.4.** Let $f : S \rightarrow \mathbb{R}_k$ be a Pfaffian function defined in a sub-Pfaffian bounded set $S \subset \mathbb{R}_k^n$. Denote by $S^{(k+1)}$ the completion of $S$ over $\mathbb{R}_{k+1}$ and by $f^{(k+1)}$ the completion of $f$. Then for any point $x \in S^{(k+1)}$ such that $B_x(r) \subset S^{(k+1)}$ for some $r > 0 < r \in \mathbb{R}_k$, the standard part $st_k(f^{(k+1)}(x)) = f(st_k(x))$. If, in addition, there do not exist $y \in S$ and $R > 0 < R \in \mathbb{R}_k$, such that $f(z) = 0$ for all $z \in B_y(R)$, and besides $f(w) \geq 0$ for all $w \in S$, then

$$st_k\{f^{(k+1)} = e_{k+1}\} = \{f = 0\}.$$

**Proof.** First, observe that any Pfaffian function is continuous. This is true for a Pfaffian function $\varphi$ definable over $\mathbb{R}$ (since $\varphi$ is analytic, see Definition A.2), then the Pfaffian formula of the language $L_0$ expressing continuity is valid for the completion $\varphi^{(\ell)}$, $\ell > 0$, due to the transfer principle, and hence it is valid as well for Pfaffian functions definable over arbitrary $\mathbb{R}_k$. The equality $st_k(f^{(k+1)}(x)) = f(st_k(x))$ and thereby the inclusion $st_k(\{f^{(k+1)} = e_{k+1}\}) \subset \{f = 0\}$ follows from the continuity of $f$ and $f^{(k+1)}$.

Now let $x \in \{f = 0\}$. Take $r, 0 < r \in \mathbb{R}_k$, such that $B_x(r) \subset S$ (cf. Definition A.2). Consider a sub-Pfaffian set $D = \{\|x - z\| : z \in S^{(k+1)}, f^{(k+1)}(z) = e_{k+1}\} \subset \mathbb{R}_{k+1}$. If it is empty, then $f^{(k+1)}$ is less than $e_{k+1}$ everywhere on the ball $B_x(r)$, by virtue of the theorem on intermediate values of continuous functions which holds for Pfaffian functions by the transfer principle, hence $f$ vanishes everywhere on the ball $B_x(r) \cap \mathbb{R}_k^n$ and we get a contradiction. Due to Corollary A.4 the set $D$ consists of a finite union of points and intervals. Denote by $u$ the minimum of these points and endpoints of these
intervals. If \( s_k(u) > 0 \) then the function \( f^{(k+1)} \) on the ball \( B_x(\sqrt{u}) \cap B_x(r) \) takes values less than \( e_{k+1} \) because of the continuity of \( f^{(k+1)} \). Therefore, \( f \) vanishes everywhere on the ball

\[
B_x(\sqrt{u}) \cap B_x(r) \cap \mathbb{R}^n = B_x(st_k(\sqrt{u})/2) \cap B_x(r) \cap \mathbb{R}^n
\]

with a positive radius from \( \mathbb{R}_k \) (cf. above). The obtained contradiction shows that \( s_k(u) = 0 \). Take any point \( w \) such that \( f^{(k+1)}(w) = e_{k+1} \) and \( ||w-x||^2 \leq u + e_{k+1} \), then \( s_k(w) = x \).

**Lemma A.5.** Let a sub-Pfaffian set \( W \subset \mathbb{R}_k^n \), defined by a Pfaffian formula \( \Pi \), be finite. Then the completion \( W^{(\ell)} \subset \mathbb{R}_\ell^n \), \( \ell > k \), of \( W \) coincides with \( W \).

**Proof.** Let \( W = \{x^{(1)}, \ldots, x^{(t)}\} \). Then the following formula of the language \( L_k \) is true over \( \mathbb{R}_k \):

\[
\forall x^{(1)} \cdots \forall x^{(t)} \left( \forall 1 \leq i \leq t \left( (X_1, \ldots, X_n) \neq x^{(i)} \Rightarrow -\Pi(X_1, \ldots, X_n) \right) \right).
\]

By the transfer principle, this formula is also true over \( \mathbb{R}_\ell \).

For a Pfaffian function \( f : G \to \mathbb{R}_k \), \( G \subset \mathbb{R}_k^n \) a point \( x \in G \) is called the critical point of \( f \) if the gradient vector \( (\partial f/\partial X_1, \ldots, \partial f/\partial X_n)(x) = 0 \). The value \( f(x) \) is called, in this case, the critical value of \( f \). The value which is not critical is called regular.

**Corollary A.5.** For a Pfaffian function \( f \) definable over \( \mathbb{R}_k \), any element \( x \in \mathbb{R}_\ell \setminus \mathbb{R}_k \) for \( \ell > k \) cannot be a critical value of \( f \).

**Proof.** Observe that the set \( \Gamma_k \subset \mathbb{R}_k \) of all critical values of \( f \) is sub-Pfaffian and definable over \( \mathbb{R}_k \).

Suppose first that \( k = 0 \). Then Corollary A.4 implies that \( \Gamma_0 \) consists of a finite number of points and segments. Moreover, by Sard’s theorem, \( \Gamma_0 \) actually consists of a finite number of points. For all sub-Pfaffian sets of the form \( \Gamma_0 \) and having a fixed format, the latter statement can be expressed by a formula of the language \( L_0 \) (taking into account that the number of points is bounded via the format). Hence, by the transfer principle the statement is true for any \( k \geq 0 \), i.e. \( \Gamma_k \) is finite.

According to Lemma A.5, the completion \( \Gamma_k^{(\ell)} = \Gamma_k \subset \mathbb{R}_k \), and, therefore, \( x \notin \Gamma_k^{(\ell)} \).

**Corollary A.6.** Let a Pfaffian function

\[
f : G \to \mathbb{R}_k, \ G \subset \mathbb{R}_k,
\]

be definable over \( \mathbb{R}_k \) and \( f \neq 0 \) on \( G \). If \( x \in \mathbb{R}_\ell \setminus \mathbb{R}_k \) for \( \ell > k \) then \( f(x) \neq 0 \).

**Proof.** According to Lemma A.1 and Corollary A.4, the set \( W \) of roots of \( f \) is finite. Apply Lemma A.5 to \( W \).
A.3. Connected components of sub-Pfaffian sets over nonstandard fields

Now we are going to extend the notion of the connected component to the sub-Pfaffian sets definable over $\mathbb{R}_k^m$, $k \geq 1$. Observe that a direct way to do this, starting with the topology on $\mathbb{R}_k^m$ with the base of all open balls, would lead to unnatural objects, e.g. the segment $[0,1] \subset \mathbb{R}_k$ is not connected in this topology. The analogous construction of connected components for semialgebraic sets over nonstandard fields was described in [14].

Let $V = \{ \Pi \}$ be a sub-Pfaffian set in $\mathbb{R}_n^m$ determined by a Pfaffian formula $\Pi$. Proposition A.3 and Corollary A.3 imply the existence of a function $\omega : \mathbb{N} \to \mathbb{N}$ such that if the elements of the 4-tuple format of $\Pi$ are bounded from above by some $\mathcal{N} \in \mathbb{N}$, then:

1. The number of connected components does not exceed $\omega(\mathcal{N})$.

2. For each connected component $V_i$ of $V$ there exists a Pfaffian formula $\Pi_i$ of a format with components not exceeding $\omega(\mathcal{N})$, such that $V_i = \{ \Pi_i \}$.

It follows that for a given positive integer $\mathcal{N}$, there exists a Pfaffian formula $\Omega_\mathcal{N}$ of the language $\mathcal{L}_0$, expressing the existence of a decomposition of any sub-Pfaffian set $V = \{ \Pi \}$ of the format of $\Pi$ less than $\mathcal{N}$ into its connected components

$$V = \bigcup_i \{ \Pi_i \},$$

such that the format of every $\Pi_i$ and the number of $\Pi_i$ are less than $\omega(\mathcal{N})$. Moreover, the formula $\Omega_\mathcal{N}$ states that for each pair of indices $i_1 \neq i_2$ the components $\{ \Pi_{i_1} \}$ and $\{ \Pi_{i_2} \}$ are “separated”, i.e. the following Pfaffian formula of the language $\mathcal{L}_0$ is valid:

$$\forall (x \in \{ \Pi_{i_1} \}) \exists z > 0 \forall (y \in \{ \Pi_{i_2} \})(\|x - y\| \geq z).$$

Besides, the formula $\Omega_\mathcal{N}$ claims the connectedness of every component $\{ \Pi_i \}$, this means that there do not exist two “separated” sub-Pfaffian subsets of $\{ \Pi_i \}$, each determined by a Pfaffian formula with format less than $\omega(\omega(\mathcal{N}))$.

Apart from that, for given positive integers $\mathcal{N}$, $\mathcal{M}$ one can verify a formula $\Omega_{\mathcal{N}, \mathcal{M}}$ of language $\mathcal{L}_0$ expressing the following statement. If a sub-Pfaffian set $\{ \Pi \}$ (where the format of $\Pi$ is less than $\mathcal{N}$) can be represented as a union of more than one and less than $\mathcal{M}$ pairwise “separated” sub-Pfaffian sets, each being determined by a Pfaffian formula of $\mathcal{L}_0$ of a format less than $\mathcal{M}$, then $\{ \Pi \}$ can be represented as a union of more than one and less than $\omega(\mathcal{N})$ pairwise “separated” connected sub-Pfaffian sets, each being determined by a Pfaffian formula of $\mathcal{L}_0$ of a format less than $\omega(\mathcal{N})$.

Applying the transfer principle to the formulas $\Omega_{\mathcal{N}, \mathcal{M}}$, $\Omega_{\mathcal{N}, \mathcal{M}}$ for all positive integers $\mathcal{N}$, $\mathcal{M}$, we conclude that any sub-Pfaffian set, defined over $\mathbb{R}_k$, $k \geq 0$, can be uniquely represented as a union of its pairwise “separated” connected components, moreover, each component is sub-Pfaffian and is connected, i.e. it cannot be represented as a union of more than one pairwise “separated” sub-Pfaffian sets.

Having defined the connected components of a sub-Pfaffian set definable over $\mathbb{R}_k$, $k \geq 0$, one can use the transfer principle to extend to this set Propositions A.3, A.4 and Corollaries A.2, A.3.
Lemma A.6. Let $V \subset \mathbb{R}^n_k$, $W \subset \mathbb{R}^n_{k+l}$ be two sub-Pfaffian sets and $V = st_k(W)$. Let

$$V = \bigcup_m V_m, \quad W = \bigcup_{i} W_i$$

be the decompositions of the sets $V, W$ into their connected components. Then, for every index $m$ there exist such indices $\ell_1, \ldots, \ell_s$ that $st(W_{\ell_1} \cup \cdots \cup W_{\ell_s}) = V_m$. Moreover, for each $\ell$ there exists the unique index $m$ such that $st(W_\ell) \subset V_m$.

**Proof.** Is almost a verbatim repetition of the proof of Lemma 1 in [14]. $\square$

For a sub-Pfaffian set $W \subset \mathbb{R}^n_k$, $k \geq 0$, we denote by $cl(W)$ its closure in the topology in $\mathbb{R}^n_k$ with the base of all open balls.

Lemma A.7 (cf. [25]). Let $W_Y = \{y\} \subset \mathbb{R}^n_{k+l}$ be a sub-Pfaffian set determined by a Pfaffian formula $v_Y$ in which the atomic Pfaffian functions are in variables $X_1, \ldots, X_n$, $Y_1, \ldots, Y_{l}$, $Z_1, \ldots, Z_s$, where the first $n + t$ variables occur free. Let, for the sequence of fields $\mathbb{R}_k \subset \mathbb{R}_{k+1} \subset \cdots \subset \mathbb{R}_{l}$, the element $e_{k+l+1}$ be infinitesimal relative to $\mathbb{R}_{k+l}$ for $0 \leq i \leq l - 1$. Denote by $v_Y$ the Pfaffian formula which is the result of the replacement of $Y_i$ by $e_{k+l}$ for every $1 \leq i \leq l$; let $W_e = \{v_Y\} \subset \mathbb{R}^n_{k+l}$. Then the set $V = st_k(W_e) \subset \mathbb{R}^n_k$ is sub-Pfaffian.

**Proof.** It is sufficient, due to Proposition A.2, to prove the lemma for the case $v_Y = \exists Z_1, \ldots, \exists Z_s(\Phi_Y)$ with quantifier-free $\Phi_Y$. Observe that $W_e = \pi(\{v_Y\})$ where $\Phi_e$ is a quantifier-free formula, being the result of the replacement of $Y_i$ by $e_{k+l}$, $1 \leq i \leq l$ in $\Phi_Y$, and $\pi$ is the linear projection map on the subspace of coordinates $X_1, \ldots, X_n$ along the coordinates $Z_1, \ldots, Z_s$.

The proof can be conducted by induction on $t$, in which an $i$th induction step proves that the set $st_{k+i-1}(W_e)$ is sub-Pfaffian. It will be obvious from formula (4) below that the output of the inductive step, namely, the set $st_{k+i-1}(W_e)$, satisfies the requirements for the set $W_e$ of the lemma, i.e. there exists a sub-Pfaffian set $W'_Y$, determined by a Pfaffian formula $v'_Y$ in variables $X_1, \ldots, X_n$, $Y_1, \ldots, Y_{l-i}$, $Z'_1, \ldots, Z'_s$, where the first $n + t - i$ variables occur free, such that $st_{k+i-1}(W_e) = \{v'_Y\}$, where $v'_Y$ is the result of the replacement of $Y_i$ by $e_{k+l}$ for every $1 \leq i \leq l - i$.

Thus, we assume that $t = 1$.

We can identify the sets $\{\Phi_Y\}$ and $\{\Phi_Y \& (Y_1 = e_{k+1})\}$.

Let us prove that

$$st_k(\{\Phi_Y \& (Y_1 = e_{k+1})\}) = cl(\{\Phi_Y \& (Y_1 > 0)\}) \cap \{Y_1 = 0\}. \quad (2)$$

Observe that the right-hand side of equality (2) is a sub-Pfaffian set.

Let $x \in st_k(\{\Phi_Y \& (Y_1 = e_{k+1})\})$, then there exists $z \in \{\Phi_Y \& (Y_1 = e_{k+1})\}$ such that $x = st_k(z)$. Hence, $x \in \{Y_1 = 0\}$. Suppose that $x \notin cl(\{\Phi_Y \& (Y_1 > 0)\})$. Then there exists an element $r$, $0 < r \in \mathbb{R}_k$, such that $B_\varepsilon(r) \cap \{\Phi_Y \& (Y_1 > 0)\} = \emptyset$. This contradicts the inclusion $z \in \{\Phi_Y \& (Y_1 = e_{k+1})\} \subset \{\Phi_Y \& (Y_1 > 0)\}$.

Suppose now that

$$x \in cl(\{\Phi_Y \& (Y_1 > 0)\}) \cap \{Y_1 = 0\},$$

i.e. $x$ belongs to the right-hand side of (2).
Let us prove the following claim: for any element \( R, 0 < R \in \mathbb{R}_k \), there exists an element \( x, 0 < x \in \mathbb{R}_k \), such that for every \( \beta, 0 < \beta \subset \mathbb{R}_k \), \( \beta < x \) the intersection
\[
B_\epsilon(R) \cap \{(\Phi_Y \& (Y_1 = \beta)\}
\]
is nonempty. Indeed, since the set \( B_\epsilon(R) \cap \{(\Phi_Y \& (Y_1 > 0)\) is sub-Pfaffian, and thus has a finite number of connected components (see the considerations preceding the lemma), there exists a connected component \( U \) of this set such that \( x \in cl(U) \). One can take as \( x \) the \( Y_1 \)-coordinate of any point from \( U \) and the claim is proved.

It follows (with the help of the transfer principle) that for every fixed \( R, 0 < R \in \mathbb{R}_k \), the intersection
\[
B_\epsilon(R) \cap \{(\Phi_Y \& (Y_1 = e_{k+1})\}
\]
is nonempty. Indeed, since the set \( B_\epsilon(R) \cap \{(\Phi_Y \& (Y_1 > 0)\) is sub-Pfaffian, and thus has a finite number of connected components (see the considerations preceding the lemma), there exists a connected component \( U \) of this set such that \( x \in cl(U) \). One can take as \( x \) the \( Y_1 \)-coordinate of any point from \( U \) and the claim is proved.

It follows (with the help of the transfer principle) that for every fixed \( R, 0 < R \in \mathbb{R}_k \), the intersection
\[
B_\epsilon(R) \cap \{(\Phi_Y \& (Y_1 = e_{k+1})\}
\]
is nonempty. Indeed, since the set \( B_\epsilon(R) \cap \{(\Phi_Y \& (Y_1 > 0)\) is sub-Pfaffian, and thus has a finite number of connected components (see the considerations preceding the lemma), there exists a connected component \( U \) of this set such that \( x \in cl(U) \). One can take as \( x \) the \( Y_1 \)-coordinate of any point from \( U \) and the claim is proved.

Observe that the set \( A = \{(z-x)^2 : z \in \{(\Phi_Y \& (Y_1 = e_{k+1})\)\} \subset \mathbb{R}_{k+1} \) is sub-Pfaffian. Due to Corollary A.4, \( A \) is a finite union of points and intervals. Let \( w \in \mathbb{R}_{k+1} \) be the minimal among these points and the endpoints of these intervals.

Suppose that \( x \notin st_k(\{(\Phi_Y \& (Y_1 = e_{k+1})\}) \), i.e. there does not exist \( z \in \{(\Phi_Y \& (Y_1 = e_{k+1})\) such that \( st_k(z) = x \). Thus, \( w > r_1^2 \) for an element \( 0 < r_1 \in \mathbb{R}_k \). It follows that \( B_\epsilon(r_1) \cap \{(\Phi_Y \& (Y_1 = e_{k+1})\} = \emptyset \). This contradicts (3) for \( R = r_1 \), and equality (2) is proved.

We have
\[
st_k(W_z) = st_k(\pi(\{(\Phi_Y \& (Y_1 = e_{k+1})\}))
\]
\[
= \pi(st_k(\{(\Phi_Y \& (Y_1 = e_{k+1})\}))
\]
\[
= \pi(cl(\{(\Phi_Y \& (Y_1 > 0)\}) \cap \{Y_1 = 0\}).
\]

The latter set is obviously sub-Pfaffian, this proves the lemma. \( \square \)

**Lemma A.8.** Let \( W \subset \mathbb{R}_{k+1}^n \) be a sub-Pfaffian set, \( V = st_k(W) \subset \mathbb{R}_k^n \). Then \( \dim(V) \leq \dim(W) \).

**Proof.** Suppose the contrary, let \( \dim W = \ell - 1, \dim(V) \geq \ell \). There exists a linear projection \( \pi : \mathbb{R}_{k+1}^n \to \mathbb{R}_k^\ell \) definable over \( \mathbb{R} \) such that \( \dim(\pi(W)) = \dim(W), \dim(\pi(V)) = \ell \), here \( \pi(V) \subset \mathbb{R}_k^n \) (actually "almost any" linear projection satisfies these properties). Using the obvious identity \( st_k(\pi(W)) = \pi(st_k(W)) \) one can assume without loss of generality that \( \dim(W) = n - 1, \dim(V) = n \). Hence \( V \) contains a ball of a certain radius \( 0 < r \in \mathbb{R}_k \).

Fix some integer \( M \) which we will specify later. Making a suitable affine transformation of the coordinates (definable over \( \mathbb{R}_k \)), we can assume that the following requirements are fulfilled (cf. Lemma 2). The set \( V \) contains \( n \)-dimensional cube \( \mathcal{X} \) with a side \( 0 < r_1 \in \mathbb{R}_k \), contained in the nonnegative octant and having the origin as one of its nodes. Moreover, we require that for each \( 0 < r_1 \leq n \) and a \( j \)-plane \( P \) being the intersection of any \( (n-j) \) hyperplanes of the form \( P_s = \{X_s = (m/M)r_1\}, 1 \leq s \leq n, 0 \leq m \leq M \), the dimension \( \dim(W \cap P) \leq j - 1 \).

Observe that the hyperplanes \( P_s \) divide \( \mathcal{X} \) in \( M^n \) small cubes with sides \( r_1/M \). Moreover, for each \( 0 \leq j \leq n \) and each \( j \)-plane \( P \) the intersection \( P \cap \mathcal{X} \) is divided by
the same way in $M^j$ $j$-facets being $j$-dimensional cubes with sides $r_1/M$ (we assume here that a facet contains its boundary). Note that the boundary of the $j$-facet is the union of $(j-1)$-facets. Denote by $v_j$ the number of $j$-facets which have common points with $W$. Denote by $\mathcal{A}_j$, $0 \leq j \leq n$, the intersection of the set $W \cap X$ with the union of all $j$-planes of the described form. Obviously, $\mathcal{A}_j$ is a sub-Pfaffian set. Denote by $\alpha_j$ the number of connected components of $\mathcal{A}_j$.

We claim that $v_j \leq 2(n-j+1)v_{j-1} + \alpha_j$, $1 \leq j \leq n$. Indeed, $v_j \leq v_j^{(0)} + v_j^{(1)}$, where $v_j^{(0)}$ is the number of $j$-facets $Q^{(0)}$ which have common points with the connected components $C^{(0)}$ of $\mathcal{A}_j$ such that $C^{(0)}$ has no common points with $j$-facets other than $Q^{(0)}$, and $v_j^{(1)}$ is the number of $j$-facets $Q^{(1)}$ not satisfying this property and $Q^{(1)} \cap W \neq \emptyset$. Obviously, $v_j^{(0)} \leq \alpha_j$. For $j$-facet $Q^{(1)}$ take any connected component $C^{(1)}$ of $\mathcal{A}_j$ such that $C^{(1)}$ has common points with some $j$-facet different from $Q^{(1)}$, then $C^{(1)}$ has a common point with a certain $(j-1)$-facet $R$ from the boundary of $Q^{(1)}$, attach to $Q^{(1)}$ any such $(j-1)$-facet $R$. Since any $(j-1)$-facet $R$ lies in the boundary of at most $2(n-j+1)$ $j$-facets, $R$ can be attached to at most $2(n-j+1)$ $j$-facets. Hence $v_j^{(1)} \leq 2(n-j+1)v_{j-1}$, which proves the claim.

Corollary A.3 implies that there exists an integer $c$ which depends only on the format of a Pfaffian formula defining the set $W$ such that the number of connected components of the intersection of $W \cap X$ with any $j$-plane does not exceed $c$. Therefore, $\alpha_j \leq c(M+1)^{n-j}$.

Clearly, $v_n = M^n$ since $st_k(W) \supset X$ (indeed, if some $n$-facet does not intersect with $W$ then its centre does not belong to $st_k(W)$). Using the bound on $\alpha_j$ and the above proved claim we prove by induction on $0 \leq j \leq n-1$ the existence of integers $c_j$ such that $v_{n-j} \geq (1/c_j)M^n$ for large enough arbitrary $M$.

On the other hand, $\mathcal{A}_1$ consists of a finite number of points (since $\dim(\mathcal{A}_1) = 0$), hence $v_0 \leq \alpha_1$, then the proved claim (for $j = 1$) entails $v_1 \leq (2n+1)v_0 \leq c'M^{n-1}$ for an appropriate integer $c'$, which leads to a contradiction for large enough $M > c'c_{n-1}$.

A.4. Degree of sub-Pfaffian transcendency

Let $1 \leq j_1 < j_2$ and the elements $\gamma_1, \ldots, \gamma_k$, $\theta_1, \ldots, \theta_{\ell} \in \mathbb{R}_{j_2}$. Denote the coordinates in $\mathbb{R}_{j_2}^{k+\ell}$ by $Y_1, \ldots, Y_{k+\ell}$.

Definition A.10. The degree of sub-Pfaffian transcendency $[(\theta_1, \ldots, \theta_{\ell}) : (\gamma_1, \ldots, \gamma_k)] = [(\theta_1, \ldots, \theta_{\ell}) : (\gamma_1, \ldots, \gamma_k)]_{\mathbb{R}_{j_1}}$ is the minimal integer $s \geq 0$ such that there exists a sub-Pfaffian set $S \subset \mathbb{R}_{j_2}^{k+\ell}$ definable over $\mathbb{R}_{j_1}$ such that $(\gamma_1, \ldots, \gamma_k, \theta_1, \ldots, \theta_{\ell}) \in S$ and $\dim(S \cap \{Y_1 = \gamma_1, \ldots, Y_k = \gamma_k\}) = s$.

When $k = 0$ we write simply $[\theta_1, \ldots, \theta_{\ell}]$.

Observe that the definition correlates with the usual notion of degree of transcendency of the fields extension $[F(\theta_1, \ldots, \theta_{\ell}, \gamma_1, \ldots, \gamma_k) : F(\gamma_1, \ldots, \gamma_k)]$ replacing $\mathbb{R}_{j_1}$ by a field $F$ and taking as $S$ an algebraic variety.
Lemma A.9. (1) \([\theta_1, \ldots, \theta_{r+1}] \leq [\theta_1, \ldots, \theta_r] + 1;\)
(2) \([e_{j_1-1}, \ldots, e_{j_2}] = j_2 - j_1\) (the infinitesimals \(e_j\) were introduced in Section A.2).

Proof. (1) Let \(S \subset \mathbb{R}_{j_2}'\) be as in the definition, then the point \((\theta_1, \ldots, \theta_{r+1})\) belongs to the cylinder \(S \times \mathbb{R}_{j_2}' \subset \mathbb{R}_{j_2}^{r+1}\).

(2) Conduct the proof by induction on \((j_2 - j_1)\). The base of induction for \(j_2 - j_1 = 0\) is trivial. For the inductive step assume the contrary and let \(S \subset \mathbb{R}_{j_2}^{j_2-j_1}\) be as in Definition A.10 such that \((e_{j_1+1}, \ldots, e_{j_2}) \in S\) and \(\dim(S) = s \leq j_2 - j_1 - 1\). Let \(Y_1, \ldots, Y_{j_2-j_1}\) be the coordinates in \(\mathbb{R}_{j_2-j_1}'\). Consider the sub-Pfaffian set \(S_0 = \{y : \dim(\{Y_1 = y_1\} \cap S) = s\} \subset \mathbb{R}_{j_2}'\). Then \(\dim(S_0) = 0\), since \(\dim(S) = s\). Observe that \(S_0\) is defined over \(\mathbb{R}_{j_1}\), hence, due to Corollary A.4, \(S_0\) consists of a finite number of points all belonging to \(\mathbb{R}_{j_1}\). Denote \(S_1 = \{Y_1 = e_{j_1+1}\} \cap S \subset \{Y_1 = e_{j_1+1}\} \approx \mathbb{R}_{j_2-j_1-1}'\). Then \(\dim(S_1) \leq s - 1\), and one can apply the inductive hypothesis to the set \(S_1\), taking into account that \((e_{j_1+2}, \ldots, e_{j_2}) \in S_1\). \(\square\)

The following lemma is an analogy of the additivity of the usual degree of transcendency: \([F_3 : F_1] = [F_3 : F_2] + [F_2 : F_1]\) for fields extensions \(F_1 \subset F_2 \subset F_3\).

Lemma A.10. \([\gamma_1, \ldots, \gamma_k, \theta_1, \ldots, \theta_r] = [\gamma_1, \ldots, \gamma_k] + [(\theta_1, \ldots, \theta_r) : (\gamma_1, \ldots, \gamma_k)]\).

Proof. Denote \([\gamma_1, \ldots, \gamma_k, \theta_1, \ldots, \theta_r] = m, [\gamma_1, \ldots, \gamma_k] = p, [(\theta_1, \ldots, \theta_r) : (\gamma_1, \ldots, \gamma_k)] = s\). First prove:

(1) \(m \leq p + s\). Let a sub-Pfaffian set \(S\) be as in Definition A.10. Consider the sub-Pfaffian set \(U_1 \subset \mathbb{R}_{j_2}^k\) consisting of all the points \((y_1, \ldots, y_k)\) for which \(\dim(S \cap \{Y_1 = y_1, \ldots, Y_k = y_k\}) \leq s\). Then \(U_1\) is definable over \(\mathbb{R}_{j_1}\). Due to Definition A.10 there exists a sub-Pfaffian set \(U \subset \mathbb{R}_{j_2}^k\) definable over \(\mathbb{R}_{j_1}\) such that \((\gamma_1, \ldots, \gamma_k) \in U\) and \(\dim U = p\).

Denote by \(\pi : \mathbb{R}_{j_2}^{k+r} \to \mathbb{R}_{j_2}^k\) the natural projection onto the subspace with the coordinates \(Y_1, \ldots, Y_k\). Consider the sub-Pfaffian set \(\mathcal{U} = S \cap ((U \cup U_1) \times \mathbb{R}_{j_2}^r) \subset \mathbb{R}_{j_2}^{k+r}\). Then \(\mathcal{U}\) is definable over \(\mathbb{R}_{j_1}\), besides \((\gamma_1, \ldots, \gamma_k, \theta_1, \ldots, \theta_r) \in \mathcal{U}\). The dimension \(\dim(\mathcal{U}) \leq p + s\), since \(\dim(\pi(\mathcal{U}) \leq \dim(U) = p\) and for any point \(y \in \pi(\mathcal{U})\) we have \(\dim(\mathcal{U} \cap \pi^{-1}(y)) \leq s\).

(2) \(m \geq p + s\). According to Definition A.10 there exists a sub-Pfaffian set \(V' \subset \mathbb{R}_{j_2}^{k+r}\) definable over \(\mathbb{R}_{j_1}\) such that \((\gamma_1, \ldots, \gamma_k, \theta_1, \ldots, \theta_r) \in V'\) and \(\dim(V') = m\). Denote \(\dim(V' \cap \{Y_1 = \gamma_1, \ldots, Y_k = \gamma_k\}) = s_1\). Obviously \(s_1 \geq s\). Consider the sub-Pfaffian set \(V_1 \subset \mathbb{R}_{j_2}^k\) consisting of all the points \((y_1, \ldots, y_k)\) for which \(\dim(V' \cap \{Y_1 = y_1, \ldots, Y_k = y_k\}) \geq s_1\). Then \(V_1\) is definable over \(\mathbb{R}_{j_1}\) and \((\gamma_1, \ldots, \gamma_k) \in V_1\), therefore \(\dim V_1 \geq p\). Arguing similarly as in (1), we get \(m \geq s_1 + \dim V_1 \geq s + p\). \(\square\)

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References