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FORBIDDEN MINORS CHARACTERIZATION OF PARTIAL 3-TREES

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A k-tree is formed from a k-complete graph by recursively adding a vertex adjacent to all vertices in an existing k-complete subgraph. The many applications of partial k-trees (subgraphs of k-trees) have motivated their study from both the algorithmic and theoretical points of view. In this paper we characterize the class of partial 3-trees by its set of four minimal forbidden minors (H is a minor of G if H can be obtained from G by a finite sequence of edge-extraction and edge-contradiction operations.)

1. Introduction

A *k-tree* is a graph that can be reduced to the k-complete graph, K_k, by a sequence of operations of "pruning a k-leaf", each being a removal of a degree k vertex with completely connected neighbors (i.e. inducing a K_k). A *partial k-tree* is any subgraph of a k-tree or, equivalently, a graph embeddable in a k-tree with the same vertex set.

The interest in partial k-trees arises from many areas of application such as reliability of communication networks in the presence of constrained line- and site-failures (Farley [7], Farley and Proskurowski [8], Neufeld and Colbourn [9], Wald and Colbourn [12]), concurrent broadcasting in a common medium network (Colbourn and Proskurowski [6]), reliability evaluation in complex systems (Arnborg [1]), and evaluation of queries in relational database systems; for a survey see Arnborg [2]. In many of these problems, the class of partial k-trees accurately captures the structure of the application. Furthermore, many problems which are NP-complete for general graphs have linear time algorithms for partial k-trees where k is fixed (Arnborg and Proskurowski [5]). Note that every graph G with n vertices is a partial $(n-1)$-tree. In [3] it is shown that the recognition of partial k-trees may be done in polynomial time for fixed k, however it is NP-hard for arbitrary k.

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These practical issues motivate our study of the theoretical structure of partial k-trees. In this paper we examine the class of partial 3-trees and characterize this class by its set of four minimal forbidden minors.

A graph H is a *minor* of a graph G iff it can be obtained from G by a finite sequence of edge-extraction and edge-contraction operations. Given a graph G and an edge e, *edge extraction* results in a graph $G-e$ with the same vertex set as G and the edge set $E(G)-\{e\}$; *edge contraction* results in a graph G/e with the vertex set obtained by replacing the end-vertices of e in G by a new "big" vertex; this new vertex inherits all the neighbors of the two replaced vertices, without introducing a self-loop or multiple edges. The edge set of G/e is otherwise equal to that of G.

Wagner's Conjecture (proved in a series of papers by Robertson and Seymour [11]), states that for every infinite set of graphs, one of its members is a minor of another. Thus, every class of graphs that is closed under minor taking has a complement with a finite set of minimal minors (although we do not know how large and how many they are).

We now show that partial k-trees are closed under minor taking and thus that the class of partial k-trees is completely characterized by a finite set of forbidden minors.

Theorem 1.1. *Every minor of a partial k-tree is a partial k-tree.*

Proof. This is obvious for edge extraction, since this operation results in a partial graph of the original graph. To prove that edge contraction preserves the property of being a partial k-tree, consider two cases:

(i) The contracted edge is incident with a k-leaf v of an embedding k-tree. The resulting graph is then identical with a subgraph of the embedding k-tree with pruned leaf v.

(ii) There is a sequence of leaf removals in an embedding k-tree, such that the end-vertex v of the contracted edge becomes a k-leaf (of the reduced k-tree) before the other end-vertex, u. Pruning v and restoring all previous k-leaves (with adjacencies to v replaced by adjacencies to u) results in a k-tree which embeds the edge-contracted minor of the original partial k-tree. □

It follows from Theorem 1.1 that the class of partial k-trees (for any fixed k) is completely characterized by a set of graphs which are forbidden as minors. In this paper we show that the set of minimal forbidden minors for partial 3-trees consists of four graphs K_5, M_6, M_8, and M_{10} (see Fig. 1).

Theorem 1.2. *The four graphs in Fig.* 1 *are minimal forbidden minors for partial 3-trees.*

Proof. By inspection, none of the four graphs is a partial 3-tree, and any edge contraction or extraction results in a partial 3-tree. □

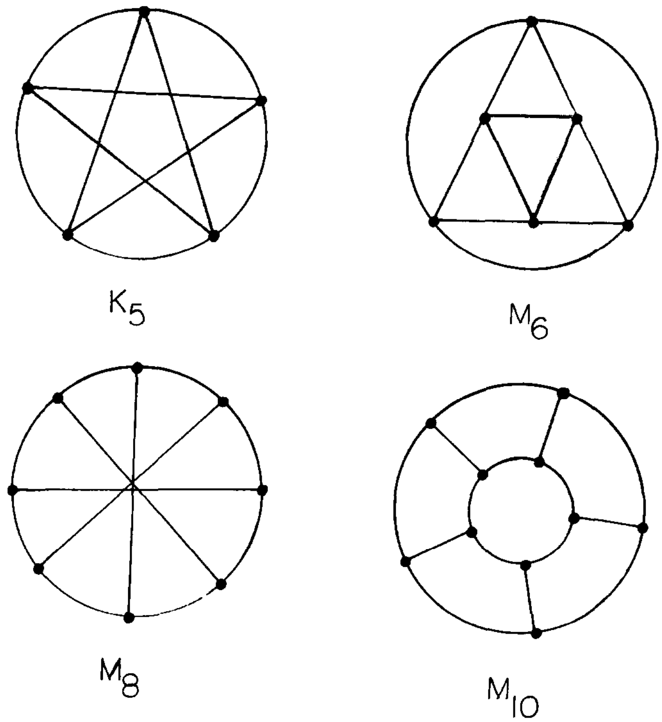

Fig. 1. The four minimal forbidden minors characterizing partial 3-trees.

The rest of the paper proves that this set is complete. In Section 2 we examine the set of safe reductions or rewriting rules which preserve membership in both the class of partial 3-trees and the class of non-partial 3-trees. The connectivity and cycle structure of the forbidden minors are studied in Section 3. Using these properties we prove that in all such forbidden minors every vertex has degree 3 or 4. Furthermore K_5 and M_6 are the only 4-regular such minors and M_8 and M_{10} are the only 3-regular such minors. In Section 4 we show that no minors with some vertices of degree 3 and others of degree 4 may exist thereby establishing that our four minors form in fact a complete set. Before presenting this material we give a few definitions.

Given a k-leaf v of a k-tree T embedding a partial k-tree G, the graph G' obtained from G by removing v and adding all edges between the vertices of the neighborhood S of v is also a partial k-tree. G' is said to be the result of a "k-star–k-complete" substitution of v in G. The k-complete subgraph induced in G' by S is denoted $K(S)$.

2. Safe reductions of partial 3-trees

Let us give an example of techniques that might be applied in a search for minimal forbidden minors for partial k-trees using the case of $k=2$.

Example. We will show that the complete graph of 4 vertices, K_4, is the only forbidden minor of partial 2-trees. Partial 2-trees are easily recognizable by reducing a graph to an edge by application of the following "rewriting rules" (cf. Fig. 2(a)): remove vertices of degree 0 or 1, and contract 2-paths ("series reduction": replace by a single edge two edges incident with a common degree 2 vertex). Applications of these rewriting rules create minors of the original graph.

By absence of vertices of degree 2 or less (which would lead to a smaller minor through a rewriting rule), a minimal minor is cubic, since deletion of any edge must create two vertices of degree 2 or less (every 2-tree has at least two 2-leaves, which are present in partial 2-trees as vertices of degree at most 2). To create two vertices of degree 2 by contraction of any edge, every edge must be in at least two triangles: take such an edge (x, y) and consider two common neighbors of x and

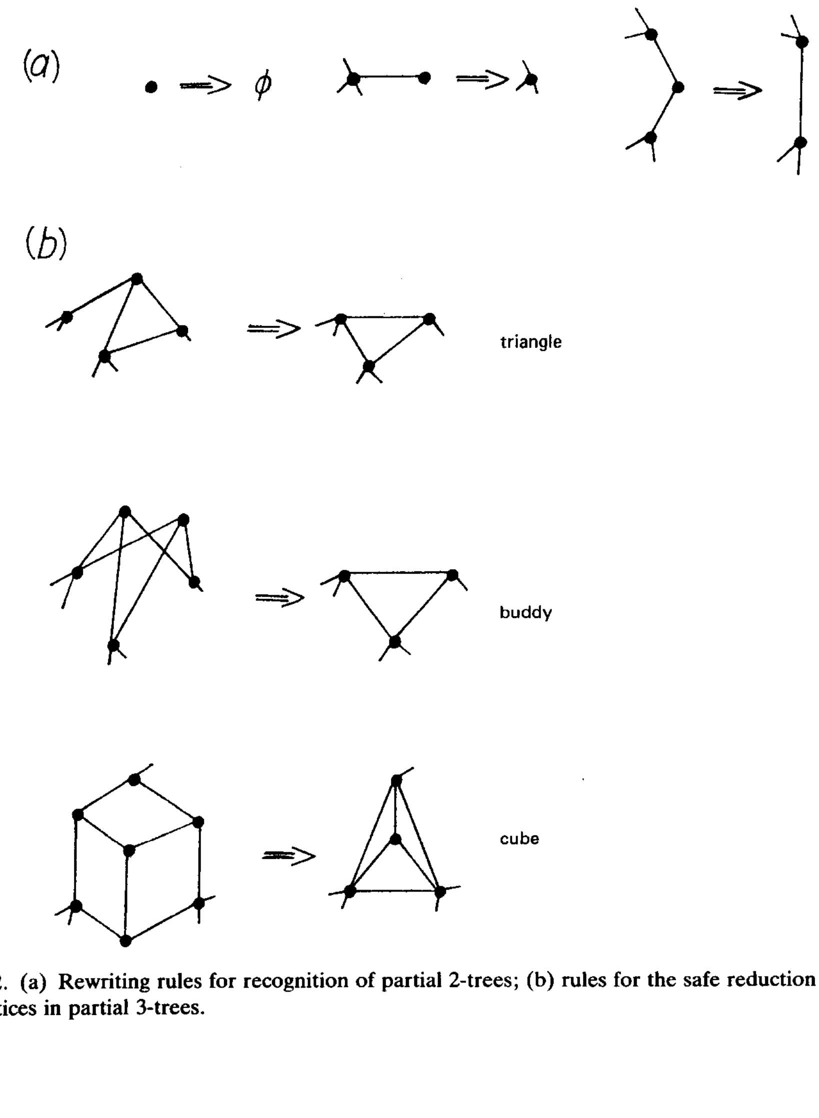

Fig. 2. (a) Rewriting rules for recognition of partial 2-trees; (b) rules for the safe reduction of degree 3 vertices in partial 3-trees.

y, u and v. Since (x, u) must be in another triangle and x has already three neighbors, the third edge incident to u must lead to v giving a K_4.

We are now ready to approach the case of $k = 3$. This is the largest case with a known set of safe *rewriting* rules, graph reduction rules which preserve membership in both the class of partial 3-trees and the class of graphs which are not partial 3-trees (Arnborg and Proskurowski [4]). Besides the rules for partial 2-trees (stated in the Example), we have three cases of degree 3 vertex reductions: the *triangle,* the *buddy,* and the *cube* rules (cf. Fig. 2). In the following, we will discuss minimal forbidden minors of partial 3-trees and call them simply "minimal forbidden minors",

Lemma 2.1. *A minimal forbidden minor has no safe reduction.*

Proof. Otherwise, it would have a forbidden minor—since every safe reduction creates a minor of the original graph—and thus would not be minimal. □

We observe that the cube-like configuration with the "hub" (vertex x in Fig. 2b) of degree greater than 3 can be safely reduced as well. This follows from the facts that (i) the 3-star-triangle substitution of the purported 3-leaves cannot transform into a partial 3-tree a graph that is not a partial 3-tree, and (ii) in this case of cube-like configuration, the 3-star-triangle substitutions produce a minor of the original graph, thus preserving membership in the class of partial 3-trees.

Corollary 2.2. *The cube-like configuration with the hub of degree greater than* 3 *must not be present in a minimal forbidden minor.*

For the remainder of this section we study properties of safe reductions in partial 3-trees. A subgraph isomorphic to a left-hand side of any of the safe rewriting rule is called *an instance of a safe reduction* and the set of 3-leaves assumed by this reduction is called *the set of reducible vertices.* An important role in our investigation is played by the notion of a *minimal separator* which is a minimal set of vertices separating the remaining vertices of the graph. After presenting a number of lemmas, we will conclude that, for any sufficiently large partial 3-tree, there is a 3-tree embedding in which two leaves correspond to vertices reducible in two separate instances of safe reductions.

Lemma 2.3. *For a partial* 3-*tree* G *and an instance of a safe reduction with the set of reducible vertices* A *there exists an embedding* H *of* G *such that vertices in* A *are leaves of* H.

Proof. Each safe reduction rule corresponds to one or more independent instances of the "3-star-triangle substitution" that mimicks pruning of a 3-leaf in an embedding 3-tree (cf. Arnborg and Proskurowski [4]). □

Lemma 2.4. *Any partial* 3*-tree* G *with an embedding* H *containing two minimal separators* T_1 *and* T_2 *that have at most one vertex in common has* (a) *a minimal separator* T_3 *such that* T_1 *and* T_2 *are in two different components* $G_1 = C_1 + K(T_3)$ *and* $G_2 = C_2 + K(T_3)$, *respectively, where the* C_i *are connected components of* $G - T_3$, (b) *an embedding* H' *with* T_3 *as a separator, and two instances of safe reductions with reducible vertex sets* A *and* B *such that* $A \subseteq C_1$ *and* $B \subseteq C_2$ *and* A *and* B *are leaves in* H'.

Proof. (a) Existence of T_3 follows from the structure of separators in k-trees (Proskurowski [10, Theorem 2.2]).

(b) Consider the partial 3-tree G' obtained from G by deletion of all 3-leaves of H; the embedding of G' corresponding to H has a 3-leaf x. This 3-leaf may be a vertex of T_1 or T_2 but not both (and not a vertex of T_3). In fact, there is such an x both in G_1 and in G_2, and we will now prove the existence in G of an instance of a safe reduction A which is not separated from x by T_3. We proceed by case analysis over the number of 3-leaves in H adjacent to x. We may assume that in G all of these have degree 3 and are not involved in triangles, or they would be instances of safe reductions (and thus H would be our required embedding H').

(i) If there is only one 3-leaf adjacent to x, then x has degree less than 3 in G, or the leaf is involved in a triangle. In the latter case, $H = H'$. In the former case, an embedding H' can be obtained from H by replacing an edge (not in G) adjacent to x with a new edge adjacent to the single leaf and to the fourth neighbor of x in H.

(ii) Assume that there are two 3-leaves adjacent to x. They either form a buddy configuration, or one of them is involved in a triangle (closed by another edge incident with x), or x has degree less than 3; the last case is similar to that of (i);

(iii) If three or more 3-leaves are adjacent to x, then they form a buddy configuration, or a cube configuration.

In the cases where the leaves contain a buddy or cube configuration, since neither x nor any of the reducible vertices are part of T_3, the whole set of safely reducible vertices in the configuration must be on the same side of separator T_3 as x.

This completes the proof of the lemma. □

Lemma 2.5. *A partial* 3*-tree* G *with at least* 9 *vertices has an embedding* H *in which two disjoint subsets of leaves correspond to two safe reductions of* G.

Proof. (by case analysis based on the number of separators in an embedding H of G.) *If* H has five or more separators, then the conditions of Lemma 2.4 are met (there are two separators with at most one vertex in common) and the two safe reductions exist on "two sides" of the separator T_3. This is also true when three or four separators do not form sides of a tetrahedron (i.e. their vertices do not induce a K_4). We thus have to consider only the remaining cases of 4 or fewer separators. As before, we may assume that all leaves have degree 3.

(i) If H has exactly one separator, then there are at least six 3-leaves of H adjacent to its vertices, creating enough instances of buddy configuration (or having degree less than 3 in G).

(ii) If H has only two separators, then the five (or more) 3-leaves form at least two buddy configurations.

(iii) If H has exactly three separators then they form three sides of a tetrahedron, or the assumptions of Lemma 2.4 hold. In the forrmer case there are at least five 3-leaves and the analysis of (ii) applies.

(iv) If H has four separators forming a tetrahedron, then three of the 3-leaves form a cube configuration, and two other 3-leaves form a buddy. □

Lemma 2.6. *In any partial* 3*-tree, there are at least two instances of safe reductions such that* (a) *neither of the two sets of reducible vertices is a subset of the other; and* (b) *there are two safely reducible vertices, one from each instance, that are not adjacent.*

Proof. By Lemma 2.4, we only have to consider partial 3-trees without embeddings containing two separators with at most one vertex in common. Such embeddings may have (i) one separator, (ii) two separators sharing an edge, (iii) three separators that form three sides of a tetrahedron, or (iv) four separators forming a tetrahedron. In these cases, there are at least (i) two, (ii) two, (iii) three, and (iv) four 3-leaves (of an appropriate embedding). In the following, we assume that no two of these 3-leaves have degree less than 3 in the partial 3-tree (or else they would be safely reducible) and consider additional adjacencies of the separator vertices. The presence of two independent vertices of degree less than 3 in the separators would imply the hypothesis of the lemma. We will show that to force degree 3 on the separator vertices there must be at least two instances of safe reductions. The edges incident with separator vertices may connect separator vertices creating a number of triangles that involve leaf vertices. Alternatively, they may be incident also with additional 3-leaves creating instances of the buddy configuration. In cases (i)–(iii), addition of new leaves or edges connecting separator vertices will not force all vertex degrees to be at least three without making two leaves reducible in two different safe reduction instances. Neither will it force all but one vertex degree to be at least three (the remaining vertex of small degree becoming one safely reducible leaf) without creating another safely reducible leaf (or leaves). In case (iv), the partial 3-tree consisting of four separators and four leaves only has two vertex-disjoint instances of the cube configuration. Although the reducible vertices of these two configurations are not independent, for every reducible vertex in one of them there is a non-adjacent reducible vertex in the other. □

Corollary 2.7. *In a minimal forbidden minor, every edge extraction and edge contraction creates at least two instances of safe reductions.*

An edge extraction in a minimal forbidden minor creates two instances of safe reductions by decreasing the degree of the edge's end-vertices. If the reductions involve a vertex of degree 2, a triangle, or a buddy configuration, then these end-vertices are safely reducible in the reductions. Since the "cube-like" configuration (Corollary 2.2) is also safely reducible, the end-vertices of the extracted edge must be safely reducible in a cube configuration as well.

Corollary 2.8. *In a minimal forbidden minor G with edge e, the end-vertices of e are reducible in $G - e$.*

3. Regular forbidden minors

We now examine various properties of forbidden minors. In particular we study the existence of certain cycles through any edge as well as possible vertex degrees. This leads to the proof that K_5 and M_6 (respectively M_8 and M_{10}) are the only 4-regular (respectively 3-regular) forbidden minors.

Lemma 3.1. *A minimal separator of size 2 in a partial 3-tree G can be extended to a 3-separator of a 3-tree which is an embedding of G.*

Proof. Since every clique separator can be extended to a separator in an embedding k-tree of a partial k-tree (Arnborg and Proskurowski [4]), we can assume that the two vertices x and y of the minimal separator of a partial 3-tree G are independent. It suffices to show that $G + (x, y)$ is a partial 3-tree. This follows from the fact that every graph G_i (that corresponds to a component C_i of $G - \{x, y\}$, includes the edge (x, y), and has all edges of G with one end-vertex in $\{x, y\}$ and the other in C_i) is a minor of G and hence a partial 3-tree. □

A direct consequence of Lemma 3.1 is that we can add the edge between the vertices of a 2-separator of a partial 3-tree and still have a partial 3-tree.

Corollary 3.2. *If $\{x, y\}$ is a minimal separator of a partial 3-tree G, then $G + (x, y)$ is also a partial 3-tree.*

In the next three lemmas we present various structural properties of minimal forbidden minors. These properties will later be used to show the non-existence of minimal minors not in the set illustrated in Fig. 1.

Lemma 3.3. *A minimal forbidden minor M for the class of partial 3-trees is 3-connected.*

Proof (by contradiction). It is obvious that M must be connected. Assume it has a minimal separator S where $S = \{x\}$ or $S = \{x, y\}$. Assume w.l.o.g. that $M - S$ has two components, C_1 and C_2, each with at least two vertices. x has a neighbor, call it z_i, in C_i for $i = 1, 2$. Contract edge (x, z_i) to get a partial 3-tree M_i with separator S. In the case $S = \{x, y\}$, edge (x, y) is added to M_i, which remain partial 3-trees by Corollary 3.2. M_i now has a clique separator, so each of the induced subgraphs on $S + C_i$ is a partial 3-tree by the k-decomposability characterization (Arnborg and Proskurowski [4]). However, M can be obtained by joining two of these induced subgraphs at S, namely the C_1 part of M_2 and the C_2 part of M_1. So M is a partial 3-tree and thus cannot be a forbidden minor for partial 3-trees. □

Lemma 3.4. *Every vertex of a minimal forbidden minor has degree* 3 *or* 4.

Proof. Extraction of an edge must result in at least two vertices of degree 3 or less, since these two vertices must be safely reducible—thus there are no vertices of degree larger than 4. Since there are no safe reductions in a forbidden minor, no vertex has degree less than 3. □

Lemma 3.5. *In a 4-regular minimal forbidden minor, every edge is shared by two triangles.*

Proof. Contraction of an edge creates a new "big" vertex of degree at least 4 unless three pairs of edges incident with its end-vertices are adjacent, i.e. the edge is shared by three triangles. Otherwise, two vertices can decrease their degrees (which is required by Lemma 2.6) only if they are adjacent to both end-vertices of the contracted edge. In both cases, the edge is in two triangles. □

Theorem 3.6. *K_5 and M_6 are the only 4-regular minimal forbidden minors.*

Proof. Let us construct a 4-regular minimal forbidden minor from a five vertex graph consisting of a vertex v of degree 4 and its neighbors. Since each of the four edges must be in two triangles (and no more edges can be incident with v) the other four vertices must induce a four-cycle, C_4, possibly with chords. The edges of this four-cycle can be in two triangles by adding two chords (which results in K_5), or by adding a number of vertices to which the five vertex graph must be connected by at least three edges (Lemma 3.3); this implies that also the fourth vertex on the cycle is adjacent to an extraneous vertex. Since these four edges must create four triangles (second triangles for the cycle edges), there is exactly one extra vertex completing the only other 4-regular minimal forbidden minor, M_6. □

Corollary 2.7 gives us an idea of what the neighborhood of every edge of a minimal forbidden minor must look like. Since an edge contraction must result in two instances of a triangle, a buddy, or a cube configuration, we need only consider the six possible combinations, each implying the presence of a particular subgraph (see Fig. 3). This facilitates analysis of regular forbidden minors.

The "big" vertex resulting from edge contraction in a minimal forbidden minor has degree at least 4 if the contracted edge has a degree 3 end-vertex (otherwise the degree 3 vertex would be in a triangle and thus safely reducible in the minor, contradicting Lemma 2.1). Thus, two of the safely reducible vertices created by this contraction are neighbors of the big vertex. This implies that this big vertex is in a triangle, in a four-cycle of a cube, or adjacent to two vertices of a buddy. Before stating this formally in the following lemma, we introduce the notion of cycle neighbors: Given an edge $e = (x, y)$ and a cycle C containing e we refer to the two endpoints of the path $C - \{x, y\}$ as the *cycle neighbors* of e.

Lemma 3.7. *In a minimal forbidden minor G, an edge e with an end-vertex of degree 3 must be included in two cycles each of which consists of either four or five vertices. In such a four cycle, there must exist a cycle neighbor of e of degree 3 and in such a five cycle, both cycle neighbors of e must be degree 3.*

Theorem 3.8. *M_8 and M_{10} are the only cubic minimal forbidden minors.*

Proof. We will analyze the six cases of an edge neighborhood in a minimal cubic forbidden minor involving two of: the triangle, the buddy, and the cube configurations resulting from the edge's contraction. After the basic configuration (a graph with between 6 and 14 vertices, some of degree 3 and some of degree 2) is laid out, we consider all possible nonisomorphic completions of the purported minimal minor, first without and then with some new vertices. In creating these basic configurations, four vertices will be "unsaturated", with explicitly constructed two (out of three) incident edges. (In Fig. 3, these unsaturated vertices are adjacent to s, t, u, v, respectively.) In our reasoning, we will use the result of Lemma 3.3 implying the existence of three vertex- disjoint paths between any two vertices of a minimal forbidden minor. Thus, any set of new vertices must be connected to the rest of the graph by at least three edges. The threes edges connecting the existing subgraph with the new vertices force the fourth edge to be incident with a new vertex as well.

(i) (triangle–triangle) The basic configuration consists of the edge e and four unsaturated vertices forming two four-cycles (see Fig. 3a). Since there is no way

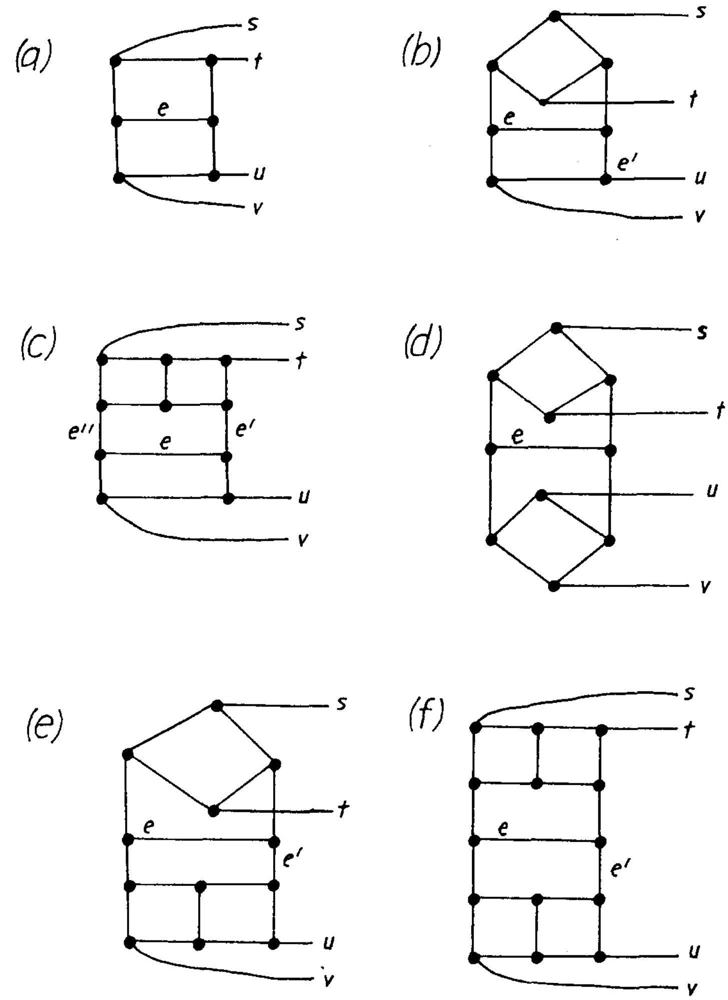

Fig. 3. The six cases of an edge neighborhood in a cubic minimal forbidden minor.

of connecting the four vertices without creating a partial 3-tree, consider the four "new" vertices, s, t, u, and v. We have to show that all four of them are unique. Indeed, identifying s and t would result in a triangle, identifying t and u gives a cube; identifying both s and u, and t and v, and connecting the two vertices results in a graph with M_8 as a minor. The only remaining case, of three new vertices, is to identify only s and u. But the existing edges adjacent to t and v cannot both be in two required four- or five-cycles, since such cycles must include both edges (s, t) and (s, v), i.e. a total of four edges incident to s. Vertices s and v, and t and u must be adjacent to ensure existence of second cycles involving edges adjacent to e. But to provide the same for the remaining original edges, s and t (and u and v) must be connected by a path of one or two edges, creating a graph with an M_{10} minor.

(ii) (triangle–buddy) The only way of connecting unsaturated vertices in the basic configuration without creating triangles (see Fig. 3b) results in the graph M_8. As before, consider the four new vertices, s, t, u, and v. Identifying s and t gives the buddy, identifying u and v gives the triangle, identifying s and u gives a graph with M_8 as a minor (there must be two vertex-disjoint paths from v to t and to the third vertex). With all four new vertices present, there is no way to include the edge e' in a four- or five-cycle, thereby contradicting the conditions spelled out in Lemma 3.7.

(iii) (triangle–cube) Connecting the unsaturated vertices results in either a graph with an M_8 minor or a graph with an M_{10} minor. Consider the four new vertices, s, t, u, and v (see Fig. 3c). Identifying s and t gives a cube, identifying u and v gives a triangle; to include the edge e' (resp. e'') in another (five-) cycle, vertices, s and v (resp. t and u) must be identical and s and t connected which gives a graph with M_{10} as a minor.

(iv) (buddy–buddy) Connecting the unsaturated vertices without creating a triangle (Fig. 3d) results in a graph with K_5 as a minor. Identifying s and t gives a buddy, identifying t and u results in a graph with a K_5 minor (since there is a path between s and v avoiding the existing vertices.) To include the four edges incident to vertices s, t, u, and v into a second (five-) cycle for each, edges (s, t) and (u, v) must exist; since there exist paths between vertices s and u, and t and v that avoid all the existing vertices, the resulting graph has a K_5 minor.

(v) (buddy–cube and cube–cube) To include the edge e' in another cycle (see Fig. 3e and 3f), the unsaturated vertices must be connected resulting in a graph with M_{10} as a minor. □

4. Minors with vertices of degrees 3 and 4

We now show that no forbidden minors may exist that have some vertices of degree 3 and others of degree 4. To this end we examine the cycle structure through an edge in such a graph. First we note a fact about a *mixed degree edge*, an edge with end-vertices of degree 3 and 4, respectively, in a purported minimal minor.

Lemma 4.1. *Contraction of a mixed degree edge of a minimal forbidden minor cannot result in the cube configuration.*

Proof. The edge e''' (see Figs. 4a, 4b, and 4c) cannot be in any five-cycle in which it would have two cycle neighbors of degree 3, since it would imply an edge closing a triangle involving a vertex of degree 3. Since e''' is only in one four-cycle, no supergraph of this graph is a minimal forbidden minor. □

We must now strengthen Lemma 3.7 when applied to a mixed degree edge:

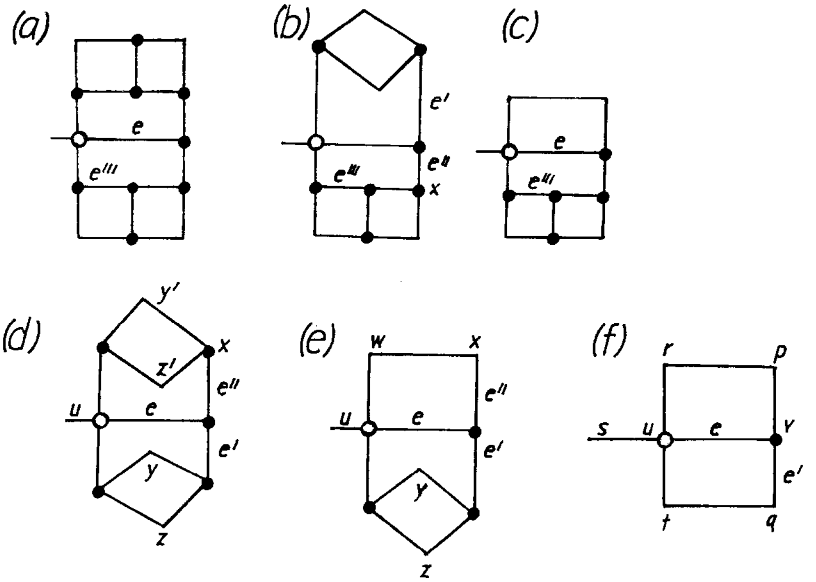

Fig. 4. The six cases of a mixed degree edge neighborhood in a purported minimal forbidden minor; vertices are coded as: ° degree 4, • degree 3.

Lemma 4.2. *In a minimal forbidden minor G, a mixed degree edge e must be included in two cycles which overlap only on e and which consist of either four or five vertices. In such a four cycle, there must exist a cycle neighbor of e of degree* 3 *and in such a five cycle, both cycle neighbors of e must be of degree* 3.

Proof. By Lemma 3.7, we need only prove that the cycles must not have another common edge adjacent to e. Let us assume that this is not true.

In the case of two overlapping four-cycles, the two vertices of the cycles that are not incident with a shared edge must be of degree 3, otherwise there would not be two non-adjacent reducible vertices after contraction of e (Lemma 2.6b).

If the other shared edge is incident with the degree 3 end-vertex v of e (see Fig. 5a) then the removal of e leaves the degree 4 end-vertex u of e (now of degree 3) without a safe reduction. This becomes apparent when analyzing each of the three safe reductions that might involve u, recalling that a minimal forbidden minor must not have a safe reduction (Lemma 2.1).

(i) The triangle is impossible because one of two neighbors of degree 3 (r or s) would have to be involved in it in G.

(ii) The buddy would imply that the two degree 3 vertices r and s are buddies in G, since they must be adjacent to the buddy of u.

(iii) The cube in which u would be a leaf can have as "the hub" (degree 3 vertex adjacent to the 3-leaves of the cube) either r (s is equivalent) or the third neighbor of u, t (cf. Fig. 5a). In the former case, v and r would have two common neighbors of degree 3 (p and the third safely reducible vertex q of the cube,

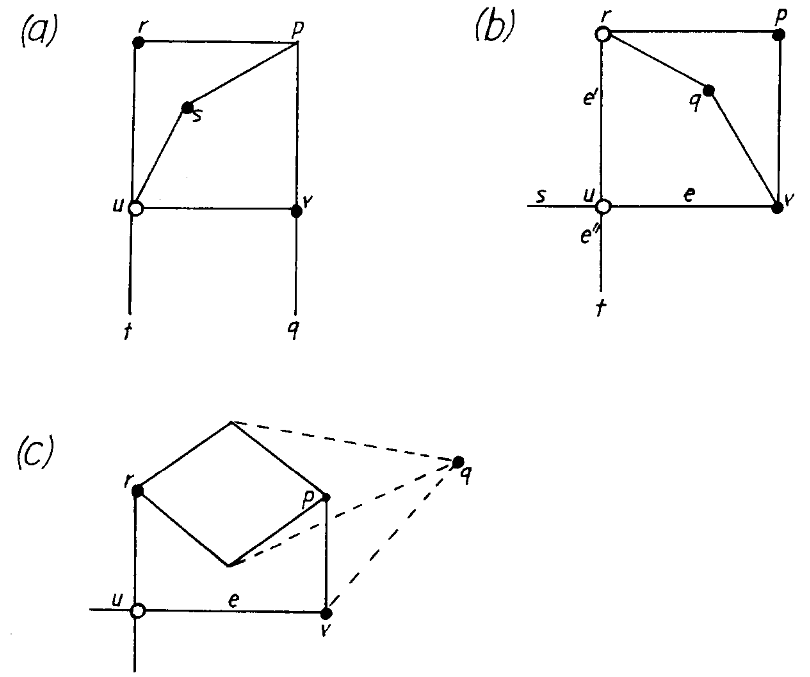

Fig. 5. Overlapping cycles in a purported minimal forbidden minor with mixed degree edges; vertices are coded as: ∘ degree 4, • degree 3.

which also shares a neighbor t with u.) This would imply existence of a two-vertex separator, $\{s, t\}$ of G. In the case of t being the hub of the newly created cube, the constraints on the degree of its neighbors and the connectivity considerations (Lemma 3.3) would imply that G have M_8 as a minor.

If the other shared edge is incident with the degree 4 end-vertex u of e, we consider the following cases of safe reductions involving vertex u in the graph with the edge e'' removed (see Fig. 5b; the vertex r has degree 4, or else it would be a buddy of v in the original graph).

(i) If u is in a triangle of $G - e''$, then (s, r) is an edge and in $G - e'$, r cannot be safely reduced (r is not in a triangle, a buddy would make p and q buddies in G, a cube with hub $q(p)$ or s would imply a triangle in G involving a vertex of degree 3).

(ii) If u has a buddy in $G - e''$, then it is p or q, which is adjacent also to s, so there is another independent four-cycle involving e.

(iii) If u is in a cube of $G - e''$, then v or s (r is of degree 4) must be the hub of the cube and again there must be an independent four-cycle involving e.

When one of the overlapping cycles is a five-cycle, contraction of e must result in a buddy and not a cube (Lemma 4.1). If there is a four-cycle overlapping the five-cycle (see Fig. 5c), then u or v must be adjacent to one of the two common neighbors of r and p, which creates a reducible vertex in a triangle configuration.

If there is an overlapping five-cycle (possible only through vertex q in Fig. 5c), then contraction of e results in the buddy configuration involving q, implying that r on p would have to be of degree 4, or p and q would be buddies in G. □

Lemma 4.3. *If a minimal forbidden minor has a mixed degree edge, then the edge is part of two non-overlapping* 4*-cycles.*

Proof. Contraction of a mixed degree edge cannot result in the cube by Lemma 4.1. Contraction of a mixed degree edge cannot result in the buddy configuration either: the edge e' (see Figs. 4b, 4d and 4e) cannot use e in a five-cycle (because u has degree 4, cf. Lemma 4.2), or in a four-cycle (because this would require a vertex of degree 3 involved in an existing triangle). Thus, all other four- and five-cycles involving e' include vertex x, and overlap also on e''.

In the case of a triangle and a buddy configuration (Fig. 4e), the required cycles around e' must include y and z, respectively. If (y, w) is an edge in one (five-) cycle, then (z, w) cannot be an edge, because then y and z would be buddies. Therefore, there is a path of one or two edges from z to x, which implies a K_5 minor. In the case of two buddies (Fig. 4d), the two cycles must be obtained through edges (y, y') and (z, z'), which also implies a K_5 minor of G. □

For the case where one of the cycles is of length five, the cycles do not overlap even if the edge has two end-vertices of degree 3.

Lemma 4.4. *For every edge e with two degree* 3 *end-vertices in a minimal forbidden minor, if at least one of the two required cycles is of length* 5, *then the two cycles do not overlap on an edge adjacent to e.*

Proof. Two such cycles of length 5 cannot overlap on an edge adjacent to e, or the degree constraints for the safely reducible vertices would be violated (see Figs. 3d, 3e and 3f). A 4-cycle overlapping with a 5-cycle leading to a buddy configuration (through contraction of e) creates a triangle involving degree 3 vertices. When a 4-cycle overlaps with a 5-cycle leading to a cube configuration, Lemma 2.6(b) requires vertex x (see Fig. 6) to have degree 3, since vertex y

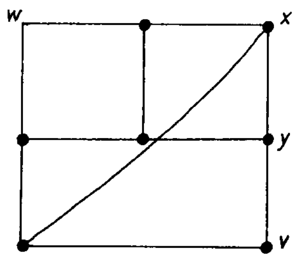

Fig. 6.

would be in the set of safely reducible vertices of the cube. This results in a 2-vertex separator (v, w), contradicting Lemma 3.3. □

Theorem 4.5. *No minimal forbidden minor has both vertices of degree* 3 *and vertices of degree* 4.

Proof. In a minor G with mixed vertex degrees there is an edge $e = (u, v)$, where u and v have degree 4 and 3, respectively. According to Lemma 4.3, e must be in two non-overlapping 4-cycles (see Fig. 7). We will show that such an edge must not exist. First, we show that the neighbors r and t of u have degree 3. Then, we show that u cannot be safely reducible both in $G - (u, v)$ and in $G - (u, s)$.

Assume that one of r and t, say t, has degree 4. This implies that q has degree 3 (to be able to take advantage of the triangle created by contraction of e), and that

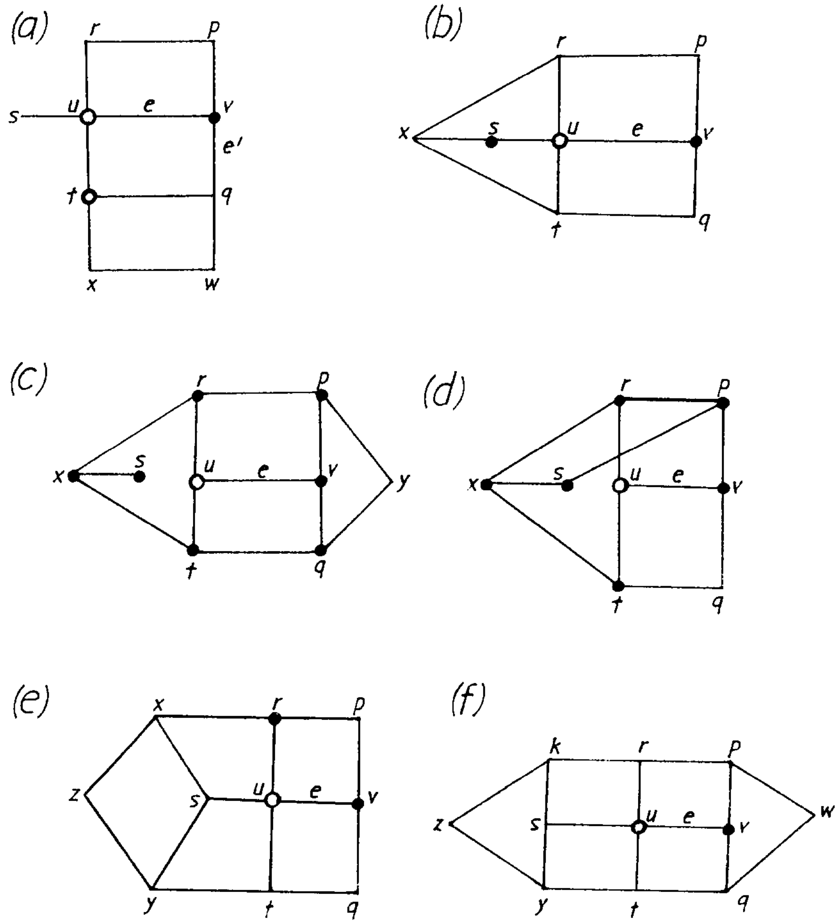

Fig. 7.

there is another non-overlapping four-cycle around the mixed degree edge (t, q) (Fig. 7a). The edge $e' = (v, q)$ must be in two four-cycles (other than (u, v, q, t)) since a five-cycle would have to share another edge with the second required cycle contradicting Lemma 4.4. (by Lemma 3.7, both neighbors—along the five-cycle—of end-vertices of e' would have to be of degree 3). But the only two four-cycles that include e' and another vertex of degree 3 can be obtained through addition of edges (u, w) and (p, t), (since (q, r) would create buddies v and r unless r has degree 4, in which case, however, (u, v, q, r) cannot be used as a required four-cycle). If these edges were present, p and w would have to be of degree 3 and $\{r, x\}$ would be a separator of G.

Since an edge between any of the neighbors of u would close a triangle involving a vertex of degree 3, $\{r, s, t, v\}$ is an independent set. This implies that if any edge incident to u is removed, u becomes reducible in a buddy or a cube.

Consider first removing the edge $e = (u, v)$. This results in either (i) a buddy or (ii) cube configuration in which u is reducible.

(i) If we have a buddy configuration, then the other reducible vertex in it is neither p nor q, since they are both adjacent to v while u is not. Call the other reducible vertex in the buddy x (Fig. 7b). Since all neighbors of u, and their neighbors are given, u cannot be reducible in a buddy configuration of $G - (u, s)$. u cannot be reducible in a cube configuration of $G - (u, s)$ since then the hub of the cube would be v or r (t is isomorphic to r). But if the hub were v (Fig. 7c), then p and q are both of degree 3 and have another neighbor in common (beside v). Since all the other vertices except s have known neighborhoods, this results in a separator of size at most 2, contradicting Lemma 3.3. If the hub were r (Fig. 7d), the edge (s, p) would have to be present and $\{s, q\}$ would be a separator of G.

(ii) If u is safely reducible in a cube configuration of $G - (u, v)$, then we consider the location of the cube's hub. If r is the hub, then the other reducible vertices of the cube (beside u) must be p (sharing a neighbor – s or t – with u) and a neighbor of r that shares another neighbor – v – with p; this vertex must be q. Thus, p and q have degree 3 which implies that $\{s, t\}$ be a separator of G. If s is the hub, then depending on the choice of the other two safely reducible vertices in the cube (thus adjacent to s), we have three subcases. One choice is of both p and q as adjacent to s; this results in s and v that are buddies in G. Another choice is p (alternatively, q) and another vertex x (*not* q) adjacent to s; these two vertices have, however, no other common neighbor (required in a cube). Finally, two "new" vertices, x and y, may be the remaining neighbors of s. This implies a common neighbor z of x and y (Fig. 7e) which cannot be identical to any earlier introduced vertex (if z were identical to p or q there would be a triangle with a vertex of degree 3). Since this is the only remaining configuration around a mixed degree edge e, we must have it also around edge (u, s) (since s—as the hub of a cube—has degree 3), as shown in Fig. 7f. In this case, G has a minor isomorphic to M_{10}. No alternatives remain around a mixed degree edge, and the proof is complete. $\square$

An immediate consequence of Lemma 3.4 and Theorems 3.5, 3.8 and 4.5 is:

Theorem 4.6. *Partial* 3*-trees are completely characterized by the forbidden minors* K_5, M_6, M_8 *and* M_{10}.

As stated in Section 1, the class of partial k-trees for any fixed k is completely characterized by a finite set of minimal forbidden minors. It is natural to wonder whether our results for $k=3$ can be extended to $k \geqslant 4$. In our proofs we depended heavily on the complete set of safe reduction rules which characterize partial 3-trees. Using a similar approach it would be difficult to extend our results to partial k-trees for higher values of k, since no such sets of rules are known for $k \geqslant 4$.

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References

[1] S. Arnborg, Reduced state enumeration—another algorithm for reliability evaluation IEEE Trans. Reliability R-27 (1978) 101–105.
[2] S. Arnborg, Efficient algorithms for combinatorial problems on graphs with bounded decomposability—a survey, BIT 25 (1985) 2–33.
[3] S. Arnborg, D.G. Corneil and A. Proskurowski, Complexity of finding embeddings in a k-tree, SIAM J. Alg. and Discr. Methods 8 (1987) 277–284.
[4] S. Arnborg and A. Proskurowski, Characterization and recognition of partial 3-trees, SIAM J. Alg. and Discr. Methods 7 (1986) 305–314.
[5] S. Arnborg and A. Proskurowski, Linear Time Algorithms for NP-hard Problems on Graphs Embedded in k-trees, TRITA-NA-8404, The Royal Institute of Technology (1984) (to appear in Discr. Appl. Math., 1989).
[6] C.J. Colbourn and A. Proskurowski, Concurrent transmissions in broadcast networks, Proc. Int. Conf. Automata, Languages, Programming (Springer-Verlag LNCS 172 1984) 128–136.
[7] A.M. Farley, Networks, immune to isolated failures, Networks 11 (1981) 255–268.
[8] A.M. Farley and A. Proskurowski, Networks immune to isolated line failures, Networks 12 (1982) 393–403.

[9] E.M. Neufeld and C.J. Colbourn, The Most Reliable Series-Parallel Networks, Dept. of Computing Science, University of Saskatchewan, TR 83–7 (1983).

[10] A. Proskurowski, Separating subgraphs in k-trees: cables and caterpillars, Discr. Math. 49 (1984) 275–285.

[11] N. Robertson and P.D. Seymour, Disjoint paths—a survey, SIAM J. Alg. Discr. Meth. 6 (1985) 300–305.

[12] A. Wald and C.J. Colbourn, Steiner trees, partial 2-trees, and minimum IFI networks, Networks 13 (1983) 159–167.