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Infinitely Divisible Distributions, Conditions for Independence, and Central Limit Theorems

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I. INTRODUCTION

The class of infinitely divisible (I.D.) random variables has been studied extensively in the statistical literature, and increasing applications are being found for I.D. random variables in various areas of Engineering. Much less has been written about vector-valued I.D. variates, although the theory for the scalar and vector cases is essentially the same. The new factor in the vector case is the possibility of statistical dependence among the vector components. Not surprisingly, the correlation matrix is no longer an adequate description of that dependence.

As shown in Section II, if $\mathbf{X}$ is a zero-mean I.D. random vector with components X_i, $i = 1,..., N$, mutual independence of the X_i's is implied by pairwise independence. In addition, variates are pairwise independent if their *squares* are uncorrelated. In the special case of $\mathbf{X}$ being either normal or positive, noncorrelation of the variates is sufficient to imply independence.

In proving the results above, a significant role is played by the parameter

$$\pi_{ij} = EX_i^2X_j^2 - EX_i^2EX_j^2 - 2(EX_iX_j)^2.$$

We show that $0 \leqslant \pi_{ij} \leqslant (\pi_{ii}\pi_{jj})^{1/2}$ and that when π_{ij} attains its upper limit, X_i and X_j can be written as different linear combinations of the same four random variables.

In Section III, some characterizations of normal and Poisson distributions are developed. It is shown that $\mathbf{X}$ is normal if and only if $\pi_{ii} = 0$ for all i, and $\mathbf{X}$ has Poisson marginals if and only if $\pi_{ii} = 2EX_i^3 - EX_i^2$ for all i. These characterizations are used to obtain central limit theorems for sequences of I.D. variates and also for sequences of sums of "small" independent random variables.

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Finally, it is perhaps worth noting that the class of scalar I.D. variates is quite broad. It includes the normal, Poisson, and compound Poisson, as well as the gamma distribution, the related exponential and chi-square distributions, and the double-exponential distributions. Some distributions which are *not* included are those with bounded support (i.e., for some positive $M < \infty, P\{|X| > M\} = 0$), those for which $\pi_{ii} < 0$, and those whose characteristic function is zero at some point on the real line.

II. Measures of Independence and Dependence for Infinitely Divisible Variates

Perhaps the first problem to arise in the modern theory of probability is that of determining the asymptotic distribution of sums of the form

$$\mathbf{S}_n = \sum_{k=1}^{r_n} \mathbf{X}_{nk}, \tag{1}$$

where the $\mathbf{X}_{nk}$'s are *row vectors* with components X_{nki}, $i = 1,..., N$, and for each $n = 1, 2,...$, the $\mathbf{X}_{nk}$'s are mutually independent with distribution functions $F_{nk}(\mathbf{x})$ and characteristic functions $f_{nk}(\mathbf{u})$. It is further assumed that $E\mathbf{S}_n = 0$, $\operatorname{Cov} \mathbf{S}_n \to \mathbf{\Gamma}$, and the $\mathbf{X}_{nk}$'s are asymptotically small (i.e., $\max_{k.i} \operatorname{Var} X_{nki} \to 0$ as $n \to \infty$).

If $\{\mathbf{S}_n\}$ has a limiting distribution function $F(\mathbf{x})$ and characteristic function $f(\mathbf{u})$, then there is a bounded measure M having no mass at the origin and such that for any Borel set $B \in R^N$ whose closure does not contain the origin,

$$M(B) = \lim_{n\to\infty} \sum_{k=1}^{r_n} \int_B \mathbf{x}\mathbf{x}'\, dF_{nk}(\mathbf{x}), \tag{2}$$

where $\mathbf{x}'$ is the transpose of $\mathbf{x}$. Furthermore,

$$\log f(\mathbf{u}) = -\mathbf{u}\mathbf{\Gamma}_1\mathbf{u}' + \int (e^{i\mathbf{u}\mathbf{x}'} - 1 - i\mathbf{u}\mathbf{x}')\,(\mathbf{x}\mathbf{x}')^{-1}\, M(d\mathbf{x}), \tag{3}$$

where $\mathbf{\Gamma}_1$ is a positive definite matrix. The two terms of Eq. (3) correspond to independent random vectors, the first of which is normal with covariance matrix $\mathbf{\Gamma}_1$ and the log of the characteristic function of the second is given by the integral of Eq. (3).

Conversely, if the limit in Eq. (2) exists for all Borel sets, B, whose closure does not contain the origin, and if

$$\sum_{k=1}^{r_n} \int \mathbf{x}'\mathbf{x}\, dF_{nk}(\mathbf{x}) \to \mathbf{\Gamma}_1 + \int \mathbf{x}'\mathbf{x}(\mathbf{x}\mathbf{x}')^{-1}\, M(d\mathbf{x}),$$

then $\{\mathbf{S}_n\}$ converges in distribution and its characteristic function is given by Eq. (3). A random variable with this characteristic function is an I.D. random variable, so called because a characteristic function can be put in this form if and only if for every integer $p \geqslant 1$ there exists a characteristic function $g_p(\mathbf{u})$ such that $f(\mathbf{u}) = (g_p(\mathbf{u}))^p$.

These conclusions are taken partly from Feller ([1], Ch. XVII and p. 245) who gives explicit formulas for the two-dimensional case and also shows how the normal component of $f(\mathbf{u})$ arises. A particularly good development for the special case of finite variances (the case we will consider) and finite fourth moments is given by Takano [2].

For any random vector $\mathbf{X}$, define

$$\pi_{ij} = EX_i^2X_j^2 - EX_i^2EX_j^2 - 2(EX_iX_j)^2. \tag{4}$$

We see that if $i = j$, then π_{ii} is the fourth cumulant of X_i, which is often written $k_4(X_i) = \pi_{ii}$. As usual, ρ_{ij} will be the correlation coefficient. If $\mathbf{X}$ has characteristic function $f(\mathbf{u})$, then

$$\pi_{ij} = \frac{\partial^4}{\partial u_i^2 \, \partial u_j^2} \log f(\mathbf{u}) \Big|_{\mathbf{u}=0} . \tag{5}$$

LEMMA 1. *If* $\mathbf{X}$ *is infinitely divisible with finite* $(2n)$th *absolute moments, then*

$$\frac{\partial^{m_i+m_j}}{\partial u_i^{m_i} \, \partial u_j^{m_j}} \log f(\mathbf{u}) \Big|_{\mathbf{u}=0} = (i)^{m_i+m_j} \int x_i^{m_i} x_j^{m_j} (\mathbf{x}\mathbf{x}')^{-1} \, M(d\mathbf{x}) \tag{6}$$

for all integers m_i *and* m_j *such that* $m_i + m_j \leqslant 2n$.

Proof. First of all, the existence of the $(2n)$th absolute moments ensures the existence of the derivatives on the left-hand side of the equation. The $(2n)$th symmetric derivative can be written ([3], p. 199)

$$\left| \frac{\partial^{2n}}{\partial u_i^{2n}} \log f(0) \right| = \lim_{h \to 0} \int \left(\frac{\sin hx_i}{hx_i} \right)^{2n} x_i^{2n} (\mathbf{x}\mathbf{x}')^{-1} \, M(d\mathbf{x}), \tag{7}$$

and by the Fatou-Lebesque theorem the right-side is greater than or equal to

$$\int x_i^{2n} (\mathbf{x}\mathbf{x}')^{-1} \, M(d\mathbf{x}).$$

Similarly, the integral of x_j^{2n} with respect to $(\mathbf{x}\mathbf{x}')^{-1} \, M(d\mathbf{x})$ is finite. Thus the integral which results from differentiation of Eq. (3) under the integral sign is absolutely convergent, and therefore the differentiation is justified.

Characterization of Independence

THEOREM 1. *Let* $\mathbf{X}$ *be I.D. and let* $E\mathbf{X} = 0$.

1. *If* $\mathbf{X}$ *is normal,* $\{X_i\}$ *are statistically independent if and only if* $\rho_{ij} = 0$ *for all* i, j *such that* $i \neq j$.

2. *If for some finite number* M, $P\{X_i > M\} = 1$ *for all* i, *then* $\{X_i\}$ *are statistically independent if and only if* $\rho_{ij} = 0$ *for all* i, j *such that* $i \neq j$.

3. *In general,* $\{X_i\}$ *are statistically independent if and only if*

$$EX_i^2X_j^2 - EX_i^2EX_j^2 = 0$$

for all i, j *such that* $i \neq j$.

4. *The nonnormal term of* $\mathbf{X}$ *has statistically independent components if and only if* $\pi_{ij} = 0$ *for all* i, j *such that* $i \neq j$.

Statement 1 is included for completeness. We also note that in all cases, pairwise independence implies mutual independence.

Consider statement 3 above. That statistical independence implies $EX_i^2X_j^2 - EX_i^2EX_j^2 = 0$ is well known. Conversely, from Eq. (6) we see that π_{ij} is always positive. Thus if $EX_i^2X_j^2 - EX_i^2EX_j^2 = 0$, then $\pi_{ij} = 0$, $EX_iX_j = 0$, and $\rho_{ij} = 0$. Consider

$$\sum_{\substack{i,j \\ i\neq j}} \pi_{ij} = \int \sum_{\substack{i,j \\ i\neq j}} x_i^2x_j^2(\mathbf{xx}')^{-1}\, M(d\mathbf{x}) = 0. \tag{8}$$

Since the integrand above is 0 only on the axes of R^N, M can have mass only on the axes. In this case, M can be written as $M = \sum_{i=1}^N M_i$, where M_i has mass only on the x_i axis. If we let $h(u_1 ,..., u_N; \mathbf{x})$ be the integrand of Eq. (3), we have

$$\begin{aligned}\log f(\mathbf{u}) + \mathbf{u}\boldsymbol{\Gamma}_1\mathbf{u}' &= \int h(u_1 ,..., u_N , \mathbf{x}) \sum_i M_i(d\mathbf{x}) \\ &= \sum_i \int h(0,..., 0, u_i , 0,..., 0; \mathbf{x})\, M_i(d\mathbf{x}) \qquad (9) \\ &= \sum_i \int h(0,..., 0, u_i , 0,..., 0; \mathbf{x})\, M(d\mathbf{x}),\end{aligned}$$

which is the sum of the log characteristic function of I.D. variates. Thus the nonnormal part of $\mathbf{X}$ has independent components, and the correlation of X_i and X_j equals the correlation of their normal components. Since $\rho_{ij} = 0$, Γ_1 is diagonal and $\{X_i\}$ are mutually independent.

Statement 4 results from the argument above when ρ_{ij} is not necessarily 0.

To prove statement 2, we note that if the X_i's are bounded from below (i.e., for some finite constant M, $P\{X_i > M\} = 1$), $\mathbf{X}$ can have no normal component; i.e., $\boldsymbol{\Gamma}_1 = 0$. Furthermore, M has mass only in the region in which all the x_i's are positive. To see this, we consider an M with a single mass point $\mathbf{x}$, where one of the components of $\mathbf{x}$ is negative. The X_i corresponding to that component is a Poisson variate multiplied by a negative number; and $P\{X_i > M\} < 1$ for all $M > -\infty$.

Finally, we consider

$$\sum_{\substack{i,j \\ i \neq j}} \rho_{ij} = \int_{x_i \geqslant 0} \cdots \int \sum_{\substack{ij \\ i \neq j}} x_i x_j (\mathbf{x}\mathbf{x}')^{-1} M(d\mathbf{x}) = 0 \tag{10}$$

and proceed as before.

A Measure of Dependence

Our next result establishes π_{ij} as a measure of dependence for I.D. random variables.

THEOREM 2. *Let X_i and X_j be jointly I.D.; let $EX_i = EX_j = 0$; and let $\boldsymbol{\Gamma}_1$ be null. Then*

$$\pi_{ij} \leqslant (\pi_{ii}\pi_{jj})^{1/2}. \tag{11}$$

Equality holds if and only if there exist independent random variables X^+ and X^- such that $EX^+ = EX^- = 0$ and the distribution of (X_i, X_j) is the same as $(X^+ + X^-, r(X^+ - X^-))$, where $r = (\pi_{jj}/\pi_{ii})^{1/4}$.

Proof. We use the inequality

$$2x_i^2 x_j^2 \leqslant r^2 x_i^4 + r^{-2} x_j^4$$

for any constant r.

Integrating both sides with respect to $(\mathbf{x}\mathbf{x}')^{-1} M$ gives $2\pi_{ij} \leqslant r^2\pi_{ii} + r^{-2}\pi_{jj}$ for all r. The right-hand side is minimum when $r^4 = \pi_{jj}/\pi_{ii}$. Substituting yields Eq. (11).

On the other hand, if $\pi_{ij}^2 = \pi_{ii}\pi_{jj}$,

$$\int (r^2 x_i^4 + r^{-2} x_j^4 - 2x_i^2 x_j^2)(\mathbf{x}\mathbf{x}')^{-1} M(d\mathbf{x}) = 0 \tag{12}$$

for $r^4 = \pi_{jj}/\pi_{ii}$. Since the integrand is positive, it must therefore be 0 almost everywhere. Thus only in the regions $x_j = \pm r x_i$ will M have positive mass. Here, M is defined on the plane R^2.

Since M has no mass at the origin it can be decomposed, $M = M^+ + M^-$, so that M^+ has mass only when $x_j = rx_i$, and M^- has mass only when $x_j = -rx_i$. On some probability space define X^+ and X^- as independent random variables having log characteristic functions

$$c^+(u) = \int [e^{iux_i} - 1 - iux_i] (x_i^2 + x_j^2)^{-1} M^+(d\mathbf{x})$$

and

$$c^-(u) = \int [e^{iux_i} - 1 - iux_i] (x_i^2 + x_j^2)^{-1} M^-(d\mathbf{x}),$$

respectively. Let

$$\begin{aligned}\log f_{ij}(u_i, u_j) &= \log E \exp[iu_i(X^+ + X^-) + iu_j r(X^+ - X^-)] \\ &= \log E \exp[i(u_i + ru_j) X^+] + \log E \exp[i(u_i - ru_j) X^-] \\ &= \int [e^{iu_i x_i + iur_j x_i} - 1 - i(u_i + ru_j) x_i] (x_i^2 + x_j^2)^{-1} M^+(d\mathbf{x}) \\ &\quad + \int [e^{iu_i x_i - iur_j x_i} - 1 - i(u_i - ru_j) x_i] (x_i^2 + x_j^2)^{-1} M^-(d\mathbf{x}). \end{aligned} \tag{13}$$

Since M^+ has mass only when $x_j = rx_i$, replace rx_i by x_j in the first term; replace $-rx_i$ by x_j in the second term. Then

$$\log f_{ij}(u_i, u_j) = \int [e^{iu_i x_i + iu_j x_j} - 1 - iu_i x_i - iu_j x_j] (x_i^2 + x_j^2)^{-1} M(d\mathbf{x}), \tag{14}$$

and the proof is complete.

Theorem 2 remains true when $\mathbf{\Gamma}_1$ is not null, except that (X_i, X_j) are distributed as $(Z_i + X^+ + X^-, Z_j + rX^+ - rX^-)$, where (Z_i, Z_j) are jointly normal. Of course, Z_i and Z_j can be written as different linear combinations of independent normal variates.

The bound in Eq. (11) makes π_{ij} a legitimate measure of dependence between variates X_i and X_j with no normal component. When $\pi_{ij}^2/\pi_{ii}\pi_{jj}$ is small compared to 1, it is reasonable to infer that the dependence between X_i and X_j is small. Without the bound, to say that π_{ij} is small or large would not be meaningful.

Of course, the correlation coefficient of X_i^2 and X_j^2 is also a measure of dependence, since when it is 0, X_i and X_j are independent, and when it is 1, there is a constant C such that $X_i^2 = CX_j^2$ with probability 1.

III. Central Limit Theorems

Before proving certain central limit theorems, we will show some characterizations of jointly normal and jointly Poisson variates.

Some Characterizations

Lemma 2. *Let* $\mathbf{X}$ *be I.D. with* $E\mathbf{X} = 0$; $\mathbf{X}$ *is normal if and only if* $\pi_{ii} = 0$ *for all* i.

Proof. Consider

$$\sum_i \pi_{ii} = \int \sum_i x_i^4 (\mathbf{xx}')^{-1} M(d\mathbf{x}) = 0.$$

Since M has no mass at the origin (Eq. (3)) and the integrand is zero only at the origin, M is the zero measure. Conversely, if $\mathbf{X}$ is normal, then $\pi_{ii} = 0$ for all i.

Notation. A zero-mean Poisson variate is one whose characteristic function is $\exp[-i\lambda + \lambda e^{iu} - \lambda]$.

Lemma 3. *Let* $\mathbf{X}$ *be I.D. with* $E\mathbf{X} = 0$; $\mathbf{X}$ *is Poisson (i.e., jointly I.D. with Poisson marginals) if and only if* $\pi_{ii} - 2EX_i^3 + EX_i^2 = 0$ *for all* i.

Proof. It is easy to show that when X_i is a zero-mean Poisson variate, $\pi_{ii} = 2EX_i^3 - EX_i^2$. Conversely, the characteristic function of X_i can be written

$$\int (e^{iu_i x_i} - 1 - iu_i x_i)\, x_i^{-2}\, dK_i(x_i),$$

where $K(x_i)$ is a monotonic function which, however, may have a step at the origin.

$$\begin{aligned} \pi_{ii} - 2EX_i^3 + EX_i^2 &= \int (x_i - 1)^2\, dK(x_i) \\ &= 0 \end{aligned}$$

implies that K increases only at $x_i = 1$, and thus X_i is a zero-mean Poisson variate.

Lemma 4. *Let* $\mathbf{X}$ *be I.D. with* $E\mathbf{X} = 0$; $\{X_i\}$ *are mutually independent and Poisson if and only if* $\pi_{ii} - 2EX_i^3 + EX_i^2 = 0$ *for all* i, *and* $EX_iX_j = 0$ *for all* i, j *such that* $i \neq j$.

Proof. Since $\pi_{ii} = 2EX_i^3 - EX_i^2$, X_i has no normal component. Consider

$$\sum_i (\pi_{ii} - 2EX_i^3 + EX_i^2) = \int \sum_i x_i^2(x_i - 1)^2 (\mathbf{xx}')^{-1} M(d\mathbf{x})$$
$$= 0.$$

Thus $M(d\mathbf{x})$ has mass only when all the x_i's are either 0 or 1. However, mass at the origin is excluded in the definition of M. Finally,

$$\sum_{\substack{i,j \\ i \neq j}} EX_iX_j = \int \sum_{\substack{i,j \\ i \neq j}} x_i x_j (\mathbf{xx}')^{-1} M(d\mathbf{x})$$
$$= 0.$$

Since the mass of M occurs only at points such that all coordinates are greater than or equal to 0, and since the integrand is positive at any point $\mathbf{x}$ where more than one coordinate is positive, $M(d\mathbf{x})$ has mass only on the axes. Now, using the same arguments as in Theorem 1, we see that $\{X_i\}$ are independent.

To characterize I.D. distributions corresponding to measures, M, more complicated than those we have considered would require more complicated moments. For example, if X is a scalar variate whose log characteristic function is

$$\int (e^{iux} - 1 - iux)\, x^{-2}\, dK(x),$$

and if k_j is the jth cumulant of X,

$$k_j = (-i)^j \frac{\partial^j}{\partial u^j} \log f_X(u) \Big|_{u=0}, \tag{15}$$

then $X = X_G + X_P$ where X_G is normal, and X_P is Poisson if and only if $k_6 - 2k_5 + k_4 = 0$. In this case, k_4 will be equal to the size of the jump of $K(x)$ at $x = 1$, which means that Var $X_P = k_4$. Similarly, Var $X_G = k_4 - 2k_3 + k_2$ or Var $X_G = k_2 - k_4$.

Central Limit Theorems

The characterizations discussed above are related to certain central limit theorems for I.D. variates and the sums of Eq. (1). Consider a sequence of I.D. random vectors $\mathbf{X}_n = (X_{n1}, ..., X_{nN})$ with characteristic function

$$\log f_n(\mathbf{u}) = -\mathbf{u}\boldsymbol{\Gamma}_{1n}\mathbf{u}' + \int (e^{i\mathbf{ux}'} - 1 - i\mathbf{ux}')\, (\mathbf{xx}')^{-1}\, M_n(d\mathbf{x}). \tag{16}$$

Takano [2] has shown that $f_n(\mathbf{u}) \to f(u)$ if $M_n(B) \to M(B)$ for all bounded Borel sets, B, whose closure does not contain the origin and

$$\mathbf{\Gamma}_{1n} + \int \mathbf{x}'\mathbf{x}(\mathbf{x}\mathbf{x}')^{-1} M_n(d\mathbf{x}) \to \mathbf{\Gamma}_1 + \int \mathbf{x}'\mathbf{x}(\mathbf{x}\mathbf{x}')^{-1} M(d\mathbf{x}).$$

Similarly, for the sequence $\mathbf{S}_n$ (Eq. (1)), $f_n(\mathbf{u}) \to f(u)$ if

$$\sum_{k=1}^{r_n} \int_B \sum_{i=1}^{N} x_i^2 \, dF_{nk}(\mathbf{x}) \to M(B)$$

for all bounded Borel sets, B, whose closure does not contain the origin and

$$\sum_{k=1}^{r_n} \int \mathbf{x}'\mathbf{x} \, dF_{nk}(\mathbf{x}) \to \mathbf{\Gamma}_1 + \int \mathbf{x}'\mathbf{x}(\mathbf{x}\mathbf{x}')^{-1} M(d\mathbf{x}).$$

THEOREM 3. $\mathbf{X}_n$, $n = 1, 2, \ldots$, *are I.D. variates, and* $\mathrm{Cov}(\mathbf{X}_n) \to \mathbf{\Gamma}$. *Then,*

1. $\mathbf{X}_n$ *is asymptotically normal if* $\pi_{ii}(X_{ni}) \to 0$ *as* $n \to \infty$.

2. $\mathbf{X}_n$ *is asymptotically Poisson if* $N \leqslant 2$, $\pi_{ii}(X_{ni}) - 2EX_{ni}^3 + EX_{ni}^2 \to 0$ *as* $n \to \infty$, *and* $\pi_{12}(X_{n1}, X_{n2}) \to 0$.

Proof of statement 1. Consider

$$\begin{aligned}\int_{|\mathbf{x}|>\epsilon} M_n(d\mathbf{x}) &\leqslant \frac{1}{\epsilon^2} \int_{|\mathbf{x}|>\epsilon} \Big(\sum x_i^2\Big)^2 (\mathbf{x}\mathbf{x}')^{-1} M_n(dx) \\ &\leqslant \frac{1}{\epsilon^2} \sum_{i,j} \Pi_{ij}(X_{ni}, X_{nj}) \\ &\leqslant \frac{1}{\epsilon^2} \sum_{i,j} (\Pi_{ii}(X_{ni}) \, \Pi_{jj}(X_{nj}))^{1/2} \\ &\to 0\end{aligned}$$

as $n \to \infty$. Thus $M_n(\{\mathbf{x} : |x| > \epsilon\}) \to 0$. The proof is complete.

Proof of Statement 2. Consider

$$R_1 = \{\mathbf{x} : |\mathbf{x}| > \epsilon \text{ and } |x_i - 1| > \epsilon \text{ for } i = 1, 2\}$$

and

$$\begin{aligned}\int_{R_1} M_n(d\mathbf{x}) &\leqslant \frac{1}{\epsilon^2} \int \sum_i x_i^2 (x_i - 1)^2 (\mathbf{x}\mathbf{x}')^{-1} M_n(d\mathbf{x}) \\ &\leqslant \frac{1}{\epsilon^2} \sum_i (\Pi_{ii} - 2EX_{ni}^3 + EX_{ni}^2) \\ &\to 0\end{aligned}$$

as $n \to \infty$. Consider also $R_2 = \{\mathbf{x} : |x_i| > \epsilon \text{ for } i = 1, 2\}$ and

$$\int_{R_2} M_n(d\mathbf{x}) \leqslant \frac{1}{\epsilon^2} \int \sum_{\substack{i,j \\ i \neq j}} x_i^2 x_j^2 (\mathbf{x}\mathbf{x}')^{-1} M_n(d\mathbf{x})$$

$$\leqslant \frac{1}{\epsilon^2} \sum_{\substack{i,j \\ i \neq j}} \pi_{ij}(X_{ni}, X_{nj})$$

$$\to 0$$

as $n \to \infty$. Therefore $\lim M_n$ is a zero measure in every region excluding the origin, except perhaps on the axes near the points $x_i = 1$.

By an argument similar to that of Lemma 3, it can be shown that $\pi_{ii}(X_{ni}) - 2EX_{ni}^3 + EX_{ni}^2 \to 0$ implies that the marginal distributions of X_{ni} are asymptotically Poisson. Therefore, $\Gamma_{1n} \to 0$ as $n \to \infty$, and

$$\int_{|x_i - 1| < \epsilon} x_i^2 (\mathbf{x}\mathbf{x}')^{-1} M_n(d\mathbf{x}) \to \operatorname{Var} X_i$$

for all $\epsilon > 0$. Since M_n is asymptotically 0 in regions in which more than one component of $\mathbf{x}$ is greater than ϵ and since the integrand is less than or equal to 1, the integral may be reduced to

$$\int_{|\mathbf{x} - e_i| < \epsilon} x_i^2 (\mathbf{x}\mathbf{x}')^{-1} M_n(d\mathbf{x}) \to \operatorname{Var} X_i ,$$

where e_i is either (0, 1) or (1, 0). Since this is true for all $\epsilon > 0$, we have shown that M_n converges to the proper limit, and the proof is complete.

Sufficient conditions for the distribution function of scalar I.D. variates $\{X_n\}$ to converge to the convolution of a normal and Poisson distribution can be obtained using the characterization discussed previously. It can be shown that if $k_6(X_n) - 2k_5(X_n) + k_4(X_n) \to 0$ and both $k_2(X_n)$ and $k_4(X_n)$ are convergent, then X_n converges in distribution to $X = X_P + X_G$, where $\operatorname{Var} X_P = \lim k_4(X_n)$ and $\operatorname{Var} X_G = \lim k_2(X_n) - \lim k_4(X_n)$.

To prove limit theorems for sums of the form of Eq. (1), we need simply apply the comparison Lemma (see ([3], p. 291) or [2]).

COMPARISON LEMMA. *Let* $\mathbf{S}_n$ *be the sum of Eq.* (1) *and let* $\mathbf{S}_n'$ *be a random vector with I.D. law*

$$\log f_n' = \int (e^{i\mathbf{u}\mathbf{x}} - 1 - i\mathbf{u}\mathbf{x}) \sum_k dF_{nk}(\mathbf{x});$$

then $E\mathbf{S}_n' = E\mathbf{S}_n$, $\operatorname{Var} \mathbf{S}_n' = \operatorname{Var} \mathbf{S}_n$, *and* $(\log f_n - \log f_n') \to 0$.

Theorem 3 provides sufficient conditions for the convergence of I.D. variates. Since these conditions involve only moments, they are also sufficient for the convergence of $\mathbf{S}_n$.

THEOREM 4. *Let* $\mathbf{S}_n$ *be sums of the form of Eq.* (1) *with* $E\mathbf{S}_n = 0$, $\operatorname{Cov} \mathbf{S}_n \to \mathbf{\Gamma}$, *and* $\max_{k,i} \operatorname{Var} X_{nki} \to 0$ *as* $n \to \infty$.

1. $\mathbf{S}_n$ *is asymptotically normal if* $\pi_{ii}(S_{ni}) \to 0$ *as* $n \to \infty$ *for all* i.
2. $\mathbf{S}_n$ *is asymptotically Poisson if* $N \leqslant 2$ *and* $\pi_{ii}(\mathbf{S}_{ni}) - 2E\mathbf{S}_{ni}^3 + E\mathbf{S}_{ni}^2 \to 0$ *as* $n \to \infty$ *for all* i *and* $\pi_{12}(S_{n1}, S_{n2}) \to 0$.

The conditions of Theorem 4 may be restated in terms of moments of $\mathbf{X}_{nk}$ as follows:

$$\pi_{ii}(S_{ni}) = \sum_{k=1}^{r_n} [EX_{nki}^4 - 3(EX_{nki}^2)^2].$$

Since $\max_{k,i} EX_{nki}^2 \to 0$ as $n \to \infty$, $\sum (EX_{nki}^2)^2 \leqslant \max_{k,i} EX_{nki}^2 \operatorname{Var}(S_{ni}) \to 0$ as $n \to \infty$. Therefore, the condition $\max_{k,i} EX_{nki}^2 \to 0$ together with

$$\sum_{k=1}^{r_n} EX_{nki}^4 \to 0 \qquad (\text{for all } i)$$

implies $\pi_{ii}(S_{ni}) \to 0$. The condition above is the Liapunov condition. In fact, one can show that the Liapunov condition implies $\max_{k,i} EX_{nki}^2 \to 0$ and thus also implies $\pi_{ii}(S_{ni}) \to 0$.

Similarly,

$$\begin{aligned}\lim_{n\to\infty}(\pi_{ii}(S_{ni}) - 2ES_{ni}^3 + ES_{ni}^2) &= \lim_{n\to\infty} \sum_{k=1}^{r_n} [\pi_{ii}(X_{nki}) - 2EX_{nki}^3 + EX_{nki}^2] \\ &= \lim_{n\to\infty} \sum_{k=1}^{r_n} EX_{nki}^2(X_{nki} - 1)^2,\end{aligned}$$

and

$$\begin{aligned}\pi_{ij}(S_{ni}, S_{nj}) &= \sum_{k=1}^{r_n} \pi_{ij}(X_{nki}, X_{nkj}), \\ &= \sum_{k=1}^{r_n} [EX_{nki}^2 X_{nkj}^2 - EX_{nki}^2 EX_{nkj}^2 - 2(EX_{nki}X_{nkj})^2].\end{aligned}$$

One can show that the last two sums are asymptotically 0, so that

$$\lim \pi_{ij}(S_{ni}, S_{nj}) = \lim \sum_{k=1}^{r_n} EX_{nki}^2 X_{nkj}^2.$$

The results of Theorems 3 and 4 are generalizations of results already reported in Pierre [4].

IV. Applications

Suppose a collection of M airborne targets are illuminated by N different radiation sources having N different frequencies. At each source is a receiver which receives only echoes from its corresponding source. The received data are often modeled as a random process (see Karp, et al. [5]):

$$\mathbf{X}(t) = \sum_{m=0}^{M} \boldsymbol{\varphi}_m(t - t_m), \tag{17}$$

where M is a Poisson-distributed random variable with mean λ, and $\{t_m\}$ are a collection of time delays reflecting the spatial distribution of the targets. The t_m's are assumed to be independent and uniformly distributed over the time period of interest. Finally $\varphi_0(t) \equiv 0$, $\boldsymbol{\varphi}_1(t)$ is the vector random process representing the signatures of the first scatterer when illuminated by the N different sources, and $\boldsymbol{\varphi}_\ell(t)$, $\ell = 1, 2,\ldots,$ are independent and identically distributed signatures corresponding to independent scatterers. For the purpose of illustration, we take the case of $N = 2$ and look at $\mathbf{X}(0)$:

$$X_1(0) = \sum_{m=0}^{M} \varphi_{1m}(t_m) = \sum_{m=0}^{M} \phi_{1m} \tag{18a}$$

and

$$X_2(0) = \sum_{m=0}^{M} \varphi_{2m}(t_m) = \sum_{m=0}^{M} \phi_{2m}\,, \tag{18b}$$

where $\phi_{im} = \varphi_{im}(t_m)$. Let $H(x_1\,, x_2)$ be the joint distribution function of $(\phi_{1m}\,, \phi_{2m})$ for $m > 0$. Then,

$$\begin{aligned}
\log f(u_1\,, u_2) &= \log Ee^{iu_1X_1(0)}e^{iu_2X_2(0)} \\
&= \log\left\{\sum_{m=0}^{\infty} [Ee^{iu_1\phi_{1m}}e^{iu_2\phi_{2m}}]^m\, P\{M = m\}\right\} \\
&= \log\{\exp \lambda[Ee^{iu_1\phi_{1m}}e^{iu_2\phi_{2m}} - 1]\} \\
&= \lambda \iint (e^{iu_1x_1+iu_2x_2} - 1)\, dH(x_1\,, x_2).
\end{aligned} \tag{19}$$

If $E\phi_{1m} = E\phi_{2m} = 0$,

$$\log f(u_1, u_2) = \lambda \iint (e^{iu_1x_1 + iu_2x_2} - 1 - iu_1x_1 - iu_2x_2)\, dH(x_1, x_2). \tag{20}$$

It is now clear that $\mathbf{X}(0)$ is jointly I.D., and $M(d\mathbf{x}) = \lambda(\mathbf{xx}')\, dH(x_1, x_2)$.

One problem which arises in this context is that of choosing the frequencies of the sources so as to maximize the amount of new information contained in each additional frequency band about any desired target. One approach to the problem would be to minimize the amount of redundant information about the target. (This is not equivalent to the original problem, of course.) If the $\mathbf{X}(t)$ processes were normal, this could be accomplished by minimizing the correlation coefficients ρ_{ij}, thus making X_i and X_j approach statistical independence. In the general nonnormal situation, minimizing ρ_{ij} may or may not be relevant. However, the results of Section II show that in the I.D. case, minimizing $EX_i^2X_j^2 - EX_i^2EX_j^2$ causes X_i and X_j to approach statistical independence.

Next, consider the situation in which the number of targets is large; i.e., λ is large. The notion of a central limit theorem is often invoked in order to verify that the random processes $X_i(t)$ are approximately normal. However, certain conditions must be satisfied in order for this to be possible. A simple physical argument will illustrate a situation in which the necessary conditions for the central limit theorem are violated. Assume that we have a large number of scatterers, but that each is highly directional. In this case, most of the scatterers will contribute very little to $X_i(t)$, and only a few will make significant contributions. The process $X_i(t)$ will look more like a shot-noise process (i.e., a small number of impulses) than a normal process. Thus $X_i(t)$ is approximately normal only if the scatterers are not highly directional. The conditions of Section III allow this constraint to be expressed mathematically.

Finally, we note that the results of Section III allow all of the results of [6] to be extended to the vector case. In ([6], Sec. III), the author considered scalar processes $X(t)$ defined by Eq. (17), except that the functions $\varphi_i(t)$ and $\varphi_j(t)$ were not required to be statistically independent. This allows for some coupling among the scatterers. In this case, $X(t)$ is called a conditionally linear process. Such conditionally linear processes are often appropriate as models of clutter from dense, statistically coupled scatterers.

References

1. W. Feller, "An Introduction to Probability Theory and its Applications, Vol. II," John Wiley and Sons Inc., New York, 1966.
2. K. Takano, Multidimensional central limit criterion in the case of bounded variances, *Ann. Inst. of Statist. Math.* **2** (1956), 81–94.

3. M. Loève, "Probability Theory," 3d ed., D. Van Nostrand Company, Princeton, New Jersey, 1963.
4. P. A. Pierre, New conditions for central limit theorems, *Ann. Math. Stat.* **40** (1969), 319–321.
5. S. Karp, R. M. Gagliardi, and I. S. Reed, Radiation models using discrete radiator ensembles, *Proc. IEEE* **56** (1968), 1704–1711.
6. P. A. Pierre, Central limit theorems for conditionally linear random processes, with applications to models of radar clutter, Report RM-6013-PR, The RAND Corp., April 1969.