Available online at www.sciencedirect.com
ScienceDirect
Journal of Approximation Theory 163 (2011) 623–639
Journal of Approximation Theory
www.elsevier.com/locate/jat

Full length articles

L^p-convergence of Fourier sums with exponential weights on $(-1, 1)$ ☆

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Received 22 November 2010; received in revised form 19 January 2011; accepted 2 February 2011
Available online 16 February 2011

Communicated by Doron S. Lubinsky

Abstract

In order to approximate functions defined on $(-1, 1)$ and having exponential singularities at the endpoints of the interval, we study the behavior of some modified Fourier Sums in an orthonormal system related to exponential weights. We give necessary and sufficient conditions for the boundedness of the related operators in suitable weighted L^p-spaces, with $1 < p < \infty$. Then, in these spaces, these processes converge with the order of the best polynomial approximation.

Keywords: Orthogonal polynomials; Fourier series; Approximation by polynomials; Exponential weights

1. Introduction

Letting $w(x) = e^{-(1-x^2)^{-\alpha}}, \alpha > 0, x \in (-1, 1)$, a Pollaczek-type weight, we denote by $\{p_m(w)\}_{m\in\mathbb{N}}$ the corresponding sequence of orthonormal polynomials with positive leading coefficients. Now, let

☆ This work was supported by Università degli Studi della Basilicata (local funds) and by PRIN 2008 "Equazioni integrali con struttura e sistemi lineari".
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0021-9045/$ - see front matter
doi:10.1016/j.jat.2011.02.003

$$S_m(w, f) = \sum_{k=0}^{m-1} c_k p_k(w), \quad c_k = \int_{-1}^{1} p_k(w) f w, \tag{1}$$

be the mth Fourier sum related to a function $f \in L^p_{\sqrt{w}}$, $1 \leq p < \infty$, i.e. f is such that $\|f\sqrt{w}\|_p^p = \int_{-1}^1 |f\sqrt{w}|^p < \infty$.

Concerning the behavior of $S_m(w, f)$ in $L^p_{\sqrt{w}}$, we note that for $p = 2$, since the Weierstrass theorem holds in $L^2_{\sqrt{w}}$ (see [6]), the system $\{p_m(w)\}_{m\in\mathbb{N}}$ is complete. While, for $p \neq 2$, setting $\|S_m(w)\|_p = \sup_{\|f\sqrt{w}\|_p=1} \|S_m(w, f)\sqrt{w}\|_p$, it is easy to prove (see Proposition 3.1) that the uniform boundedness of $\|S_m(w)\|_p$ implies a strong restriction on the parameter p, i.e. $p \in (4/3, 4)$. Furthermore, the assumption $p \in (4/3, 4)$ seems to be not sufficient in order to get $\sup_{m\in\mathbb{N}} \|S_m(w)\|_p < \infty$ (see the proof of Lemma 4.1 with $\sigma = w$, $u = \sqrt{w}$ and $4/3 < p < 4$). This is concordant with a classical result of Nevai (see [12, Cor.14, p. 155]), who proved that if $w(x) = \mathrm{e}^{-\frac{1}{\sqrt[4]{1-x^2}}}$, then $\sup_{m\in\mathbb{N}} \|S_m(w)\|_p < \infty$ implies $p = 2$. Therefore, at the moment only the functions belonging to $L^2_{\sqrt{w}}$ can be represented by a Fourier series in the system $\{p_m(w)\}_{m\in\mathbb{N}}$.

On the other hand, the polynomial approximation of functions defined on $(-1, 1)$ and increasing exponentially at the endpoints ± 1 is required in various contexts, for instance in numerical quadrature and derivation or in the numerical treatment of functional equations. To overcome this gap, we are going to modify the Fourier operator $S_m(w)$, in order to obtain another operator which has a wider application. To this aim we introduce the weights

$$\sigma(x) = v^\lambda(x) w(x) = (1 - x^2)^\lambda \mathrm{e}^{-(1-x^2)^{-\alpha}}$$

and

$$u(x) = v^\mu(x)\sqrt{w(x)} = (1 - x^2)^\mu \mathrm{e}^{-\frac{1}{2}(1-x^2)^{-\alpha}}$$

with $\alpha > 0$ and $\lambda, \mu \geq 0$. We are going to prove some properties of these weights and, also applying an idea used in [5,8], for any function $f \in L^p_u$, we define the sum $S_m(\sigma, f_{\mathcal{B}})$, where $f_{\mathcal{B}}$ is the function f, "truncated" in $\mathcal{B} = \mathcal{B}_m$ (i.e. $f_{\mathcal{B}}(x) = 0$ if $x \notin \mathcal{B}$) and $\mathcal{B}$ is a suitable subset of the Mhaskar–Saff interval. Under necessary and sufficient assumptions, we are going to show that, for a wide class of functions, the sequence

$$\{\chi_{\mathcal{B}} S_m(\sigma, f_{\mathcal{B}})\}_{m\in\mathbb{N}},$$

with $\chi_{\mathcal{B}}$ denoting the characteristic function of $\mathcal{B}$, converges to $f \in L^p_u$, $1 < p < \infty$, with the order of the best polynomial approximation.

2. Preliminary results

Let us first introduce some notation. In the sequel $\mathcal{C}$ will stand for a positive constant that could assume different values in each formula and we shall write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ when $\mathcal{C}$ is independent of $a, b, \ldots$. Furthermore $A \sim B$ will mean that if A and B are positive quantities depending on some parameters, then there exists a positive constant $\mathcal{C}$ independent of these parameters such that $(A/B)^{\pm 1} \leq \mathcal{C}$. Moreover, we denote by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most m.

With ϱ a weight function in $(-1, 1)$, we denote by $\{p_m(\varrho)\}_{m\in\mathbb{N}}$ the corresponding sequence of orthonormal polynomials with leading coefficients $\gamma_m(\varrho) > 0$. Let us define a class of weights.

Following Levin and Lubinsky in [2, p. 5], we will say that the weight $\varrho(x) = \mathrm{e}^{-q(x)}$, $|x| < 1$, belongs to the class $\hat{\mathcal{W}}$, and write $\varrho \in \hat{\mathcal{W}}$, if and only if the function $q : (-1, 1) \to \mathbb{R}$ fulfills the following conditions:

(i) q is even and twice continuously differentiable, with $\lim_{x\to 1} q(x) = +\infty$;
(ii) $q'(x) \geq 0$, $q''(x) \geq 0$ for $x \in (0, 1)$;
(iii) the function

$$T(x) = 1 + \frac{xq''(x)}{q'(x)}$$

is increasing in $[0, 1)$ with $T(0) > 1$;
(iv) for some $A \in (0, 1)$, the function T satisfies $T(x) \sim \frac{q'(x)}{q(x)}$ for $x \in [A, 1)$.

The Mhaskar–Rahmanov–Saff number $a_m = a_m(\varrho)$, related to the weight ϱ, is implicitly defined as the positive root of the equation

$$m = \frac{2}{\pi}\int_0^1 a_m t q'(a_m t)\frac{\mathrm{d}t}{\sqrt{1-t^2}}. \tag{2}$$

Also the equivalence (see [4])

$$q'(a_m) \sim m\sqrt{T(a_m)} \tag{3}$$

can lead to an approximation of a_m.

By a slight abuse of notation, we denote by L^∞ the space of all continuous function f on $(-1, 1)$, with the norm $\|f\|_\infty = \sup_{x\in(-1,1)} |f(x)|$. Then, with regards to the number $a_m = a_m(\varrho)$, where $\varrho \in \hat{\mathcal{W}}$, for any polynomial $P_m \in \mathbb{P}_m$, we have

$$\|P_m\varrho\|_\infty = \|P_m\varrho\|_{L^\infty[-a_m,a_m]}. \tag{4}$$

Moreover, for $1 \leq p \leq \infty$, the restricted range inequality

$$\|P_m\varrho\|_p \leq \mathcal{C}\|P_m\varrho\|_{L^p[-a_m,a_m]} \tag{5}$$

and also the inequality

$$\|P_m\varrho\|_{L^p\{|x|\geq a_{sm}\}} \leq \mathcal{C}\mathrm{e}^{-AmT(a_m)^{-1/2}}\|P_m\varrho\|_{L^p[-a_m,a_m]}, \quad s > 1, \tag{6}$$

hold with $\mathcal{C}$ and A positive constants independent of P_m (see [2, Th. 1.7, p. 12] and [4, Lemma 2.3]).

The following remark is sometimes useful. Letting ϱ be a weight belonging to $\hat{\mathcal{W}}$, consider another weight $\bar{\varrho}(x) = v^\gamma(x)\varrho(x)$, with $v^\gamma(x) = (1-x^2)^\gamma$, $\gamma > 0$. Denote by $a_m = a_m(\varrho)$ and $\bar{a}_m = a_m(\bar{\varrho})$ the related Mhaskar–Rahmanov–Saff numbers. If the parameter γ of $\bar{\varrho}$ is a positive integer number, then, using the equality (4), for any polynomial $P_m \in \mathbb{P}_m$, we have

$$\|P_m\bar{\varrho}\|_\infty = \|P_m v^\gamma\varrho\|_\infty = \|P_m v^\gamma\varrho\|_{L^\infty[-a_{m+2\gamma},a_{m+2\gamma}]}. \tag{7}$$

In the general case $\gamma \in \mathbb{R}$, the following proposition holds.

Proposition 2.1. *Let* $1 \leq p \leq \infty$, $\varrho \in \hat{\mathcal{W}}$ *and* $v^\gamma(x) = (1-x^2)^\gamma$, $\gamma > 0$. *Then, for any* $P_m \in \mathbb{P}_m$, *we have*

$$\left\|P_m v^\gamma\varrho\right\|_p \leq \mathcal{C}\left\|P_m v^\gamma\varrho\right\|_{L^p[-a_m,a_m]} \tag{8}$$

where $a_m = a_m(\varrho)$ *and* $\mathcal{C}$ *is independent of* m *and* P_m.

In the particular case $\varrho(x) = \tau(x) = (1-x^2)^\nu e^{-(1-x^2)^{-\alpha}}$, $\nu, \alpha > 0$, *setting* $v^{-\gamma}(x) = (1-x^2)^{-\gamma}$, $\gamma > 0$, *for any* $P_m \in \mathbb{P}_m$, *we get*

$$\left\| P_m v^{-\gamma} \tau \right\|_p \leq \mathcal{C} \left\| P_m v^{-\gamma} \tau \right\|_{L^p[-a_{sm}, a_{sm}]} \tag{9}$$

for some $s > 1$, *where* $a_{sm} = a_{sm}(\tau)$ *and* $\mathcal{C}$ *is independent of* m *and* P_m.

Proof. We first prove inequality (8). By the restricted range inequality (5), with $a_m = a_m(\varrho)$, we have

$$\begin{aligned}\left(\int_{a_m<|x|<1} |P_m(x) v^\gamma(x) \varrho(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} &\leq v^\gamma(a_m) \left(\int_{a_m<|x|<1} |P_m(x)\varrho(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \\ &\leq \mathcal{C} v^\gamma(a_m) \left(\int_{-a_m}^{a_m} |P_m(x)\varrho(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \\ &\leq \mathcal{C} \left(\int_{-a_m}^{a_m} |P_m(x)\bar{\varrho}(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}},\end{aligned}$$

and (8) follows, since v^γ is decreasing on $(0, 1)$.

Now, in order to prove (9), since

$$\left\| P_m v^{-\gamma} \tau \right\|_p \leq \left\| P_m v^{-\gamma} \tau \right\|_{L^p[-a_{sm}, a_{sm}]} + \left\| P_m v^{-\gamma} \tau \right\|_{L^p\{|x| \geq a_{sm}\}}, \; a_{sm} = a_{sm}(\tau),$$

it suffices to show that the second norm at the right-hand side is bounded by $\mathcal{C} \left\| P_m v^{-\gamma} \tau \right\|_{L^p[-a_{sm}, a_{sm}]}$, $s > 1$.

Let us assume $\gamma > (\alpha+\nu)(1-1/s)$. Then the function $v^{-\gamma}\tau^{1-1/s}$, which is even on $(-1, 1)$, is nonincreasing on $[a_{sm}, 1)$ for a sufficiently large m. Hence, using the restricted range inequality (5), with $a_m(\tau^{1/s}) = a_{sm}(\tau)$, we get

$$\begin{aligned}\left\| P_m v^{-\gamma} \tau \right\|_{L^p\{|x| \geq a_{sm}\}} &= \left\| P_m v^{-\gamma} \tau^{1/s} \tau^{1-1/s} \right\|_{L^p\{|x| \geq a_{sm}\}} \\ &\leq v^{-\gamma}(a_{sm}) \tau^{1-1/s}(a_{sm}) \left\| P_m \tau^{1/s} \right\|_{L^p\{|x| \geq a_{sm}\}} \\ &\leq \mathcal{C} v^{-\gamma}(a_{sm}) \tau^{1-1/s}(a_{sm}) \left\| P_m \tau^{1/s} \right\|_{L^p[-a_{sm}, a_{sm}]}.\end{aligned}$$

Now, taking into account that, for m sufficiently large, the function $v^{-\gamma}\tau^{1-1/s} : [-a_{sm}, a_{sm}] \to \mathbb{R}$ attains its absolute minimum at a_{sm}, we have

$$\left\| P_m v^{-\gamma} \tau \right\|_{L^p\{|x| \geq a_{sm}\}} \leq \mathcal{C} \left\| P_m v^{-\gamma} \tau^{1-1/s} \tau^{1/s} \right\|_{L^p[-a_{sm}, a_{sm}]},$$

which was our claim.

We omit the proof for $\gamma \leq (\alpha + \nu)(1 - 1/s)$, which is simpler than the previous case, being $v^{-\gamma}\tau^{1-1/s}$ a nonincreasing function on $(0, 1)$. □

Let us consider the weights $\varrho \in \hat{\mathcal{W}}$ and $\varphi^2\varrho$, $\varphi^2(x) = 1 - x^2$, and the corresponding systems $\{p_m(\varrho)\}_{m\in\mathbb{N}}$ and $\{p_m(\varphi^2\varrho)\}_{m\in\mathbb{N}}$ of orthonormal polynomials with positive leading coefficients $\{\gamma_m(\varrho)\}_{m\in\mathbb{N}}$ and $\{\gamma_m(\varphi^2\varrho)\}_{m\in\mathbb{N}}$, respectively. The following proposition is useful in different contexts, concerning Lagrange interpolation and Fourier sums.

Proposition 2.2. *If the weights ϱ and $\varphi^2\varrho$ belong to the class $\hat{\mathcal{W}}$, then the equivalence*

$$\frac{\gamma_m(\varrho)}{\gamma_{m+1}(\varphi^2\varrho)} \sim 1 \tag{10}$$

holds with the constants in " $\sim$ " independent of m.

We observe that equivalence (10) is well-known if ϱ satisfies the Szegő condition, i.e. $\int_{-1}^{1} \frac{\log \varrho(x)}{\sqrt{1-x^2}}\,\mathrm{d}x > -\infty$, but in general the weights of the class $\hat{\mathcal{W}}$ violate this condition.

Moreover, we remark that Proposition 2.2 holds also with $\varphi^2(x) = 1 - x^2$ replaced by $v^\gamma(x) = (1-x^2)^\gamma$, where $\gamma > 0$ is an integer number or with $\gamma_{m+1}(\varphi^2\varrho)$ replaced by $\gamma_{m-1}(\varphi^2\varrho)$ (see the proof in Section 4).

Let us introduce now the weight

$$\sigma(x) = v^\lambda(x)w(x) = (1-x^2)^\lambda \mathrm{e}^{-\frac{1}{(1-x^2)^\alpha}}, \tag{11}$$

with $|x| < 1, \alpha > 0, \lambda \geq 0$ and the corresponding orthonormal system $\{p_m(\sigma)\}_{m\in\mathbb{N}}$. It is well-known that the weight w belongs to the class $\hat{\mathcal{W}}$, while σ can be considered as a logarithmic perturbation of w.

Proposition 2.3. *The weight σ, defined by (11), belongs to the class $\hat{\mathcal{W}}$ and fulfills the further condition:*

$$T(x) > \frac{A}{1-x^2}, \tag{12}$$

for some $A > 2$ and for x close enough to 1.

From Propositions 2.2 and 2.3 it follows that $\varphi^2\sigma \in \hat{\mathcal{W}}$ and then

$$\frac{\gamma_m(\sigma)}{\gamma_{m+1}(\varphi^2\sigma)} \sim 1.$$

For a function $f \in L^1_\sigma$, i.e. $\int_{-1}^{1} |f\sigma| < \infty$, we consider the mth Fourier sum in the system $\{p_m(\sigma)\}_{m\in\mathbb{N}}$, given by

$$S_m(\sigma, f) = \sum_{k=0}^{m-1} c_k p_k(\sigma), \quad c_k = \int_{-1}^{1} p_k(\sigma) f\sigma.$$

In Section 3 we will state some results concerning the behavior of $S_m(\sigma, f)$ in some suitable function spaces. To this aim, we introduce the weight function

$$u(x) = v^\mu(x)\sqrt{w(x)} = (1-x^2)^\mu \mathrm{e}^{-\frac{1}{2(1-x^2)^\alpha}}, \tag{13}$$

where $\mu \geq 0, \alpha > 0$ and $|x| < 1$, and denote by L^p_u, $1 < p < \infty$, the collection of all measurable functions f such that

$$\|f\|_{L^p_u} = \|fu\|_p = \left(\int_{-1}^{1} |f(x)u(x)|^p\,\mathrm{d}x\right)^{1/p} < \infty.$$

Letting σ and u be the weights in (11) and (13), since $u = v^{\mu-\lambda/2}\sqrt{\sigma}$, from Proposition 2.1 we deduce

$$\|P_m u\|_p \le \mathcal{C} \begin{cases} \|P_m u\|_{L^p[-a_{sm},a_{sm}]}, & s > 1, \quad \text{if } \mu - \lambda/2 < 0 \\ \|P_m u\|_{L^p[-a_m,a_m]}, & \text{otherwise} \end{cases} \tag{14}$$

where $a_m = a_m(\sqrt{\sigma})$ satisfies

$$1 - a_m \sim m^{-\frac{1}{\alpha+1/2}}, \tag{15}$$

by (3).

3. Main results

First of all we prove the following

Proposition 3.1. *With the previous notation, if*

$$\|S_m(w, f)\sqrt{w}\|_p \le \mathcal{C}\|f\sqrt{w}\|_p, \quad \mathcal{C} \ne \mathcal{C}(m, f), \tag{16}$$

then $p \in (4/3, 4)$.

Proof. The bound (16) implies

$$\sup_m \left\|p_m(w)\sqrt{w}\right\|_p \left\|p_m(w)\sqrt{w}\right\|_q \le \mathcal{C}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Since $w \in \hat{\mathcal{W}}$, the estimates

$$\left\|p_m(w)\sqrt{w}\right\|_r \sim \begin{cases} 1, & 1 \le r < 4, \\ (\log m)^{1/4}, & r = 4, \\ (mT(a_m))^{\frac{2}{3}\left(\frac{1}{4}-\frac{1}{r}\right)}, & 4 < r < \infty, \end{cases}$$

hold (see [2, Th. 1.8, p. 12]), where $a_m = a_m(\sqrt{w})$ and $T(a_m) \sim (1 - a_m^2)^{-1} \sim m^{\frac{1}{\alpha+1/2}}$, by (12) and (15). The proposition is completely proved. □

Therefore, as already mentioned in the Section 1, taking also into account the mentioned result of Nevai, the polynomial approximation by means of the sum $S_m(w, f)$ concerns only a restricted class of functions.

To overcome this drawback, we are going to modify the operator $S_m(w)$. In Section 2, we have introduced the weight $\sigma = v^\lambda w$, the related orthonormal system $\{p_m(\sigma)\}_{m\in\mathbb{N}}$, and, for a function $f \in L^1_\sigma$, the mth Fourier sum $S_m(\sigma, f)$. Now, let $\theta \in (0, 1)$ and $\chi_\theta = \chi_{\theta,m}$ be the characteristic function of the interval $[-a_{\theta m}, a_{\theta m}]$, $a_m = a_m(\sqrt{\sigma})$. Setting $f_\theta = \chi_\theta f$, we consider the sequence $\{\chi_\theta S_m(\sigma, f_\theta)\}_{m\in\mathbb{N}}$ in the function space L^p_u, where $u(x) = v^\mu\sqrt{w}$, $\mu \ge 0$.

Denoting by $E_m(f)_{u,p} = \inf_{P\in\mathbb{P}_m} \|(f - P)u\|_p$ the error of best polynomial approximation in L^p_u, we can state the following theorems.

Theorem 3.2. *Let* $1 < p < \infty$*. Then, for any function* $f \in L^p_u$ *and fixed* $\theta \in (0, 1)$*, the bound*

$$\|\chi_\theta S_m(\sigma, f_\theta)u\|_p \le \mathcal{C}_\theta\|f_\theta u\|_p, \tag{17}$$

with $\mathcal{C}_\theta \neq \mathcal{C}_\theta(m, f)$ and $\mathcal{C}_\theta = \mathcal{O}\left(\log^{-1/2}(1/\theta)\right)$, holds if and only if

$$\frac{v^\mu}{\sqrt{v^\lambda \varphi}} \in L^p \quad \textit{and} \quad \frac{1}{v^\mu}\sqrt{\frac{v^\lambda}{\varphi}} \in L^q, \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{18}$$

Moreover, under the assumption (18), we have

$$\|[f - \chi_\theta S_m(\sigma, f_\theta)]\, u\|_p \leq \mathcal{C}_\theta \left\{E_M(f)_{u,p} + \mathrm{e}^{-AM^\gamma} \|fu\|_p\right\}, \tag{19}$$

where $M = \left\lfloor \frac{\theta m}{2(\theta+1)} \right\rfloor$, $\gamma = \frac{2\alpha}{2\alpha+1}$, $A \neq A(M, f)$, $\mathcal{C}_\theta(m, f) \neq \mathcal{C}_\theta = \mathcal{O}\left(\log^{-1/2}(1/\theta)\right)$.

If we consider the sequence $\{S_m(\sigma, f_\theta)\}_{m\in\mathbb{N}}$ in L^p_u, then Theorem 3.2 holds with $1 < p < 4$, namely we state

Theorem 3.3. *With the notation of Theorem 3.2, for any $f \in L^p_u$, $1 < p < 4$, and $\theta \in (0, 1)$, we have*

$$\|S_m(\sigma, f_\theta) u\|_p \leq \mathcal{C}_\theta \|f_\theta u\|_p, \tag{20}$$

with $\mathcal{C}_\theta(m, f) \neq \mathcal{C}_\theta = \mathcal{O}\left(\log^{-1/4}(1/\theta)\right)$, if and only if

$$\frac{1}{4} - \frac{1}{p} < \mu - \frac{\lambda}{2} < \frac{3}{4} - \frac{1}{p}. \tag{21}$$

Moreover, the conditions (18) imply the estimate

$$\|[f - S_m(\sigma, f_\theta)]\, u\|_p \leq \mathcal{C}_\theta \left\{E_M(f)_{u,p} + \mathrm{e}^{-AM^\gamma} \|fu\|_p\right\}, \tag{22}$$

where M and γ are as in Theorem 3.2, $A \neq A(M, f)$, $\mathcal{C}_\theta \neq \mathcal{C}_\theta(m, f)$ and $\mathcal{C}_\theta$ is as above.

Therefore, under the assumptions on the weights and the parameter p, the two proposed sequences converge in L^p_u to the function f with the same order of the best polynomial approximation. The parameter θ is crucial for the convergence and it cannot assume the value 1; in other words, the "truncation of the function" seems to be essential.

We also remark that, denoting by $S_m(v^\lambda, f)$ the mth Fourier sum with respect to the Jacobi weight $v^\lambda(x) = (1 - x^2)^\lambda$, for any $f \in L^p_{v^\mu}$, it is well-known (see [13,14,9]) that the bound

$$\|S_m(v^\lambda, f) v^\mu\|_p \leq \mathcal{C}\|f v^\mu\|_p, \quad \mathcal{C} \neq \mathcal{C}(m, f),$$

is equivalent to the conditions

$$\frac{v^\mu}{\sqrt{v^\lambda \varphi}} \in L^p \quad \text{and} \quad \frac{v^\lambda}{v^\mu}, \frac{1}{v^\mu}\sqrt{\frac{v^\lambda}{\varphi}} \in L^q, \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

which are the assumption (18), if we exclude the condition $\frac{v^\lambda}{v^\mu} \in L^q$ (see the proof of Theorem 3.2). Then, the behavior of the sequence $\{\chi_\theta S_m(\sigma, f_\theta)\}_{m\in\mathbb{N}}$ in L^p_u is reduced to that of the sequence $\left\{S_m(v^\lambda, f)\right\}_{m\in\mathbb{N}}$ in $L^p_{v^\mu}$.

4. Proofs

Proof of Proposition 2.2. Having set $\varrho(x) = e^{-q(x)}$, we have $\bar\varrho(x) := \varphi^2(x)\varrho(x) = e^{-\left(q(x)+\log\frac{1}{1-x^2}\right)}$. Since $\varrho, \bar\varrho \in \hat{\mathcal{W}}$, the following asymptotic estimates hold for m sufficiently large (see [3, p. 25])

$$\gamma_m(\varrho) = \frac{1}{\sqrt{2\pi}}\left(\frac{2}{a_m}\right)^{m+1/2} \exp\left(\frac{2}{\pi}\int_0^{a_m}\frac{q(s)}{\sqrt{a_m^2-s^2}}\,ds\right)(1+o(1))$$

and

$$\gamma_{m+1}(\bar\varrho) = \frac{1}{\sqrt{2\pi}}\left(\frac{2}{\bar a_{m+1}}\right)^{m+3/2} \exp\left(\frac{2}{\pi}\int_0^{\bar a_{m+1}}\frac{q(s)+\log\frac{1}{1-s^2}}{\sqrt{\bar a_{m+1}^2-s^2}}\,ds\right)(1+o(1))$$

with $a_m = a_m(\sqrt{\varrho})$ and $\bar a_{m+1} = a_{m+1}(\sqrt{\bar\varrho})$. Taking into account that, by (7), $a_{m+1}(\sqrt{\bar\varrho}) = a_{2m+2}(\bar\varrho) = a_{2m+4}(\varrho) = a_{m+2}(\sqrt{\varrho}) = a_{m+2}$ and setting

$$c_m = \exp\left(\frac{2}{\pi}\int_0^{a_{m+2}}\frac{\log\frac{1}{1-s^2}}{\sqrt{a_{m+2}^2-s^2}}\,ds\right),$$

it follows that

$$\begin{aligned}\frac{\gamma_m(\varrho)}{\gamma_{m+1}(\bar\varrho)} &= \left(\frac{a_{m+2}}{a_m}\right)^{m+1/2}\frac{a_{m+2}}{2}\frac{1}{c_m}\frac{\exp\left(\frac{2}{\pi}\int_0^{a_m}\frac{q(s)}{\sqrt{a_m^2-s^2}}\,ds\right)}{\exp\left(\frac{2}{\pi}\int_0^{a_{m+2}}\frac{q(s)}{\sqrt{a_{m+2}^2-s^2}}\,ds\right)}(1+o(1))\\ &=: \left(\frac{a_{m+2}}{a_m}\right)^{m+1/2}\frac{a_{m+2}}{2}\frac{1}{c_m}R_m\,(1+o(1)).\end{aligned}$$

Concerning R_m, we have:

$$\begin{aligned}R_m &= \exp\left(-\frac{2}{\pi}\int_0^1\frac{q(a_{m+2}s)-q(a_m s)}{\sqrt{1-s^2}}\,ds\right)\\ &=: \exp\left(-\frac{2}{\pi}\int_0^1\frac{q'(\tau_s)s(a_{m+2}-a_m)}{\sqrt{1-s^2}}\,ds\right),\end{aligned}$$

with $a_m s < \tau_s < a_{m+2}s$. Since q' is an increasing function, we get

$$\left(\frac{a_{m+2}}{a_m}-1\right)(a_m s)q'(a_m s) \le q'(\tau_s)s(a_{m+2}-a_m) \le \left(1-\frac{a_m}{a_{m+2}}\right)(a_{m+2}s)q'(a_{m+2}s)$$

and then, by (2), we obtain

$$e^{-\left(1-\frac{a_m}{a_{m+2}}\right)(m+2)} \le R_m \le e^{-\left(\frac{a_{m+2}}{a_m}-1\right)m}.$$

Since the following estimates hold (see [3, formula (3.5.3), p. 81] and [2, p. 27])

$$\left(1-\frac{a_m}{a_{m+2}}\right) \le \frac{c}{T(a_m)}\log\left(1+\frac{2}{m}\right), \quad c > 0, \tag{23}$$

$$\left(\frac{a_{m+2}}{a_m}-1\right) \geq \frac{\log\left(1+\frac{2}{m}\right)}{T(a_{m+2})}, \tag{24}$$

and

$$\frac{2}{m}-\frac{2}{m^2} < \log\left(1+\frac{2}{m}\right) < \frac{2}{m}, \tag{25}$$

we deduce

$$e^{-\frac{2c}{T(a_m)}} \leq R_m \leq e^{-\frac{1}{T(a_{m+2})}}.$$

Being $1/T(a_m) = o(1)$ for $m \to \infty$, it follows that $R_m \sim 1$.

Let us now consider $(a_{m+2}/a_m)^{m+1/2}$. By (23)–(25), we have

$$1+\frac{1}{mT(a_{m+2})} \leq \frac{a_{m+2}}{a_m} \leq 1+\frac{\mathcal{C}}{mT(a_m)},$$

and then $(a_{m+2}/a_m)^{m+1/2} \sim 1$, taking into account again that $1/T(a_{m+2}) \sim 1-a_{m+2} = o(1)$ for $m \to \infty$.

Finally, we get

$$\int_0^{1/2} \log\frac{1}{1-s^2}\,\mathrm{d}s < \int_0^{a_{m+2}} \frac{\log\frac{1}{1-s^2}}{\sqrt{a_{m+2}^2-s^2}}\,\mathrm{d}s = \int_0^1 \frac{\log\frac{1}{1-a_{m+2}^2 s^2}}{\sqrt{1-s^2}}\,\mathrm{d}s$$
$$< \int_0^1 \frac{\log\frac{1}{1-s^2}}{\sqrt{1-s^2}}\,\mathrm{d}s$$

and then $c_m \sim 1$. Since $a_m \sim 1$, the proof is completed.

We observe that if $\gamma_{m+1}(\bar{\varrho})$ is replaced by $\gamma_{m-1}(\bar{\varrho})$, the proof is simpler, since $a_{m-1}(\sqrt{\bar{\varrho}}) = a_m(\sqrt{\varrho})$, by (7).

Finally, we remark that the proof still works with $\varphi^2\varrho$ replaced by $v^\gamma\varrho$, where $v^\gamma(x) = (1-x^2)^\gamma$, $\gamma > 0$ is an integer number, if we take into account that, by (7), $a_{m+1}(\sqrt{v^\gamma\varrho}) = a_{m+\gamma+1}(\sqrt{\varrho})$. □

Proof of Proposition 2.3. Let us set $\bar{Q}(x) = (1-x^2)^{-\alpha}$, $Q(x) = \bar{Q}(x) + \lambda\log\frac{1}{1-x^2}$ and $\sigma(x) = e^{-Q(x)}$. Since

$$Q'(x) = \frac{2\lambda x}{1-x^2}\left(1+\frac{\alpha}{\lambda}\bar{Q}(x)\right)$$

and

$$Q''(x) = \frac{2\lambda}{1-x^2}\left(1+\frac{\alpha}{\lambda}\bar{Q}(x)\right) + \frac{4\lambda x^2}{(1-x^2)^2}\left(1+\frac{\alpha(\alpha+1)}{\lambda}\bar{Q}(x)\right),$$

we get that Q is an even function with Q' and Q'' positive on $(0, 1)$ and $\lim_{x\to 1^-} Q(x) = +\infty$. Moreover, calculation leads to

$$T(x) = 1+\frac{xQ''(x)}{Q'(x)} = 2+\frac{2x^2}{1-x^2}\left(1+\frac{\frac{\alpha^2}{\lambda}\bar{Q}(x)}{1+\frac{\alpha}{\lambda}\bar{Q}(x)}\right)$$

whence T is increasing on $[0, 1)$, with $T(0) = 2$ and

$$T(x) \sim \frac{1}{1-x^2}, \quad x \in (0, 1),$$

where the constants in "$\sim$" are independent of x. In particular, if $x \in \left[\frac{1}{\sqrt[4]{1+\alpha^2/(\lambda+\alpha)}}, 1\right)$, since $\left(1 + \frac{\frac{\alpha^2}{\lambda}\bar{Q}(x)}{1+\frac{\alpha}{\lambda}\bar{Q}(x)}\right) \geq \left(1 + \frac{\alpha^2}{\lambda+\alpha}\right)$, we conclude that

$$T(x) > \frac{2\left(1+\alpha^2/(\lambda+\alpha)\right)x^2}{1-x^2} \geq \frac{2\sqrt{1+\alpha^2/(\lambda+\alpha)}}{1-x^2}$$

whence (12) follows.

Finally, for $x \in [1/2, 1)$, we have

$$\frac{Q'(x)}{Q(x)} = \frac{1}{1-x^2}\left[\frac{2\lambda\left(1+\frac{\alpha}{\lambda}\right)x\bar{Q}(x)}{\bar{Q}(x)} + \lambda \log \frac{1}{1-x^2}\right] =: \frac{A}{1-x^2}.$$

Since

$$\frac{\lambda+\alpha}{1+\lambda} \leq A \leq 2(\lambda+\alpha),$$

we deduce

$$\frac{Q'(x)}{Q(x)} \sim \frac{1}{1-x^2} \sim T(x), \quad x \geq \frac{1}{2},$$

which completes the proof. □

We recall that, if f belongs to L^1_σ, its mth Fourier sum $S_m(\sigma, f)$ is defined as

$$S_m(\sigma, f, x) = \sum_{k=0}^{m-1} c_k(\sigma, f)p_k(\sigma, x) = \int_{-1}^{1} K_m(\sigma, x, t)f(t)\sigma(t)\,dt,$$

where $c_k(\sigma, f) = \int_{-1}^{1} p_k(\sigma, t)f(t)\sigma(t)\,dt$ is the kth Fourier coefficient of f in the system $\{p_m(\sigma)\}_{m\in\mathbb{N}}$ and

$$\begin{aligned} K_m(\sigma, x, t) &= \sum_{k=0}^{m-1} p_k(\sigma, x)p_k(\sigma, t) \\ &= \frac{\gamma_{m-1}(\sigma)}{\gamma_m(\sigma)} \frac{p_m(\sigma, x)p_{m-1}(\sigma, t) - p_{m-1}(\sigma, x)p_m(\sigma, t)}{x-t} \end{aligned} \tag{26}$$

is the Christoffel–Darboux kernel. By using the Pollard formula, this kernel can be written as

$$\begin{aligned} K_m(\sigma, x, t) &= -\alpha_m p_m(\sigma, x)p_m(\sigma, t) \\ &\quad + \beta_m \frac{p_m(\sigma, x)p_{m-1}(\varphi^2\sigma, t)\varphi^2(t) - p_{m-1}(\varphi^2\sigma, x)\varphi^2(x)p_m(\sigma, t)}{x-t} \end{aligned} \tag{27}$$

where $\varphi^2(t) = 1 - t^2$,

$$\alpha_m = \left(1 + \frac{\gamma_{m+1}(\varphi^2\sigma)\gamma_{m-1}(\varphi^2\sigma)}{\gamma_m(\sigma)^2}\right)^{-1} \frac{\gamma_{m-1}(\varphi^2\sigma)}{\gamma_m(\sigma)}$$

and

$$\beta_m = \left(1 + \frac{\gamma_{m+1}(\varphi^2\sigma)\gamma_{m-1}(\varphi^2\sigma)}{\gamma_m(\sigma)^2}\right)^{-1} \frac{\gamma_{m+1}(\varphi^2\sigma)\gamma_{m-1}(\varphi^2\sigma)}{\gamma_m(\sigma)^2}.$$

By Proposition 2.2 we have $\gamma_{m-1}(\varphi^2\sigma)/\gamma_m(\sigma) \sim 1 \sim \gamma_{m+1}(\varphi^2\sigma)/\gamma_m(\sigma)$ and then the terms α_m and β_m fulfill $\alpha_m \sim 1 \sim \beta_m$.

Concerning the polynomials $\{p_m(\sigma)\}_{m\in\mathbb{N}}$, the equivalences

$$\sup_{x\in(-1,1)} \left| p_m(\sigma, x)\sqrt{\sigma(x)}\sqrt[4]{|a_m^2 - x^2|}\right| \sim 1 \tag{28}$$

and

$$\sup_{x\in(-1,1)} \left| p_m(\sigma, x)\sqrt{\sigma(x)}\right| \sim (mT(a_m))^{1/6}, \tag{29}$$

have been proved in [2, formulae (1.38) and (1.39), p. 10], where $a_m = a_m\left(\sqrt{\sigma}\right)$ and $T(a_m) \sim (1-a_m)^{-1} \sim m^{\frac{1}{\alpha+1/2}}$, by Proposition 2.3 and (15).

Let $\theta \in (0, 1)$. Then for any $x \in [-a_{\theta m}, a_{\theta m}]$ we have

$$\mathcal{C}_\theta(1 - x^2) \le a_m^2 - x^2 \le 1 - x^2, \tag{30}$$

where the constant $\mathcal{C}_\theta$ depends only on θ. In fact, by (24), we get

$$\begin{aligned} 1 &\le \frac{1-x^2}{a_m^2 - x^2} \le 1 + \frac{1-a_m^2}{\frac{a_m}{a_{\theta m}} - 1} \le 1 + \frac{(1-a_m^2)T(a_m)}{\log(1/\theta)} \\ &\le 1 + \frac{c}{\log(1/\theta)}, \end{aligned}$$

where c is a constant independent of θ.

Hence, by (28) and (30), we deduce the inequality

$$|p_m(\sigma, x)|\sqrt{\sigma(x)\varphi(x)} \le \mathcal{C}_\theta, \quad |x| \le a_{\theta m}. \tag{31}$$

Above,

$$\mathcal{C}_\theta = \mathcal{C}\left(1 + \frac{1}{\log(1/\theta)}\right)^{1/4} \tag{32}$$

with $\mathcal{C}$ independent of m and θ.

In the sequel, we will denote by $\mathcal{H}(f)$ the Hilbert transform of a function f in $(-1, 1)$, i.e. the Cauchy principal value integral

$$\mathcal{H}(f, x) = \int_{-1}^{1} \frac{f(t)}{t-x}\,\mathrm{d}x, \quad x \in (-1, 1).$$

It is well-known that, letting v be a weight function and $1 < p < \infty$, the bound

$$\|\mathcal{H}(f)v\|_p \le \mathcal{C}\|fv\|_p, \quad \mathcal{C} \ne \mathcal{C}(f), \tag{33}$$

holds if and only if $v \in A_p(-1,1)$, (see [10,1,11]), namely for any interval $I \subset (-1,1)$ and with $1 < p < \infty$, v satisfies

$$\left(\frac{1}{|I|}\int_I v^p(x)\,\mathrm{d}x\right)^{1/p}\left(\frac{1}{|I|}\int_I v^{-q}(x)\,\mathrm{d}x\right)^{1/q} \le \mathcal{C} \ne \mathcal{C}(I), \qquad \frac{1}{p}+\frac{1}{q}=1, \tag{34}$$

where $|I|$ is the measure of I. If v is a generalized Jacobi weight of the form

$$\prod_{i=1}^{N}|x-x_i|^{\gamma_i}, \quad \gamma_i > -1,\ x, x_i \in [-1,1],$$

the conditions $v \in L^p(-1,1)$ and $v^{-1} \in L^q(-1,1)$, $1/p+1/q=1$, imply $v \in A_p(-1,1)$. As a consequence, for $v(x)=(1-x^2)^\nu$, if $-\frac{1}{p} < \nu < 1-\frac{1}{p}$ then $v \in A_p(-1,1)$ (see [7]).

Moreover, in the proofs, the following lemma will be useful.

Lemma 4.1. *Let σ and u be the weights in (11) and (13). Then for any $f \in L^p_u$, $1 < p < \infty$, we have*

$$\|S_m(\sigma,f)\,u\|_p \le \mathcal{C}m^\nu\|fu\|_p, \tag{35}$$

for some $\nu > 0$, where $\mathcal{C}$ is independent of m and f.

Proof. Taking into account that $u = v^{\mu-\lambda/2}\sqrt{\sigma}$, let us first assume $\mu-\lambda/2 < 0$. So, using the restricted range inequality (14), with $a_{sm}=a_{sm}(\sqrt{\sigma})$, $s>1$, we get

$$\left\|S_m(\sigma,f)\,v^{\mu-\lambda/2}\sqrt{\sigma}\right\|_p \le \left\|S_m(\sigma,f)\,v^{\mu-\lambda/2}\sqrt{\sigma}\right\|_{L^p[-a_{sm},a_{sm}]}.$$

Then, using the Pollard formula (27) and the restricted range inequality (14), with $a_{sm} = a_{sm}(\sqrt{\sigma})$, $s>1$, we have

$$\begin{aligned}\|S_m(\sigma,f)\,u\|_p \le\ & \mathcal{C}\Big\{\|p_m(\sigma)u\|_{L^p[-a_{sm},a_{sm}]}\,\|p_m(\sigma)f\sigma\|_1\\ &+\left\|p_m(\sigma)\mathcal{H}\left(p_{m-1}(\varphi^2\sigma)\varphi^2 f\sigma\right)u\right\|_{L^p[-a_{sm},a_{sm}]}\\ &+\left\|p_{m-1}(\varphi^2\sigma)\varphi^2\mathcal{H}\left(p_m(\sigma)f\sigma\right)u\right\|_{L^p[-a_{sm},a_{sm}]}\Big\}\\ =:\ & \mathcal{C}\left\{I_1+I_2+I_3\right\}.\end{aligned} \tag{36}$$

For the first term, using the Hölder inequality, with $1/p+1/q=1$, and inequality (14), by (29), we obtain

$$\begin{aligned}I_1 &\le \mathcal{C}\|p_m(\sigma)u\|_{L^p[-a_{sm},a_{sm}]}\left\|p_m(\sigma)\frac{\sigma}{u}\right\|_{L^q[-a_m,a_m]}\|fu\|_p\\ &\le \mathcal{C}\,(mT(a_m))^{\frac{1}{3}}\left(\int_{-a_{sm}}^{a_{sm}}v^{(\mu-\lambda/2)p}(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{-a_m}^{a_m}v^{(\lambda/2-\mu)q}(x)\mathrm{d}x\right)^{\frac{1}{q}}\|fu\|_p\\ &\le \mathcal{C}\,m^{1/3}T(a_m)^{1/3+\lambda/2-\mu}\|fu\|_p,\end{aligned} \tag{37}$$

since $T(a_m)\sim(1-a_{sm}^2)^{-1}$.

Now, consider the term I_2. By using (29) and (33), and then inequality (14), since $T(a_m)\sim(1-a_{sm}^2)^{-1}$ and $\mu-\lambda/2<0$, we get

$$I_2 \le \mathcal{C}m^{1/6}T(a_m)^{1/6+\lambda/2-\mu}\left\|\mathcal{H}\left(p_{m-1}(\varphi^2\sigma)\varphi^2 f\sigma\right)\right\|_{L^p[-a_{sm},a_{sm}]}$$

$$\leq \mathcal{C} m^{1/6} T(a_m)^{1/6+\lambda/2-\mu} \left\| p_{m-1}(\varphi^2\sigma)\varphi^2 f\sigma \right\|_p$$
$$\leq \mathcal{C} m^{1/6} T(a_m)^{1/6+\lambda/2-\mu} \left\| p_{m-1}(\varphi^2\sigma)\varphi^2 \frac{\sigma}{u} \right\|_{L^\infty[-a_m,a_m]} \|fu\|_p$$
$$\leq \mathcal{C} m^{1/3} T(a_m)^{1/3+\lambda/2-\mu} \|fu\|_p. \tag{38}$$

Finally, in order to estimate the term I_3, we can proceed as was done for I_2. We obtain

$$I_3 \leq \mathcal{C} m^{1/3} T(a_m)^{1/3+\lambda/2-\mu+1} \|fu\|_p. \tag{39}$$

Combining (37)–(39) in (36), and taking into account that $T(a_m) \sim (1-a_m)^{-1} \sim m^{\frac{1}{2\alpha+1}}$, our claim follows for $\mu - \lambda/2 < 0$. We omit the proof for $\mu - \lambda/2 \geq 0$, which is similar to the previous one. □

The next proposition can be useful in different contexts.

Proposition 4.2. *Let $0 < a < 1$, u be the weight in (13), with parameter $\alpha > 0$, and $1 \leq p \leq \infty$. There exists an integer $M \geq 1$ such that, for any function $f \in L^p_u$, we have*

$$\|fu\|_{L^p\{|x|\geq a\}} \leq \mathcal{C} \left\{ E_M(f)_{u,p} + \mathrm{e}^{-AMT(a_M)^{-1/2}} \|fu\|_p \right\}, \tag{40}$$

where $\mathcal{C}$, A are positive constants independent of M and f, and $MT(a_M)^{-1/2} \sim M^{\frac{2\alpha}{2\alpha+1}}$.

To complete Proposition 4.2, if $f_\theta = \chi_\theta f$, with χ_θ the characteristic function of $[-a_{\theta m}(\sqrt{\sigma}), a_{\theta m}(\sqrt{\sigma})]$, where σ is the weight in (11), for m sufficiently large we can estimate the L^p_u-distance between f and f_θ by (40) with $M = \left\lfloor \frac{\theta m}{2(\theta+1)} \right\rfloor$, taking into account Proposition 2.1.

Proof of Proposition 4.2. Let $P_M \in \mathbb{P}_M$ be the polynomial of best approximation of f in L^p_u-metric. Since for any $a \in (0,1)$ we can choose M such that $a_{sM} \leq a$, for some $s > 1$. Then, by (6), we get

$$\begin{aligned}\|fu\|_{L^p\{|x|\geq a\}} &\leq \|(f-P_M)u\|_{L^p\{|x|\geq a\}} + \|P_M u\|_{L^p\{|x|\geq a\}} \\ &\leq E_M(f)_{u,p} + \|P_M u\|_{L^p\{|x|\geq a_{sM}\}} \\ &\leq E_M(f)_{u,p} + \mathcal{C}\mathrm{e}^{-AMT(a_M)^{-1/2}} \|P_M u\|_p,\end{aligned}$$

from which our claim follows. □

Proof of Theorem 3.2. Let us first prove that assumptions (18) imply inequality (17). By using the Pollard formula (27), with $\alpha_m \sim 1 \sim \beta_m$, we have

$$\begin{aligned}\|\chi_\theta S_m(\sigma, f_\theta) u\|_p \leq\ & \mathcal{C}\Big\{ \|\chi_\theta p_m(\sigma) u\|_p \, \|p_m(\sigma) f_\theta \sigma\|_1 \\ &+ \left\| \chi_\theta p_m(\sigma) \mathcal{H}\left(p_{m-1}(\varphi^2\sigma)\varphi^2 f_\theta \sigma\right) u \right\|_p \\ &+ \left\| \chi_\theta p_{m-1}(\varphi^2\sigma)\varphi^2 \mathcal{H}\left(p_m(\sigma) f_\theta \sigma\right) u \right\|_p \Big\} \\ =:\ & \mathcal{C}\{A_1 + A_2 + A_3\}.\end{aligned} \tag{41}$$

By using (31), with $\mathcal{C}_\theta = \mathcal{O}\left(\log^{-1/4}(1/\theta)\right)$, and the Hölder inequality, with $1/p + 1/q = 1$, the first term can be estimated as

$$\begin{aligned} A_1 &\le \mathcal{C}_\theta \left\| \frac{\chi_\theta v^\mu}{\sqrt{v^\lambda \varphi}} \right\|_p \|f_\theta u\|_p \left\| \chi_\theta p_m(\sigma) \frac{\sigma}{u} \right\|_q \\ &\le \mathcal{C}_\theta^2 \left\| \frac{v^\mu}{\sqrt{v^\lambda \varphi}} \right\|_p \|f_\theta u\|_p \left\| \frac{1}{v^\mu} \sqrt{\frac{v^\lambda}{\varphi}} \right\|_q \\ &\le \mathcal{C}_\theta^2 \|f_\theta u\|_p, \end{aligned} \tag{42}$$

recalling (18).

Concerning the term A_2, we note that, holding (18), we can use (33) with the weight $v^{\mu-\lambda/2-1/4}$ and, by (31), we obtain

$$\begin{aligned} A_2 &\le \mathcal{C}_\theta \left\| \frac{v^\mu}{\sqrt{v^\lambda \varphi}} \mathcal{H}\left(p_{m-1}(\varphi^2\sigma)\varphi^2 f_\theta \sigma \right) \right\|_p \\ &\le \mathcal{C}_\theta \left\| \frac{\chi_\theta v^\mu}{\sqrt{v^\lambda \varphi}} p_{m-1}(\varphi^2\sigma)\varphi^2 f \sigma \right\|_p \\ &\le \mathcal{C}_\theta^2 \|f_\theta u\|_p. \end{aligned} \tag{43}$$

(We have used again (31) with σ replaced by $\varphi^2\sigma$.)

For the term A_3, proceeding as done for A_2, using (33) with the weight $v^{\mu-\lambda/2+1/4}$, we get

$$A_3 \le \mathcal{C}_\theta^2 \|f_\theta u\|_p, \tag{44}$$

and combining (42)–(44) in (41), inequality (17) follows.

Now we prove that (17) implies (18). From (17), we deduce

$$\|\chi_\theta S_{m+1}(\sigma, f_\theta) u\|_p \le \mathcal{C}_\theta \|f_\theta u\|_p, \qquad \mathcal{C}_\theta(m, f) \ne \mathcal{C}_\theta = \mathcal{O}\left(\log^{-1/2}(1/\theta)\right),$$

where $\chi_\theta = \chi_{\theta,m}$ is the characteristic function of $[-a_{\theta m}, a_{\theta m}]$, $a_{\theta m} = a_{\theta m}(\sqrt{\sigma})$, and $f_\theta = \chi_\theta f$. Hence we get

$$\left\| \chi_\theta \left[S_{m+1}(\sigma, f_\theta) - S_m(\sigma, f_\theta) \right] u \right\|_p \le 2\mathcal{C}_\theta \|f_\theta u\|_p,$$

i.e.

$$\|p_m(\sigma) u\|_{L^p[-a_{\theta m}, a_{\theta m}]} \left| \int_{-a_{\theta m}}^{a_{\theta m}} p_m(\sigma, t) f(t) \sigma(t)\, \mathrm{d}t \right| \le 2\mathcal{C}_\theta \|f_\theta u\|_p,$$

and then

$$\sup_m \|p_m(\sigma) u\|_{L^p[-a_{\theta m}, a_{\theta m}]} \left\| p_m(\sigma) \frac{\sigma}{u} \right\|_{L^q[-a_{\theta m}, a_{\theta m}]} \le 2\mathcal{C}_\theta,$$

with $1/p + 1/q = 1$.

Now, let x_k, $k = 1, \ldots, m$, be the zeros of $p_m(\sigma)$ and $\Delta x_k = x_{k+1} - x_k$. If $x \in [-a_{\theta m}, a_{\theta m}] \cap I_k$, where

$$I_k = \left[x_k + \frac{\Delta x_k}{8}, x_{k+1} - \frac{\Delta(x_k)}{8} \right],$$

since $\Delta x_k \sim \varphi(x_k)/m$ and $(1-x^2) \sim (1-x_k^2) \sim (a_m^2 - x_k^2)$, from [2, formula (12.7), p. 134], i.e.

$$|p_m(\sigma, x)| \sqrt{\sigma(x)} \sim \frac{m}{\Delta x_k} |x - x_k| \left| a_m^2 - x_k^2 \right|^{-1/4},$$

we deduce

$$|p_m(\sigma, x)| \sqrt{\sigma(x)\varphi(x)} \sim 1, \quad x \in I_k \cap [-a_{\theta m}, a_{\theta m}].$$

Hence, denoting by $\mathcal{I}_m = \bigcup_k I_k \cap [-a_{\theta m}, a_{\theta m}]$ we get

$$\begin{aligned} \|p_m(\sigma) u\|^p_{L^p[-a_{\theta m}, a_{\theta m}]} &\geq \mathcal{C} \int_{[-a_{\theta m}, a_{\theta m}] \setminus \mathcal{I}_m} \left| \frac{u(x)}{\sqrt{\sigma(x)\varphi(x)}} \right|^p \mathrm{d}x \\ &\geq \mathcal{C} \left\{ \int_{[-a_{\theta m}, a_{\theta m}]} \left| \frac{u(x)}{\sqrt{\sigma(x)\varphi(x)}} \right|^p \mathrm{d}x - \int_{\mathcal{I}_m} \left| \frac{u(x)}{\sqrt{\sigma(x)\varphi(x)}} \right|^p \mathrm{d}x \right\} \\ &=: \mathcal{C} \{S_1 - S_2\}. \end{aligned}$$

Since the measure of the subset $\mathcal{I}_m$ is less than $\mathcal{C}/m$, by the absolute continuity of the integral, for m sufficiently large, $S_2 \leq \frac{1}{2} S_1$. Whence we get

$$\begin{aligned} \sup_m \|p_m(\sigma) u\|^p_{L^p[-a_{\theta m}, a_{\theta m}]} &\geq \mathcal{C} \sup_m \int_{-a_{\theta m}}^{a_{\theta m}} \left| \frac{u(x)}{\sqrt{\sigma(x)\varphi(x)}} \right|^p \mathrm{d}x \\ &= \mathcal{C} \int_{-1}^{1} \left| \frac{u(x)}{\sqrt{\sigma(x)\varphi(x)}} \right|^p \mathrm{d}x. \end{aligned}$$

So we obtain $\frac{u}{\sqrt{\sigma\varphi}} \in L^p$. Analogously

$$\begin{aligned} \sup_m \left\| p_m(\sigma) \frac{\sigma}{u} \right\|^q_{L^q[-a_{\theta m}, a_{\theta m}]} &\geq \mathcal{C} \sup_m \int_{-a_{\theta m}}^{a_{\theta m}} \left| \frac{1}{u(x)} \sqrt{\frac{\sigma(x)}{\varphi(x)}} \right|^q \mathrm{d}x \\ &= \mathcal{C} \int_{-1}^{1} \left| \frac{1}{u(x)} \sqrt{\frac{\sigma(x)}{\varphi(x)}} \right|^q \mathrm{d}x, \end{aligned}$$

whence $\frac{1}{u}\sqrt{\frac{\sigma}{\varphi}} \in L^q$.

Finally, to prove inequality (19), let $P_M \in \mathbb{P}_M$ the polynomial of best approximation of f in L^p_u-metric. By inequality (17), Lemma 4.1 and Proposition 4.2, for m sufficiently large, we have

$$\begin{aligned} \|[f - \chi_\theta S_m(\sigma, f_\theta)] u\|_p &\leq \|(f - P_M) u\|_p + \|\chi_\theta S_m(\sigma, f_\theta - \chi_\theta P_M) u\|_p \\ &\quad + \|S_m(\sigma, P_M - \chi_\theta P_M) u\|_p + \|(P_M - \chi_\theta P_M) u\|_p \\ &\leq \mathcal{C}_\theta E_M(f)_{u,p} + \mathcal{C}(m^\nu + 1) \|(P_M - \chi_\theta P_M) u\|_p \\ &\leq \mathcal{C}_\theta \left\{ E_M(f)_{u,p} + \mathrm{e}^{-AMT(a_M)^{-1/2}} \|P_M u\|_p \right\} \\ &\leq \mathcal{C}_\theta \left\{ E_M(f)_{u,p} + \mathrm{e}^{-AMT(a_M)^{-1/2}} \|f u\|_p \right\}, \end{aligned}$$

which was our claim, taking into account $T(a_M) \sim (1 - a_M)^{-1} \sim M^{\frac{1}{\alpha + 1/2}}$. □

Proof of Theorem 3.3. To prove that, for $1 < p < 4$, assumptions (18) imply inequality (20), we can proceed similarly to the proof of Theorem 3.2, but with some more details. First of all, since $u = v^{\mu-\lambda/2}\sqrt{\sigma}$, we use inequality (14) with $a_{sm} = a_{sm}(\sqrt{\sigma}),\ s > 1$. Then, by the Pollard formula, we get

$$\begin{aligned}\|S_m(\sigma, f_\theta)u\|_p &\le \|S_m(\sigma, f_\theta)u\|_{L^p[-a_{sm},a_{sm}]}\\ &\le \mathcal{C}\Big\{\|p_m(\sigma)u\|_{L^p[-a_{sm},a_{sm}]}\|p_m(\sigma)f_\theta\sigma\|_1\\ &\quad+\left\|p_m(\sigma)\mathcal{H}\left(p_{m-1}(\varphi^2\sigma)\varphi^2 f_\theta\sigma\right)u\right\|_{L^p[-a_{sm},a_{sm}]}\\ &\quad+\left\|p_{m-1}(\varphi^2\sigma)\varphi^2\mathcal{H}\left(p_m(\sigma)f_\theta\sigma\right)u\right\|_{L^p[-a_{sm},a_{sm}]}\Big\}\\ &=: \mathcal{C}\{B_1+B_2+B_3\}. \end{aligned}\tag{45}$$

For the term B_1, by (28), (31) and the Hölder inequality, with $1/p + 1/q = 1$, we have

$$\begin{aligned}B_1 &\le \mathcal{C}\left\|\frac{v^\mu}{\sqrt{v^\lambda\sqrt{|a_m^2-\cdot^2|}}}\right\|_{L^p[-a_{sm},a_{sm}]}\|f_\theta u\|_p\left\|\chi_\theta p_m(\sigma)\frac{\sigma}{u}\right\|_q\\ &\le \mathcal{C}_\theta\left\|\frac{v^\mu}{\sqrt{v^\lambda\sqrt{|a_m^2-\cdot^2|}}}\right\|_{L^p[-a_{sm},a_{sm}]}\|f_\theta u\|_p\left\|\frac{1}{v^\mu}\sqrt{\frac{v^\lambda}{\varphi}}\right\|_q,\end{aligned}$$

with $\mathcal{C}_\theta = \mathcal{O}\left(\log^{-1/4}(1/\theta)\right)$. Now, consider the first norm at the right-hand side. For $\mu-\lambda/2 \ge 0$, the assumption $p < 4$ implies that this norm is bounded. While, for $\mu - \lambda/2 < 0$, we have

$$\begin{aligned}&\left\|v^{\mu-\lambda/2}|a_m^2-\cdot^2|^{-1/4}\right\|_{L^p[-a_{sm},a_{sm}]}\\ &\le \left\|v^{\mu-\lambda/2}|a_m^2-\cdot^2|^{-1/4}\right\|_{L^p[-a_m,a_m]}+\left\|v^{\mu-\lambda/2}|a_m^2-\cdot^2|^{-1/4}\right\|_{L^p\{a_m<|x|<a_{sm}\}}\\ &\le \left\||a_m^2-\cdot^2|^{\mu-\lambda/2-1/4}\right\|_{L^p[-a_m,a_m]}+\left\||a_m^2-\cdot^2|^{\mu-\lambda/2-1/4}\right\|_{L^p\{a_m<|x|<a_{sm}\}},\end{aligned}$$

since, for $a_m < |x| < a_{sm}$,

$$1-x^2 \ge 1-a_{sm}^2 \sim T(a_{sm})^{-1} \sim a_{sm}^2-a_m^2 \ge \mathcal{C}|a_m^2-x^2|.$$

Hence, recalling also (21), it follows that

$$B_1 \le \mathcal{C}_\theta\|f_\theta u\|_p, \quad \mathcal{C}_\theta = \mathcal{O}\left(\log^{-1/4}(1/\theta)\right).\tag{46}$$

In order to estimate the term B_2, in analogy with (43), we use the boundedness of the Hilbert transform related to the interval $[-a_{sm}, a_{sm}]$ with the weight $v^{\mu-\lambda/2}|a_m - \cdot^2|^{-1/4}$. Note that the assumptions (21) and $1 < p < 4$ imply $v^{\mu-\lambda/2}|a_m - \cdot^2|^{-1/4} \in A_p[-a_{sm}, a_{sm}]$. In fact, we have already seen that $v^{\mu-\lambda/2}|a_m - \cdot^2|^{-1/4} \in L^p[-a_{sm}, a_{sm}]$, and similar arguments apply to show that $v^{\lambda/2-\mu}|a_m - \cdot^2|^{1/4} \in L^q[-a_{sm}, a_{sm}]$. Hence, using (33), by (28) and (31), we obtain

$$B_2 \le \mathcal{C}\left\|\frac{v^\mu}{\sqrt{v^\lambda\sqrt{|a_m^2-\cdot^2|}}}\mathcal{H}\left(p_{m-1}(\varphi^2\sigma)\varphi^2 f_\theta\sigma\right)\right\|_{L^p[-a_{sm},a_{sm}]}$$

$$\leq \mathcal{C} \left\| \frac{\chi_\theta v^\mu}{\sqrt{v^\lambda \sqrt{|a_m^2 - \cdot^2|}}} p_{m-1}(\varphi^2\sigma)\varphi^2 f\sigma \right\|_{L^p[-a_{sm},a_{sm}]} \tag{47}$$

$$\leq \mathcal{C}_\theta \| f_\theta u \|_p, \tag{48}$$

with $\mathcal{C}_\theta = \mathcal{O}\left(\log^{-1/4}(1/\theta)\right)$.

Proceeding as above, distinguishing the two cases $\mu - \lambda/2 + 1/2 \geq 0$ and $\mu - \lambda/2 + 1/2 < 0$, one can show that assumptions (21) and $1 < p < 4$ imply $v^{\mu-\lambda/2+1/2}|a_m - \cdot^2|^{-1/4} \in A_p[-a_{sm}, a_{sm}]$ and then the boundedness (33) of the Hilbert transform in $[-a_{sm}, a_{sm}]$ holds with the weight $v^{\mu-\lambda/2+1/2}|a_m - \cdot^2|^{-1/4}$. Hence, by (28) and (31), we get

$$B_3 \leq \mathcal{C}_\theta \| f_\theta u \|_p, \quad \mathcal{C}_\theta = \mathcal{O}\left(\log^{-1/4}(1/\theta)\right), \tag{49}$$

and, combining this estimate with (46), (48) and (45), inequality (20) follows.

We omit the proofs of (21) and (22), since they follow from (20) proceeding as was done in the proofs of (18) and (19), respectively. □

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