On consecutive happy numbers

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Abstract

Let \( e \geq 1 \) and \( b \geq 2 \) be integers. For a positive integer \( n = \sum_{j=0}^{k} a_j \times b^j \) with \( 0 \leq a_j < b \), define

\[
S_{e,b}(n) = \sum_{j=0}^{k} a_j^e.
\]

\( n \) is called \((e, b)\)-happy if \( S_{e,b}^r(n) = 1 \) for some \( r \geq 0 \), where \( S_{e,b}^r \) is the \( r \)th iteration of \( S_{e,b} \). In this paper, we prove that there exist arbitrarily long sequences of consecutive \((e, b)\)-happy numbers provided that \( e - 1 \) is not divisible by \( p - 1 \) for any prime divisor \( p \) of \( b - 1 \).

\( \text{MSC: primary 11A63; secondary 11A07, 11B05} \)

1. Introduction

For an arbitrary positive integer \( n \), let \( S(n) \) be the sum of the squares of decimal digits of \( n \). That is, if we write \( n = \sum_{j=0}^{k} a_j \times 10^j \) with \( 0 \leq a_1, a_2, \ldots, a_k < 10 \), then \( S(n) = \sum_{j=0}^{k} a_j^2 \). Let \( S^r \) denote the \( r \)th iteration of \( S \), i.e.,

\[
S^r(n) = S(S(\cdots S(n) \cdots))^{r \text{ times}}.
\]

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In particular, we set $S^0(n) = n$. If $S$ is iteratively applied to $n$, it is easy to see (cf. [8]) that we either get 1 or fall into a cycle

$$4 \rightarrow 16 \rightarrow 37 \rightarrow 58 \rightarrow 89 \rightarrow 145 \rightarrow 42 \rightarrow 20 \rightarrow 4.$$  

We say that $n$ is a happy number if we get 1 by applying $S$ to $n$ iteratively, i.e., $S^r(n) = 1$ for some positive integer $r$.

In [2], Grundman and Teeple introduced a generalization of happy numbers. Suppose that $e \geq 1$ and $b \geq 2$. Let $S_{e,b}(n)$ denote the sum of the $e$th powers of base $b$ digits of $n$, i.e.,

$$S_{e,b} \left( \sum_{j=0}^{k} a_j \times b^j \right) = \sum_{j=0}^{k} a_j^e,$$

where $0 \leq a_j < b$. Then $n$ is called $(e,b)$-happy provided that there exists $r \geq 0$ such that $S_{e,b}^r(n) = 1$. Observe that

$$S_{e,b}(n) < (b - 1)^e (\log_b n + 1).$$

So if we iteratively apply $S_{e,b}$ to $n$, the process must reach some fixed point or cycle. The fixed points and cycles of $S_{2,b}$ and $S_{3,b}$ ($2 \leq b \leq 10$) have been listed in [2]. And the $(2,10)$- and $(3,10)$-happy numbers of least heights were calculated in [3].

In the second edition of his famous book [7, Problem E34], Guy asked whether there exist arbitrarily long sequences of consecutive $(2,10)$-happy numbers. For example, the least five consecutive $(2,10)$-happy numbers are

$$44488, 44489, 44490, 44491, 44492.$$  

In [1], El-Sedy and Siksek gave an affirmative answer to Guy’s question. A key of El-Sedy and Siksek’s proof is to find $h > 0$ such that $h + x$ is $(2,10)$-happy for each $x \in \{1, 4, 16, 20, 37, 42, 58, 89, 145\}$.

Naturally we may ask whether there exist arbitrarily long sequences of consecutive $(e,b)$-happy numbers for arbitrary power $e$ and base $b$. In [4], Grundman and Teeple determined the existence of arbitrarily long $d$-consecutive sequences of $(e,b)$-happy numbers for powers $e \geq 2$, bases $b \leq 5$, and $d$ a specified function of $e$ and $b$. (A $d$-consecutive sequence is an arithmetic sequence with the difference $d$.) Furthermore, recently they also investigated arbitrarily long $d$-consecutive sequence of $(e,b)$-happy numbers for $e = 2, 3$ [5] and $e = 5$ [6].

Let $p$ be a prime divisor of $b - 1$. In view of Fermat’s little theorem (cf. [2, Theorem 10]), it is not difficult to show that $S_{e,b}(n) \equiv n \pmod{p}$ for every $n$, provided that $e \equiv 1 \pmod{p - 1}$. However, in this paper, we shall prove the following result.

**Theorem 1.1.** Let $e \geq 1$ and $b \geq 2$ be integers. Suppose that for each prime divisor $p$ of $b - 1$, $e \not\equiv 1 \pmod{p - 1}$. Then for each positive integer $m$, there exists $l > 0$ such that $l + 1, l + 2, \ldots, l + m$ are all $(e,b)$-happy.

By Theorem 1.1, we know that there exist arbitrarily long sequences of consecutive $(2,b)$-happy numbers if $b$ is even. For example, the least nine consecutive $(2,16)$-happy numbers are

$$65988605 + i, \quad i = 0, 1, \ldots, 8.$$  

The proof of Theorem 1.1 will be given in the next section.
2. Proof of Theorem 1.1

Let \( \mathbb{Z}^+ \) denote the set of all positive integers. Since \( 2 - 1 = 1 \mid e - 1 \), below we always assume that \( b \) is even. And for convenience, we abbreviate ‘(e, b)-happy’ to ‘happy’ since \( e \) and \( b \) are always fixed. The following lemma is motivated by El-Sedy and Siksek’s proof in [1].

**Lemma 2.1.** Let \( x \) and \( m \) be arbitrary positive integers. Then for each \( r \geq 1 \), there exists a positive integer \( l \) such that

\[
S_{e,b}^r(l + y) = S_{e,b}^r(l) + S_{e,b}^r(y) = x + S_{e,b}^r(y)
for each \( 1 \leq y \leq m \).

**Proof.** We use induction on \( r \). When \( r = 1 \), choose a positive integer \( s \) such that \( bs > m \) and let

\[
l_1 = \sum_{j=0}^{x-1} b^{s+j}.
\]

Clearly

\[
S_{e,b}(l_1 + y) = S_{e,b}(l_1) + S_{e,b}(y) = x + S_{e,b}(y)
for any \( 1 \leq y \leq m \).

Now assume \( r > 1 \) and the assertion of Lemma 2.1 holds for the smaller values of \( r \). Since \( S_{e,b}(n) \leq (b - 1)^e (\log_b n + 1) \), there exists an \( m' \) satisfying that \( S_{e,b}(y) \leq m' \) for all \( 1 \leq y \leq m \). Thus by the induction hypothesis, there exists an \( l_{r-1} \) such that

\[
S_{e,b}^{r-1}(l_{r-1} + S_{e,b}(y)) = S_{e,b}^{r-1}(l_{r-1}) + S_{e,b}^{r-1}(S_{e,b}(y)) = x + S_{e,b}^r(y),
\]

whenever \( 1 \leq y \leq m \).

Let

\[
l_r = \sum_{j=0}^{l_{r-1}-1} b^{s+j},
\]

where \( s \) satisfies that \( bs > m \). Then

\[
S_{e,b}^r(l_r) = S_{e,b}^{r-1}(S(l_r)) = S_{e,b}^{r-1}(l_{r-1}) = x
\]

and for each \( 1 \leq y \leq m \)

\[
S_{e,b}^r(l_r + y) = S_{e,b}^{r-1}(S_{e,b}(l_r + y)) = S_{e,b}^{r-1}(S_{e,b}(l_r) + S_{e,b}(y)) = S_{e,b}^{r-1}(l_{r-1} + S_{e,b}(y)) = S_{e,b}^{r-1}(l_{r-1}) + S_{e,b}^r(y) = S_{e,b}^r(l_r) + S_{e,b}^r(y).
\]

Suppose that a subset \( D_{e,b} \) of positive integers satisfies that:
(1) For any $n \in \mathbb{Z}^+$, there exists $r \geq 0$ such that $S_{e,b}^r(n) \in D_{e,b}$.
(2) For any $x \in D_{e,b}$, $S_{e,b}(x) \in D_{e,b}$.
(3) For any $x \in D_{e,b}$, there exists $r \geq 1$ such that $S_{e,b}^r(x) = x$.

Then we say that $D_{e,b}$ is a cycle set for $S_{e,b}$. It is not difficult to see that $D_{e,b}$ is finite and uniquely determined by $e$ and $b$. For example, $D_{2,10} = \{1, 4, 16, 20, 37, 42, 58, 89, 145\}$.

**Corollary 2.1.** Let $D_{e,b}$ be the cycle set for $S_{e,b}$. Assume that there exists $h \in \mathbb{Z}^+$ such that $h + x$ is happy for each $x \in D_{e,b}$. Then for arbitrary $m \in \mathbb{Z}^+$, there exists $l \in \mathbb{Z}^+$ such that $l + 1, l + 2, \ldots, l + m$ are all happy.

**Proof.** By the definition of cycle sets, there exists $r \in \mathbb{Z}^+$ such that $S_{e,b}^r(y) \in D_{e,b}$ for all $1 \leq y \leq m$. Applying Lemma 2.1, there exists an $l \in \mathbb{Z}^+$ such that $S_{e,b}^r(l + y) = h + S_{e,b}^r(y)$, whenever $1 \leq y \leq m$. Thus by noting that $x$ is happy if and only if $S_{e,b}^r(x)$ is happy, we are done. □

However, in general, it is not easy to find such an $h$ for $D_{e,b}$. With the help of computers, when $e = 2$ and $b = 10$, El-Sedy and Siksek found

$$h = \sum_{r=1}^{233192} 9 \times 10^{r+4} + 20958$$

by noting that $233192 \times 9^2 + 2^2 + S_{2,10}(958 + x)$ is happy for any $x \in D_{2,10}$. Fortunately, the following lemma eliminates the requirement of finding such a highly constrained $h$.

**Lemma 2.2.** Let $D_{e,b}$ be the cycle set for $S_{e,b}$. Assume that for each $x \in D_{e,b}$, there exists $h_x \in \mathbb{Z}^+$ such that both $h_x + 1$ and $h_x + x$ are happy. Then there exists $h \in \mathbb{Z}^+$ such that $h + x$ is happy for each $x \in D_{e,b}$.

**Proof.** We shall prove that under the assumptions of Lemma 2.2, for any subset $X$ of $D_{e,b}$ with $1 \in X$, there exists $h_X \in \mathbb{Z}^+$ such that $h_X + x$ is happy for each $x \in X$.

The cases $|X| = 1$ and $|X| = 2$ are trivial. Assume that $|X| > 2$ and the assertion holds for any smaller value of $|X|$. Fix some $x$ in $X$, $x \neq 1$. Since $h_x + 1$ and $h_x + x$ are happy, there exists $r \in \mathbb{Z}^+$ such that

$$S_{e,b}^r(h_x + 1) = S_{e,b}^r(h_x + x) = 1$$

and

$$S_{e,b}^r(h_x + y) \in D_{e,b} \quad \text{for all } y \in X,$$

by the definition of the cycle set. Let

$$X^* := \{S_{e,b}^r(h_x + y) : y \in X\}.$$

Then $1 \in X^* \subseteq D_{e,b}$ and $|X^*| < |X|$. Thus by the induction hypothesis, there exists $h_{X^*} \in \mathbb{Z}^+$ such that $h_{X^*} + S_{e,b}(h_x + y)$ is happy for each $y \in X$. Further, in view of Lemma 2.1, there exists $l \in \mathbb{Z}^+$ satisfying that

$$S_{e,b}(l + h_x + y) = h_{X^*} + S_{e,b}(h_x + y)$$

provided that $y \in X$. It follows that $(l + h_x) + y$ is happy for each $y \in X$. Letting $h_X = l + h_x$ completes the induction. 

**Lemma 2.3.** Suppose that for each integer $a$, there exists a happy number $h$ such that

$$h \equiv a \pmod{(b - 1)^e}.$$ 

Then for each $x \in \mathbb{Z}^+$, there exists a happy number $l$ such that $l + x$ is also happy.

**Proof.** Choose $s \in \mathbb{Z}^+$ satisfying that $b^s > x$ and let $x^* = b^s - x$. Suppose that $h$ is the happy number such that

$$h \equiv S_{e,b}(x^*) \pmod{(b - 1)^e}.$$

Note that $hb^{\phi((b - 1)^e)}$ is also happy and

$$hb^{\phi((b - 1)^e)} \equiv h \pmod{(b - 1)^e},$$

where $\phi$ is the Euler totient function. We may assume that $h > S_{e,b}(x^*)$. Write $h = (b - 1)^ek + S_{e,b}(x^*)$. Let

$$l = x^* + \sum_{j=0}^{k-1} (b - 1)b^s + j.$$

Then

$$S_{e,b}(l) = k(b - 1)^e + S_{e,b}(x^*) = h$$

and

$$S_{e,b}(l + x) = S_{e,b}\left(b^s + \sum_{j=0}^{k-1} (b - 1)b^s + j\right) = S_{e,b}(b^s + k) = 1.$$

It follows that both $l$ and $l + x$ are happy. □

**Lemma 2.4.** Let $n$ be a positive odd integer. Then for each $a$ with $a \equiv 1 \pmod{n}$ and positive integer $k$, there exists $r \in \mathbb{Z}^+$ such that

$$(n + 1)^r \equiv a \pmod{n^k}.$$
Proof. It is trivial for \( n = 1 \), so we may assume that \( n > 1 \). Suppose that \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} \) where \( p_1, p_2, \ldots, p_s \) are distinct odd primes and \( \alpha_1, \alpha_2, \ldots, \alpha_s \geq 1 \). For \( 1 \leq i \leq s \), let \( g_i \) be a primitive root of \( p_i^{\alpha_i} \). Assume that

\[
n + 1 \equiv g_i^{\beta_i} \pmod{p_i^{\alpha_i}} \quad \text{and} \quad a \equiv g_i^{\gamma_i} \pmod{p_i^{\alpha_i}}
\]

for each \( 1 \leq i \leq s \). Clearly both \( \beta_i \) and \( \gamma_i \) are divisible by \( \phi(p_i^{\alpha_i}) \) since \( n + 1 \equiv a \equiv 1 \pmod{p_i^{\alpha_i}} \). So we only need to find \( r \) satisfying that

\[
\beta_i r \equiv \gamma_i \pmod{p_i^{\alpha_i}}
\]

for all \( i \), or equivalently,

\[
(\beta_i / \phi(p_i^{\alpha_i}))r \equiv \gamma_i / \phi(p_i^{\alpha_i}) \pmod{p_i^{\alpha_i(k-1)}}.
\]

Note that \( p_i \nmid \beta_i / \phi(p_i^{\alpha_i}) \) since \( n + 1 \not\equiv 1 \pmod{p_i^{\alpha_i+1}} \). Thus such \( r \) always exists in view of the Chinese remainder theorem. \( \square \)

**Corollary 2.2.** Assume that for each integer \( a \), there exists a happy number \( h \) such that

\[
h \equiv a \pmod{b - 1}.
\]

Then there exists a happy number \( h' \) such that

\[
h' \equiv a \pmod{(b - 1)^e}.
\]

Proof. Note that

\[
\sum_{j=1}^{h-1} b^j \equiv h - 1 \equiv a - 1 \pmod{b - 1},
\]

So there exists \( k_1 \in \mathbb{Z} \) such that

\[
\sum_{j=1}^{h-1} b^j \equiv k_1(b - 1) + a - 1 \pmod{(b - 1)^e}.
\]

Similarly there exists an integer \( k_2 \) such that

\[
-k_1(b - 1) + 1 \equiv b^k(k_2(b - 1) + 1) \pmod{(b - 1)^e}.
\]

In light of Lemma 2.4, there exists \( r \in \mathbb{Z}^+ \) such that

\[
b^r \equiv k_2(b - 1) + 1 \pmod{(b - 1)^e}.
\]
Therefore
\[ \sum_{j=1}^{h-1} b^j + b^{h+r} \equiv \sum_{j=1}^{h-1} b^j + bh (k_2 (b - 1) + 1) \equiv a \pmod{(b - 1)^e}, \]
which is apparently happy. \( \square \)

**Lemma 2.5.** Let \( a \) be a positive integer. Assume that there exists a happy number \( l \) and a positive integer \( h^* \) with \( h^* \equiv a \pmod{b - 1} \) such that
\[ l \equiv S_{e,b}(h^*) \pmod{b - 1}. \]
Then there exists a happy number \( h \) such that
\[ h \equiv a \pmod{b - 1}. \]

**Proof.** Multiplying by a power of \( b \), if necessary, we may assume that \( l > S_{e,b}(h^*) \). Let
\[ h = \sum_{j=0}^{l-S_{e,b}(h^*)-1} b^{s+j} + h^*, \]
where we choose \( s \) such that \( b^s > h^* \). Clearly
\[ h \equiv l - S_{e,b}(h^*) + h^* \equiv a \pmod{b - 1}. \]
And
\[ S_{e,b}(h) = l - S_{e,b}(h^*) + S_{e,b}(h^*) = l. \]
Thus \( h \) is the desired happy number. \( \square \)

**Proof of Theorem 1.1.** Write \( b - 1 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} \) where \( p_1, p_2, \ldots, p_s \) are distinct odd primes and \( \alpha_1, \alpha_2, \ldots, \alpha_s \) are positive integers. Let \( 1 \leq g_i \leq p_i^{\alpha_i} \) be a primitive root of \( p_i^{\alpha_i} \) for \( 1 \leq i \leq s \). For each positive integer \( 0 \leq a \leq b - 1 \), let \( L(a) \in \{0, 1, \ldots, b - 1\} \) be the integer such that for each \( 1 \leq i \leq s \),
\[ L(a) = \begin{cases} a - g_i + g_i^e \pmod{p_i^{\alpha_i}} & \text{if } a \not\equiv 1 \pmod{p_i^{\alpha_i}}, \\ 1 \pmod{p_i^{\alpha_i}} & \text{if } a \equiv 1 \pmod{p_i^{\alpha_i}}. \end{cases} \]
Let \( r_a \geq 0 \) be the minimal integer such that \( L^{r_a}(a) = 1 \), where \( L^r \) denotes \( r \)th iteration of \( L \) (in particular \( L^0(a) = a \)). Since \( e \not\equiv 1 \pmod{p_i - 1} \), we have \( g_i^e - g_i \) is prime to \( p_i \) for every \( i \). Hence \( r_a \) always exists.

Combining Corollary 2.1, Lemmas 2.2, 2.3 and Corollary 2.2, now it suffices to show that for each integer \( 0 \leq a \leq b - 1 \) there exists a happy number \( h \) such that
\[ h \equiv a \pmod{b - 1}. \]
We use induction on \( r_a \). If \( r_a = 0 \), then \( a = 1 \), and we can take \( h = 1 \). Now assume that \( r_a \geq 1 \) and the assertion holds for any \( a' \) with \( r_{a'} < r_a \). Clearly \( r_{L(a)} = r_a - 1 \). Hence by the induction hypothesis, there exists a happy number \( l \) such that

\[
l \equiv L(a) \pmod{b - 1}.
\]

Let \( 0 \leq g \leq b - 1 \) be the integer such that for each \( 1 \leq i \leq s \),

\[
g = \begin{cases} 
  g_i \pmod{p_i^{a_i}} & \text{if } a \not\equiv 1 \pmod{p_i^{a_i}}, \\
  1 \pmod{p_i^{a_i}} & \text{if } a \equiv 1 \pmod{p_i^{a_i}}.
\end{cases}
\]

And let

\[
h^* = \sum_{j=1}^{a+b-1-g} b^j + g.
\]

Then

\[
h^* \equiv a + b - 1 - g + g \equiv a \pmod{b - 1}.
\]

Further,

\[
S_{e,b}(h^*) \equiv a + b - 1 - g + g^e = \begin{cases} 
  a - g_i + g_i^e \pmod{p_i^{a_i}} & \text{if } a \not\equiv 1 \pmod{p_i^{a_i}}, \\
  1 \pmod{p_i^{a_i}} & \text{if } a \equiv 1 \pmod{p_i^{a_i}}
\end{cases}
\]

for each \( 1 \leq i \leq s \). Hence

\[
S_{e,b}(h^*) \equiv L(a) \equiv l \pmod{b - 1}.
\]

Thus in light of Lemma 2.5, we are done. \( \Box \)

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References