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Note

Covering Tori with Squares*

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We find that an $n \times n$ toroidal checkerboard can be covered with $\lceil n/k \lceil n/k \rceil \rceil$ $k \times k$ squares, and no fewer. To prove this result we also need to prove that the unit Euclidean torus can be covered with $\lceil \alpha^{-1} \lceil \alpha^{-1} \rceil \rceil$ squares of side α, and no fewer.

1. INTRODUCTION

In this paper we answer the question "how many $k \times k$ squares are required to cover an $n \times n$ toroidal checkerboard?"

Let Z_n^2 be the torus $\{(i, j);\ i, j \text{ integers mod } n\}$, and let $\mathscr{S}_k$ be the class of $k \times k$ squares on Z_n^2 of the form $\{(i, j): i_0 \leqslant i \leqslant i_0 + k - 1, j_0 \leqslant j \leqslant j_0 + k - 1\}$. We shall show that Z_n^2 can be covered with $\lceil \frac{n}{k} \lceil \frac{n}{k} \rceil \rceil$ squares from $\mathscr{S}_k$, and that no fewer will do. In order to prove this, we introduce the Euclidean torus $T^2 = \{(x, y): x, y \text{ reals mod } 1\}$ and the set $\mathscr{S}_\epsilon$ of squares of the form $\{(x, y): x_0 \leqslant x \leqslant x_0 + \epsilon, y_0 \leqslant y \leqslant y_0 + \epsilon\}$; we show that T^2 can be covered with $\lceil \epsilon^{-1} \lceil \epsilon^{-1} \rceil \rceil$ squares from $\mathscr{S}_\epsilon$, and no fewer.

In Section 2 we give a brief sketch of the problem in information theory which motivated our problem. In Section 3 we show that the two covering problems are so closely related as to be almost indistinguishable, and give a lower bound to the number of squares in a cover. The results of Sec-

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tion 3 are proved for tori of any dimension. In Section 4 we present a construction which achieves the bound derived in Section 3 (in two dimensions) and in Section 5 give a detailed example of this construction. In Section 6 we conclude with some remarks about the generalization of the problem to higher dimensions, and its relationship to the corresponding packing problem.

2. Background from Information Theory

Let X be any set, and let $\mathscr{S}$ be any collection of subsets of X with $\bigcup \mathscr{S} = X$. Regard the $x \in X$ as "data points" and the $S \in \mathscr{S}$ as "subsets of allowed uncertainty," such that, when a data point is selected, one is satisfied with knowing an S which contains it. We assume that the x's are selected according to a uniform probability distribution on X.

Then, if M is the smallest number of S's which can cover X, one needs, according to Shannon's noiseless coding theorem, at least $\log_2 M$ bits on the average to identify an S which contains x.

Now suppose that we have the capability of storing m data points before we transmit information about them. Let X^m be the Cartesian m-th power of X, and let $\mathscr{S}^m$ be the class of subsets of X^m of the form $S_1 \times \cdots \times S_m$ for some $S_i \in \mathscr{S}$. Here, if M_m is the smallest number of sets from $\mathscr{S}^m$ needed to cover X^m then, again according to Shannon's theorem, $\log_2 M_m$ bits are needed on the average to specify an m-dimensional data point $(x_1, x_2, ..., x_m)$ up to the uncertainty allowed by $\mathscr{S}^m$. And it is clear that knowing a set from $\mathscr{S}^m$ which contains $(x_1, x_2, ..., x_m)$ is the same as knowing an S_i which contains x_i for each i, so that with a storage of m samples only $1/m \log_2 M_m$ bits will be needed to identify an $x \in X$ up to the allowed uncertainty. Typically (but not always) the function $1/m \log M_m$ will decrease as m increases; this reflects the fact that data storage allows more efficient data compression.

Normally information-theorists are interested in the quantity $\lim_{m\to\infty}(1/m) \log M_m$, which represents the absolute minimum number of bits per sample required when arbitrarily many samples can be stored. The methods used are usually probabilistic and give neither very accurate bounds for M_m nor explicit methods for covering X^m with sets from $\mathscr{S}^m$. Thus it is the object of this paper to compute exactly, and in a constructive manner, the value of M_2 in two important special cases.

Our two torus-covering problems fit this description if we let $X = Z_n =$ integers (mod n) and $\mathscr{S}_k =$ subsets of the form $\{i, i+1, ..., i+k-1\}$, or $X = T =$ real numbers (mod 1), and $\mathscr{S}_\epsilon =$ closed intervals of length ϵ.

3. Upper Bounds for M_m

In this section we will consider the higher-dimensional problem of covering $Z_n{}^m$ with sets from $\mathscr{S}_k{}^m$ and T^m with sets from $\mathscr{S}_\epsilon{}^m$. Let $M_m(n, k)$ and $M_m(\epsilon)$ be the smallest number of sets required in each case.

Theorem 1. $M_m(n, k) = M_m(k/n)$.

Proof. In the torus T^m consider the lattice of n^m points of the form $(n_1/n, n_2/n,..., n_m/n)$, $0 \leqslant n_i < n$. This lattice is isomorphic to $Z_n{}^m$. Note that every point of T^m is in at least one of the cubes

$$\{(x_1, ..., x_m): n_i/n \leqslant x_i \leqslant \frac{n_i + 1}{n}\}.$$

Now if this version of $Z_n{}^m$ is covered with M subsets of the form $\{(n_1/n,..., n_m/m): b_i \leqslant n_i \leqslant b_i + k - 1 \pmod n\}$ we can get a cover of T^m with M sets from $\mathscr{S}_{k/n}^m$ by associating with each set from $\mathscr{S}_k{}^m$ the set $\{(x_1, ..., x_m): b_i/n \leqslant x_i \leqslant (b_i + k)/n\}$. This shows that $M_m(k/n) \leqslant M_m(n, k)$.

On the other hand if T^m is covered with M cubes from $\mathscr{S}_{k/n}^m$, then by translating the cover if necessary we can assume so that no point of the embedded copy of $Z_n{}^m$ lies on an edge of one of the cubes from $\mathscr{S}_{k/n}^m$. Then each cube from $\mathscr{S}_{k/n}^m$ will contain exactly a $k \times k \times \cdots \times k$ cubic array of lattice points, i.e., a cube from $\mathscr{S}_k{}^m$, and these cubes will cover $Z_n{}^m$. This shows that $M_m(n, k) \leqslant M_m(k/n)$ and completes the proof of Theorem 1.

Theorem 1 is very useful for calculating $M_m(n, k)$. It shows, for example, that $M_m(n, k)$ is a monotone decreasing function of k/n, since $M_m(\epsilon)$ is obviously a decreasing function of ϵ. This fact is not easy to prove directly. In the next section we will give a very simple construction which covers T^2 with $\lceil \epsilon^{-1} \lceil \epsilon^{-1} \rceil \rceil$ squares from $\mathscr{S}_\epsilon^2$. By Theorem 1 this shows that $Z_n{}^2$ can be covered with $\lceil {}^n/_k \lceil {}^n/_k \rceil \rceil$ squares from $\mathscr{S}_k{}^2$. Our original proof of this fact was quite long and involved several special cases.

To state Theorem 2 we need to adopt the following notation. If x is a real number, define $\lceil x \rceil^{(1)} = \lceil x \rceil =$ least integer $\geqslant x$, and $\lceil x^{(m)} \rceil = \lceil x \lceil x \rceil^{(m-1)} \rceil$.

Theorem 2. $M_m(\epsilon) \geqslant \lceil \epsilon^{-1} \rceil^{(m)}$.

Corollary. $M_m(n, k) \geqslant \lceil n/k \rceil^{(m)}$.

The corollary follows from Theorem 1.

Proof of Theorem 2. For fixed ϵ we induct on m, the case $m = 1$ being easy since at least $\lceil \epsilon^{-1} \rceil$ intervals of length ϵ are needed to cover T, which has length 1. For $m > 1$ we suppose T^m is covered with M ϵ-cubes. For each $a \in T$ the intersection of one of these cubes with the hyperplane $P_a = \{(a, x_2, ..., x_m)\}$ is either empty or an $(m - 1)$-dimensional ϵ-cube. Since these intersections must cover P_a, if $M(a)$ represents their number then $M(a) \geqslant \lceil \epsilon^{-1} \rceil^{(m-1)}$ by induction. Each $(m - 1)$ ϵ-cube has $(m - 1)$ dimensional volume ϵ^{m-1} and so

$$\int_0^1 \epsilon^{m-1} M(a)\, da = \epsilon^m M,$$

the total volume of the M m-dimensional ϵ-cubes. Hence

$$M = \epsilon^{-1} \int_0^1 M(a)\, da \geqslant \epsilon^{-1} \lceil \epsilon^{-1} \rceil^{(m-1)}$$

and so

$$M \geqslant \lceil \epsilon^{-1} \rceil^{(m)}.$$

4. Two Dimensional Constructions

THEOREM 3. $M_2(\epsilon) = \lceil \epsilon^{-1} \rceil^{(2)}, \qquad M_2(n, k) = \lceil n/k \rceil^{(2)}.$

REMARK. The special case $k = 2$ of this result also is a special case of a theorem of Hales [2, Theorem 2.14].

Proof. Because of Theorems 1 and 2 it will be sufficient to show that $M_2(\epsilon) \leqslant \lceil \epsilon^{-1} \rceil^{(2)}$. Represent T^2 as the set $\{(x, y) \bmod 1\}$, and let L be the line $y = \lceil \epsilon^{-1} \rceil x$. We place $\lceil \epsilon^{-1} \rceil^{(2)}$ squares of side ϵ on T^2, with centers on the line L and equally spaced. These squares will cover T^2.

For let $X_a = \{(x, y) \mid x = a\}$, $a \in T$. Those squares whose centers' x coordinates lie in the interval $[a - \epsilon/2, a + \epsilon/2]$ (mod 1) will have non-empty intersection (which will necessarily be a line segment of length ϵ) with X_a. Since the distance between x coordinates of consecutive squares along L is $1/\lceil \epsilon^{-1} \rceil^{(2)}$, we are guaranteed at least

$$\lfloor \epsilon \cdot \lceil \epsilon^{-1} \rceil^{(2)} \rfloor \geqslant \lfloor \epsilon \cdot \epsilon^{-1} \lceil \epsilon^{-1} \rceil \rfloor = \lceil \epsilon^{-1} \rceil$$

contributing squares. On the other hand the distance between y coordinates of consecutive squares along L is $\lceil \epsilon^{-1} \rceil / \lceil \epsilon^{-1} \rceil^{(2)} \leqslant \epsilon$ so that no "gaps" are left between adjacent line segments. In addition with spacing

$\lceil\epsilon^{-1}\rceil/\lceil\epsilon^{-1}\rceil^{(2)}$ it is clear that $\lceil\lceil\epsilon^{-1}\rceil^{(2)}/\lceil\epsilon^{-1}\rceil\rceil$ segments will suffice to cover X_a and this is $\lceil\epsilon^{-1}\rceil$ since

$$\left\lceil \frac{\lceil \epsilon^{-1}\lceil\epsilon^{-1}\rceil\rceil}{\lceil\epsilon^{-1}\rceil} \right\rceil = \left\lceil \frac{\epsilon^{-1}\lceil\epsilon^{-1}\rceil}{\lceil\epsilon^{-1}\rceil} \right\rceil = \lceil\epsilon^{-1}\rceil$$

(using the fact that $\lceil\lceil x\rceil/m\rceil = \lceil x/m\rceil$ for real x, integral m). But we have already seen that there will be at least this many segments available, so X_a is covered for each a. This proves Theorem 3.

5. An Example of the Construction

In Figure 1 we illustrate the construction of Theorem 3, for $n = 13$, $k = 5$. The large square represents the torus T^2, and since $\lceil 13/5\rceil = 3$, we have drawn the line $y = 3x$ with $\lceil 3 \cdot 13/5\rceil = 8$ equally spaced points (the black points) on it. According to Theorem 3, the $5/13 \times 5/13$ squares centered at these points will cover T^2. One of the squares has been drawn in with broken lines.

Superimposed on T^2 we have drawn a 13×13 square grid. The problem is to cover this grid, Z_{13}^2, with 8 5×5 "grid squares," i.e., squares whose

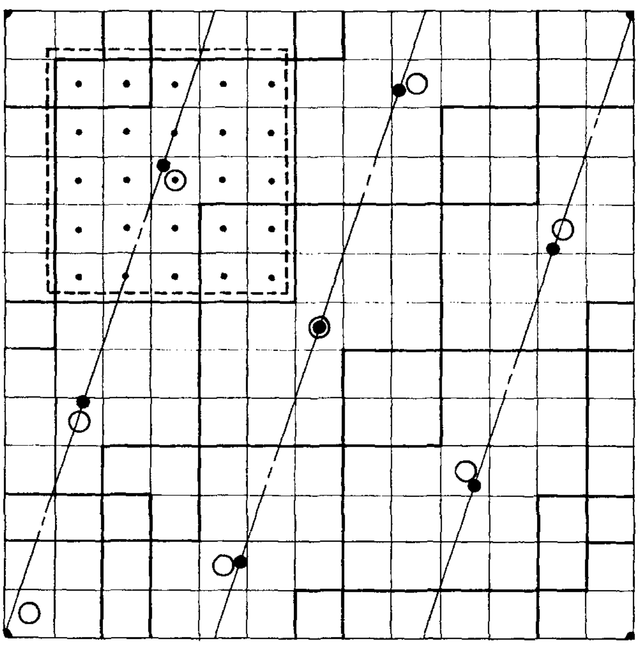

Fig. 1. $M_2(13, 5) = 8$.

edges lie on the grid lines. Clearly the squares centered at the black dots will not do. However, if we place a lattice point at the center of each grid square, then each square centered at a black dot will "catch" a 5×5 square array of these lattice points. Finally, since each lattice point will be "caught," the 5×5 grid squares which contain these 5×5 arrays of lattice points will cover Z_{13}^2. The centers of these squares are marked with small circles, and their boundaries with heavy lines.

6. Concluding Remarks

One obvious question is: "What happens in dimensions $m \geqslant 3$?" We do not know. The situation is certainly not as pleasant as for $m = 2$, since we have been able to show that, while $M_3(k/n) = \lceil n/k \rceil^{(3)}$ for $n + k \leqslant 9$, $M_3(\frac{3}{7}) > 17 = \lceil 7/3 \rceil^{(3)}$. It is possible to show that

$$\lim_{m \to \infty} M_m(\epsilon)^{1/m} = \epsilon^{-1}$$

(see [1], for example), but this is a long way from the exact calculation of the quantities M_m.

Another possible question is: "What about *packing* tori with cubes?" The answer is that all of the theorems and constructions can be dualized with no trouble, and that in two dimensions the maximum number of ϵ-squares which can be packed into T^2 without overlap is $\lfloor \epsilon^{-1} \lfloor \epsilon^{-1} \rfloor \rfloor$. However the result has already been obtained by Hales [2]. (He considers only the combinatorial torus Z_n^2, however.) And the packing problem has been considered in higher dimensions, both by Hales and by Baumert *et al.* [3]. It may be that some or all of the higher-dimensional results obtained by these authors have analogs for coverings. It is somewhat surprising that the covering problem is apparently no easier than the packing problem, because the calculation of the limit $\lim_{m \to \infty} 1/m\ M_m(\epsilon)$ in the packing case has not been done except in the trivial case where ϵ is the reciprocal of an integer, whereas, as we have mentioned, the limit in the covering case has been calculated (and is not particularly difficult).

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