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Palindromic primitives and palindromic bases in the free group of rank two ☆

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Abstract

The present paper records more details of the relationship between primitive elements and palindromes in F_2, the free group of rank two. We characterize the conjugacy classes of palindromic primitive elements as those in which cyclically reduced words have odd length. We identify large palindromic subwords of certain primitives in conjugacy classes which contain cyclically reduced words of even length. We show that under obvious conditions on exponent sums, pairs of palindromic primitives form palindromic bases for F_2. Further, we note that each cyclically reduced primitive element is either a palindrome, or the concatenation of two palindromes.

Keywords: Free groups; Primitive elements; Palindromes

1. Introduction

Notation 1. For each natural number $n \geqslant 2$, let F_n denote the nonabelian free group of rank n, which we identify with the set of reduced words in the alphabet $A_n := \{x_1, \ldots, x_n\}^{\pm}$. For elements $w, v \in \mathrm{F}_n$, we write $w \equiv_n v$ if w and v are equal words, and $w =_n v$ if w and v are equal elements of F_n. We write wv $(w.c.)$ for the concatenation

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of the words w and v and wv for the product of w and v in F_n. We write $|w|_n$ for the word-length of w in A_n. Let $\Psi_n : \mathrm{F}_n \to \mathrm{F}_n$ be the map which reverses each word in F_n. For convenience, we usually omit the subscript n from $\equiv_n$, $=_n$, Ψ_n, $|\cdot|_n$, and we write $x := x_1$ and $y := y_1$ (so F_2 is the free group on two generators x and y).

Recall that an element $w \in \mathrm{F}_n$ is said to be a *palindrome* if $\Psi(w) = w$ (that is, "w reads the same forwards and backwards") and *primitive* if it is an element of some basis for F_n. Much is known about the structure of primitive elements in F_2 (see, for example, [2,4–7]) and indeed primitive elements in free groups of rank greater than two (see, for example, [8, pp. 162–169], [1,3,6]). A newly emerging theme in the study of primitive elements in free groups is the relationship between primitive elements and palindromes (see [6,7]). The present paper records more details of this relationship.

Bardakov, Shpilrain and Tolstykh noted in [6, p. 581] that each conjugacy class of primitive elements contains an element w such that either xwy^{-1} is a palindrome or $x^{-1}wy$ is a palindrome. It is possible to make a rather more explicit statement concerning the existence of palindromes, and 'large' palindromic subwords, in conjugacy classes of primitive elements.

Theorem 1. *Let p be a primitive element in F_2. Let X be the exponent sum of x in p and let Y be the exponent sum of y in p. Then*:

(1) *$X + Y$ is odd if and only if the conjugacy class of p contains exactly one palindrome* (*a* palindromic primitive);
(2) *if $X + Y$ is even then the conjugacy class of p does not contain a palindrome but does contain*:
  (a) *exactly one element of the form $x^{\epsilon}w$ $(w.c.)$, and*
  (b) *exactly one element of the form $y^{\delta}v$ $(w.c.)$,*
  where $\epsilon, \delta \in \{\pm 1\}$, the sign of ϵ (*respectively δ*) *matches the sign of X* (*respectively Y*), *and $w, v \in \mathrm{F}_2$ are palindromes of length $|X| + |Y| - 1$.*

Osborne and Zieschang [4] have recorded an efficient algorithm for writing down a primitive element in F_2 with a given relatively prime pair of exponent sums. Theorem 1 is proved by observing the symmetries of a diagrammatic expression of Osborne and Zieshang's construction, concerning palindromic primitives. The examination of conjugacy classes of primitive elements via the corresponding exponent sums goes back to Nielsen's work of the early 20th century, which includes the well-known result that conjugacy classes of primitive elements in F_2 are in one-to-one correspondence with the set of ordered pairs of integers which are relatively prime, via the map which takes $w \in \mathrm{F}_2$ to the pair of exponent sums (see, for example, [8, p. 169], [4]).

Our second theorem demonstrates that, provided some obvious conditions on the exponent sum pairs are satisfied, pairs of palindromic primitives form bases for F_2.

Theorem 2. *Let A, B, X and Y be integers such that $AY - BX \in \{\pm 1\}$, $A + B$ is odd and $X + Y$ is odd. The unique palindromic primitive p with exponent sum pair (A, B) and the* *unique palindromic primitive q with exponent sum pair (X, Y) form a basis $\{p, q\}$ of* F_2 (*a* palindromic basis).

The proof of Theorem 2 also involves an examination of Osborne and Zieschang's construction.

It is trivial to check that if $w \in F_2$ is a product of at most two palindromes, then the image of w under an inner automorphism is also a product of at most two palindromes. Thus Theorem 1 supplies another proof of the following:

Lemma 3 *(Bardakov, Shpilrain, and Tolstykh, [6, Lemma 1.6, p. 579]). Each primitive element in* F_2 *is the product of at most two palindromes.*

Our third theorem indicates a way in which Lemma 3 is manifest in the reduced words spelling primitive elements in F_2.

Theorem 4. *For each primitive element $w \in F_2$ one of the following holds*:

(1) *w is a palindromic primitive*;
(2) *$w \equiv pq$ $(w.c.)$ for nontrivial palindromes p, q*;
(3) *$w \equiv apa^{-1}$ $(w.c.)$ for a nontrivial palindrome p and a nontrivial word $a \in F_2$*;
(4) *$w \equiv apqa^{-1}$ $(w.c.)$ for nontrivial palindromes p, q and a nontrivial word $a \in F_2$.*

Theorem 4 follows immediately from Lemma 3 and the following result:

Lemma 5. *For each natural number $n \geqslant 2$, a nonpalindromic cyclically reduced element $w \in F_n$ is a product of two palindromes in F_n if and only if $w \equiv pq$ $(w.c.)$ for palindromes $p, q \in F_n$.*

As yet, there is no known algorithm to determine the palindromic (or primitive length) of an element in F_2 (or more generally, F_n) [6, Problems 1 and 2, p. 576]. It follows from Lemma 5 that it is easy to determine whether or not the palindromic length of an element in F_2 is zero, one or two.

The structure of the present paper is simple: Theorem 1 is the subject of Section 2, Theorem 2 is the subject of Section 3 and Lemma 5 is the subject of Section 4.

After acceptance of this paper, the author became aware that Theorem 2 will also appear in a paper by Kassel and Reutenauer (Ann. Mat. Pura Appl., in press).

2. Palindromic primitives

Recall the following simple procedure, due to Osborne and Zieschang [4], for writing down a primitive element in F_2 with a given relatively prime pair of exponent sums X and Y.

Construction 6 *(Osborne and Zieschang, [4, §1.1]).* Draw $|X| + |Y|$ equally spaced distinguished points $p_1, p_2, \dots, p_{|X|+|Y|}$ (with indices read around the unit circle in the clockwise direction) on the unit circle in $\mathbb{R}^2$. Let l_1 be x if $X \geqslant 0$ and x^{-1} if $X < 0$. Let l_2 be y if $Y \geqslant 0$ and y^{-1} if $Y < 0$. Label with l_1 the points $p_1, p_2, \ldots, p_{|X|}$, and label with l_2 the remaining distinguished points. Let i be an integer such that $1 \leqslant i \leqslant |X| + |Y|$ (we call p_i the *first point*). Let $q_1 := p_i$. Inductively define q_j for $j = 2, \ldots, |X| + |Y|$ as follows: let q_j be the $|X|$th distinguished point around the circle from q_{j-1} in the clockwise direction. For each $j = 1, \ldots, |X| + |Y|$, let a_j be the label on the point q_j. The word $a_1 a_2 \ldots a_{|X|+|Y|}$ is primitive with exponent sum pair $X + Y$.

Note that each of the $|X| + |Y|$ distinct cyclically reduced primitive elements in the conjugacy class corresponds to a particular choice of first point in Construction 6.

Remark 7. It appears to have gone unremarked that a very fast algorithm for determining whether or not a given cyclically reduced word $w \in \mathrm{F}_2$ is primitive follows from Construction 6. We may assume that $|X| + |Y| = |w|$ (since otherwise w is not primitive [5]). Place the letters of w around the unit circle, placing the jth letter so that, traveling around the unit circle in the clockwise direction, the distance from $(1, 0)$ to the jth letter is

$$\frac{2\pi j \min\{|X|, |Y|\}}{|X| + |Y|} \text{ units.}$$

If two letters are placed at the same point, then w is not primitive (since X and Y are not relatively prime). Otherwise, w is primitive if and only if the occurrences of x (or x^{-1} if $X < 0$) lie in consecutive places as read around the circle.

Proof of Theorem 1. First consider the case that $X + Y$ is odd. The result is obvious in the case that $X = 0$ or $Y = 0$, so we may assume that $X, Y \neq 0$. Consider first the case that X is odd and Y is even. Consider the diagram of Construction 6 and the line $\mathcal{L}$ which passes through the Origin and the distinguished point $p_{(|X|+1)/2}$. The diagram is symmetric about $\mathcal{L}$, and $p_{(|X|+1)/2}$ is the only distinguished point which falls on $\mathcal{L}$. Symmetry about $\mathcal{L}$ ensures that the sequence constructed by reading the label on every $|X|$th distinguished point around the circle from $p_{(|X|+1)/2}$ in the clockwise direction, and the sequence constructed by reading the label on every $|X|$th distinguished point around the circle from $p_{(|X|+1)/2}$ in the anti-clockwise direction, must be identical. It follows that if we choose the first point in Construction 6 such that $q_{(|X|+|Y|+1)/2} = p_{(|X|+1)/2}$, the result of the construction is a palindrome. The uniqueness of the palindrome in the conjugacy class is immediate from the fact that $\mathcal{L}$ is the unique line of symmetry in the diagram. The case that Y is odd is handled similarly.

Now consider the case that $|X| + |Y|$ is even. Since X and Y are relatively prime, both X and Y are odd. A cyclically reduced element in the conjugacy class of p must have even length. Palindromes are cyclically reduced. A palindrome of even length must have even exponent sums, hence there is no palindrome in the conjugacy class of p. Again consider the diagram of Construction 6 and the line $\mathcal{L}$ which passes through the Origin and the distinguished point $p_{(|X|+1)/2}$. This time $\mathcal{L}$ also passes through the distinguished point $p_{|X|+(|Y|+1)/2}$. Again, $\mathcal{L}$ is a line of symmetry in the diagram. As above, it follows that for each choice of first point in Construction 6 such that $q_{(|X|+|Y|+2)/2} \in \{p_{(|X|+1)/2}, p_{|X|+(|Y|+1)/2}\}$, the result of the construction is a word of length $|X| + |Y|$ such that the terminal subword of length $|X| + |Y| - 1$ is a palindrome. The fact that there are only two such words in the conjugacy class is immediate from the fact that $\mathcal{L}$ is the unique line of symmetry in the diagram. □

Remark 8. In case $|X| + |Y|$ is odd and $|X|$ is odd, the choice of first point which gives the unique palindromic primitive is p_k such that

$$k \equiv \frac{|X|+1}{2} - \frac{(|X|+|Y|-1)\min\{|X|,|Y|\}}{2} \mod\big(|X|+|Y|\big).$$

In case $|X| + |Y|$ is odd and $|Y|$ is odd, the choice of first point which gives the unique palindromic primitive is p_k such that

$$k \equiv |X| + \frac{|Y|+1}{2} - \frac{(|X|+|Y|-1)\min\{|X|,|Y|\}}{2} \mod\big(|X|+|Y|\big).$$

3. Palindromic bases

In this section we prove our result concerning palindromic bases of F_2.

Proof of Theorem 2. The theorem is easily verified in case one of A, B, X or Y is 0, so we may assume that each is nonzero. In fact, allowing for the action of automorphisms of F_2, we may assume without loss of generality that $0 < A < B$, $0 < X < Y$ and $A + B > X + Y$. Note that

$$AY - BX = 1 \quad \Rightarrow \quad A(X+Y) - (A+B)X = 1. \tag{1}$$

Let $p_1, p_2, \ldots, p_{X+Y}$ be the distinguished points when Construction 6 is applied to X and Y, and let $w_{X,Y}$ be the result when p_1 is chosen as the first point. Let $r_1, r_2, \ldots, r_{A+B}$ be the distinguished points when Construction 6 is applied to A and B, and let $w_{A,B}$ be the result when r_1 is chosen as the first point. By [4, Theorem 1.2, p. 18], $\{w_{A,B}, w_{X,Y}\}$ is a basis for F_2. By [4, Theorem 1.3, p. 18], $w_{X,Y}$ is an initial subword of $w_{A,B}$.

Let j be the integer such that $1 \leqslant j \leqslant X + Y$ and

$$j \equiv 1 + \frac{(A+B)-(X+Y)}{2} \mod(X+Y).$$

Let $v_{X,Y}$ be the result when $p_{1+jX \bmod(X+Y)}$ is chosen as the first letter when Construction 6 is applied to X and Y. The $\frac{X+Y-1}{2}$th distinguished point visited is p_k such that

$$k \equiv 1 + jX + \frac{(X+Y-1)X}{2} \mod(X+Y).$$

Now

$$\begin{aligned}2k &\equiv 2+2jX+(X+Y-1)X \quad \operatorname{mod}(X+Y)\\ &\equiv 2+\bigl(2+(A+B)-(X+Y)\bigr)X+(X+Y-1)X \quad \operatorname{mod}(X+Y)\\ &\equiv 2+X+(A+B)X \quad \operatorname{mod}(X+Y)\\ &\equiv 1+X \quad \operatorname{mod}(X+Y)\end{aligned} \tag{2}$$

(the final congruence holds since Eq. (1) implies that $(A+B)X \equiv -1 \quad \operatorname{mod}(X+Y)$).

Let $v_{A,B}$ be the result when $q_{1+jA \quad \operatorname{mod}(A+B)}$ is chosen as the first letter when Construction 6 is applied to A and B. The $\frac{A+B-1}{2}$th distinguished point visited is q_ℓ such that

$$\ell \equiv 1+jA+\frac{(A+B-1)A}{2} \quad \operatorname{mod}(A+B).$$

Now

$$\begin{aligned}2\ell &\equiv 2+2jA+(A+B-1)A \quad \operatorname{mod}(A+B)\\ &\equiv 2+\bigl(2+(A+B)-(X+Y)\bigr)A+(A+B-1)A \quad \operatorname{mod}(A+B)\\ &\equiv 2+A-(X+Y)A \quad \operatorname{mod}(A+B)\\ &\equiv 1+A \quad \operatorname{mod}(A+B)\end{aligned} \tag{3}$$

(the final congruence holds since Eq. (1) implies that $(X+Y)A \equiv 1 \quad \operatorname{mod}(A+B)$).

The fact that $X+Y$ is odd implies that Eq. (2) has a unique solution for k in the range $1 \leqslant k \leqslant X+Y$. The fact that $A+B$ is odd implies that Eq. (3) has a unique solution for ℓ in the range $1 \leqslant \ell \leqslant A+B$. If X is odd, it follows (from $AY-BX=1$) that A is even, $k=\frac{X+1}{2}$ and $\ell = A+\frac{B+1}{2}$. If X is even, it follows that A is odd, $k = X+\frac{Y+1}{2}$ and $\ell=\frac{A+1}{2}$. In either case, p_k and q_ℓ are the unique distinguished points that lie on the line of symmetry in the respective diagrams from Construction 6. It follows that $v_{X,Y}$ and $v_{A,B}$ are palindromic primitives and

$$(v_{X,Y}, v_{A,B}) = \bigl(c^{-1}w_{X,Y}c, c^{-1}w_{A,B}c\bigr),$$

where c is the initial subword of $w_{X,Y}$ (and $w_{A,B}$) of length j. Since $\{w_{X,Y}, w_{A,B}\}$ is a basis of F_2, so is $\{v_{X,Y}, v_{A,B}\}$. □

4. Products of two palindromes

In this section we prove Lemma 5, which combines with Lemma 3 to give Theorem 4. We first record an obvious lemma.

Lemma 9. *Let $p \in \mathrm{F}_n$ be a palindrome and $w \in \mathrm{F}_n$ a word:*

(1) *if pw (w.c.) is a reduced palindrome and $|p| < |w|$ then $w \equiv qp$ (w.c.) for some palindrome q;*

(2) *if* pw $(w.c.)$ *is a reduced palindrome and* $|p| = |w|$ *then* $w \equiv p$ $(w.c.)$;
(3) *if* wp $(w.c.)$ *is a reduced palindrome and* $|p| < |w|$ *then* $w \equiv pq$ $(w.c.)$ *for some palindrome* q;
(4) *if* $wp(w.c.)$ *is a reduced palindrome and* $|p| = |w|$ *then* $w \equiv p$ $(w.c.)$.

Lemma 10. *Let* $p \in \mathrm{F}_n$ *be a palindrome and* $w \in F$ *a word. If* pw $(w.c.)$ *is a reduced palindrome, then one of the following statements holds*:

(1) $p \equiv r(qr)^m$ *and* $w \equiv qr$ $(w.c.)$ *for some palindromes* q, r *and some* $m \geqslant 0$;
(2) w *is a palindrome and* $p \equiv w^m$ $(w.c.)$ *for some* $m \in \mathbb{N}$.

If wp $(w.c.)$ *is a reduced palindrome, then one of the following statements holds*:

(3) $w \equiv rq$ *and* $p \equiv r(qr)^m$ $(w.c.)$ *for some palindromes* q, r *and some* $m \geqslant 0$;
(4) w *is a palindrome and* $p \equiv w^m$ $(w.c.)$ *for some* $m \in \mathbb{N}$.

Proof. Let a be the quotient when $|p|$ is divided by $|w|$. We prove the lemma using induction on a. By Lemma 9 the result holds in case $a = 0$. Assume the result holds when $a = k$ for some nonnegative integer k. Consider the case that $a = k + 1$. First we prove the inductive step in case pw $(w.c.)$ is a palindrome. Now $|p| > |w|$ and pw $(w.c.)$ a reduced palindrome implies that $p \equiv \Psi(w)p_1$ $(w.c.)$ for some palindrome p_1 such that $|p_1| < |p|$. Thus we have p_1 a reduced palindrome, $\Psi(w)p_1$ $(w.c.)$ a reduced palindrome and the quotient when $|p_1|$ is divided by $|w|$ is k. The inductive hypothesis implies that either

- $\Psi(w) \equiv rq$ $(w.c.)$, $p_1 \equiv r(qr)^b$ $(w.c.)$ for some palindromes q, r and some $b \geqslant 0$ and hence $p \equiv r(qr)^{b+1}$ $(w.c.)$—statement (1) holds with $m = b + 1$; or
- w is a palindrome, $p_1 \equiv w^b$ $(w.c.)$ for some $b > 0$ and hence $p \equiv w^{b+1}$ $(w.c.)$—statement (2) holds with $m = b + 1$.

The inductive step is proved similarly in case wp $(w.c.)$ is a palindrome. □

We are now ready to prove Lemma 5.

Proof of Lemma 5. We prove only the nontrivial direction of implication. Let $w \in \mathrm{F}_n$ be a nonpalindromic product of nontrivial palindromes r and s, that is, $w = rs$.

If the first letter of r and the last letter of s survive the cancellation between r and s, the fact that w is cyclically reduced implies that the first letter of r and the last letter of s are not mutually inverse. It follows that the last letter of r and the first letter of s are not mutually inverse, and no cancellation occurs between r and s.

If the first letter of r does not survive the cancellation between r and s, then $s \equiv r^{-1}w$ $(w.c.)$. It follows from Lemma 10 that either $w \equiv r^{-n}$ for some $n > 0$, or $r \equiv (cd)^m c$ $(w.c.)$ and $w \equiv cd$ $(w.c.)$ for some $m \geqslant 0$ and some palindromes c and d. The hypothesis that w is not a palindrome implies that $w \equiv cd$ $(w.c.)$ for nontrivial palindromes c and d. The case that the last letter of s does not survive the cancellation between r and s is handled similarly. □

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