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Reversible Relative Periodic Orbits

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We study the bundle structure near reversible relative periodic orbits in reversible equivariant systems. In particular we show that the vector field on the bundle forms a skew product system, by which the study of bifurcation from reversible relative periodic solutions reduces to the analysis of bifurcation from reversible discrete rotating waves. We also discuss possibilities for drifts along group orbits. Our results extend those recently obtained in the equivariant context by B. Sandstede *et al.* (1999, *J. Nonlinear Sci.* **9**, 439–478) and C. Wulff *et al.* (2001, *Ergodic Theory Dynam. Systems* **21**, 605–635).

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1. INTRODUCTION

Symmetry is a frequently occurring property of dynamical systems. In recent years, bifurcation in the presence of symmetries has been studied extensively, in particular for equivariant dynamical systems (i.e., systems with *time-preserving* symmetries).

In addition to equivariance, dynamical systems may also possess *time-reversal* symmetries. Such symmetries are well known to occur in many applications of interest, many of which—but not all!—are Hamiltonian. Examples include mechanical systems, and differential equations obtained by reduction from certain types of equivariant PDEs, see Lamb and Roberts [24] for a recent survey.

In the context of equivariant dynamical systems, group theoretical tools have been employed to develop a comprehensive understanding of bifurcation theory, see for instance [13]. The work presented in this paper is part of a program to extend the existing theory for equivariant systems to reversible equivariant systems.

The aim of this paper is to extend recently obtained results of Sandstede *et al.* [28] and Wulff *et al.* [34] on the bundle structure near relative periodic orbits in equivariant dynamical systems, to reversible equivariant dynamical systems. In effect, the bundle structure is a key tool in reducing the problem of bifurcation from reversible periodic orbits to that of reversible discrete rotating waves, which in turn reduces to the problem of bifurcation from reversible equivariant equilibria, cf [22].

A systematic development of a local bifurcation theory for equilibria of reversible equivariant is currently being pursued, see for instance Buono *et al.* [5] for an account of reversible equivariant steady state bifurcation. Earlier studies on bifurcations in reversible systems focussed mostly on the *purely* reversible case (in the absence of equivariance). For instance, the papers [3, 7, 29, 31, 32] discuss various results on branching of reversible periodic orbits without additional symmetry, including period doubling and period preserving bifurcations as well as subharmonic bifurcations. Earlier results on reversible equivariant systems include partial results on equivariant reversible Hopf bifurcation and Lyapunov centre theorems [2, 8, 12, 17, 30], and certain examples of bifurcation from reversible discrete rotating waves, see, e.g., [20]. We refer to [24] for a more extended discussion and additional references (also on other aspects of reversible systems).

Apart from their independent importance, the results of this paper form a first step towards understanding the bundle structure near relative periodic orbits in reversible equivariant Hamiltonian systems, which is discussed in the paper [35] (with Roberts).

2. SETTING OF THE PROBLEM

We consider differential equations of the form

$$\dot{u} = f(u), \tag{2.1}$$

where f is a smooth vector field defined on a finite-dimensional smooth connected manifold M. We denote the flow of (2.1) by Φ_t.

Let Γ_ρ be a finite-dimensional (possibly noncompact) Lie group acting smoothly and properly on M, possessing a normal subgroup Γ of index two, i.e. $\Gamma_\rho/\Gamma \cong \mathbb{Z}_2$. Then, taking ρ to be any element of $\Gamma_\rho \setminus \Gamma$, we have

$$\Gamma_\rho = \langle \Gamma, \rho \rangle = \Gamma \cup \rho\Gamma. \tag{2.2}$$

We assume that our differential equation (2.1) is Γ-equivariant and ρ-reversible, i.e., the flow Φ_t satisfies $\gamma\Phi_t(u) = \Phi_t(\gamma u)$ and $\gamma\rho\Phi_{-t}(u) = \Phi_t(\gamma\rho u)$ for all $\gamma \in \Gamma$ and $u \in M$. In other words, whenever $u(t)$ is a solution of (2.1) then so are $\gamma u(t)$ and $\gamma\rho u(-t)$ for all $\gamma \in \Gamma$. In terms of the vector field f in (2.1) these equivariance and reversibility conditions read

$$\gamma f(u) = f(\gamma u) \qquad \text{and} \qquad \rho f(u) = -f(\rho u) \qquad \text{for all} \quad \gamma \in \Gamma. \tag{2.3}$$

Note that from this definition it follows that (2.1) is not only reversible with respect to ρ, but also with respect to all elements in $\rho\Gamma$. We call the elements of Γ *symmetries* and the elements of $\rho\Gamma$ *reversing (or time-reversal) symmetries* of f. The above imposed structure on Γ_ρ naturally follows from the composition properties of symmetries and reversing symmetries. It should be noted that $\rho\Gamma$ need not contain an element of order two (a so-called *involution*), so that reversible equivariant systems need not possess an involutory reversing symmetry.[2]

We say that a point u_0 lies on a *relative periodic orbit* (RPO) $\mathcal{P}$ if there exists a least $T > 0$ (the relative period) such that $\Phi_T(u_0) = \sigma u_0$ for some $\sigma \in \Gamma$. The relative periodic orbit $\mathcal{P}$ is defined to be

$$\mathcal{P} = \{\gamma\Phi_t(u_0) \mid \gamma \in \Gamma, t \in \mathbb{R}\}. \tag{2.4}$$

DEFINITION 2.1. A relative periodic orbit $\mathcal{P}$ is called *reversible* if there exists a reversing symmetry ρ such that $\rho\mathcal{P} = \mathcal{P}$.

[2] We note that in the literature, there has been some confusion about the observation of Sevryuk [29] that *linear* reversible systems always possess an involutory reversing symmetry. This fact applies only to linear systems, and not to nonlinear ones. However, even in the context of linear reversible equivariant systems this observation is not particularly useful, as the reversing involution may change discontinuously when parameters are varied [25].

Note that indeed $\rho\mathscr{P}$ is always a relative periodic orbit if $\mathscr{P}$ is. Namely,

$$\Phi_T(\rho u_0) = \rho\Phi_{-T}(u_0) = \rho\sigma^{-1}\rho^{-1}(\rho u_0) = \tilde{\sigma}(\rho u_0). \tag{2.5}$$

for some $\tilde{\sigma} \in \Gamma$, since Γ is normal in Γ_ρ.

Before we discuss reversible relative periodic orbits we briefly recall the results by Wulff *et al.* [34] on relative periodic solutions in the (non-reversible) equivariant case.

Let $\mathscr{P}$ be a relative periodic orbit of relative period 1; i.e., there is some $u_0 \in \mathscr{P}$ and some $\sigma \in \Gamma$ so that $\sigma^{-1}\Phi_1(u_0) = u_0$. Let $\Delta \subset \Gamma$ denote the isotropy subgroup of the point u_0 and let

$$\Sigma = \langle \sigma, \Delta \rangle = \overline{\{\gamma \in \Gamma \mid \Phi_\theta(u_0) = \gamma u_0 \text{ for some } \theta \in \mathbb{R}\}}$$

denote the spatio-temporal symmetry group of $\mathscr{P}$ with respect to u_0. Note that due to the proper group action of Γ, the isotropy subgroup Δ must be compact.

In [28, 34] it was shown that one can parameterize a neighborhood U of $\mathscr{P}$ as

$$U \equiv (\Gamma \times V \times \mathbb{R})/(\Delta \rtimes \mathbb{Z}). \tag{2.6}$$

Here V is a Δ-invariant section transversal to $\mathscr{P}$ in u_0 and 1 is the relative period of $\mathscr{P}$. Note that V is a Poincaré section which is transverse not only to the time orbit of u_0, but also to the group orbit of u_0. The variable $\theta \in \mathbb{R}$ plays the role of the phase along the relative periodic solution.

Let (γ, v, θ) be coordinates in $\Gamma \times V \times \mathbb{R}$. Then the quotient by $\Delta \rtimes \mathbb{Z}$ corresponds to identifications due to the isotropy Δ and spatiotemporal symmetry Σ (both defined with respect to some fixed $u_0 \in \mathscr{P}$):

$$(\gamma, v, \theta) = (\gamma\delta^{-1}, \delta v, \theta) = (\gamma\sigma^{-1}, Qv, \theta + 1). \tag{2.7}$$

Here $Q^{-1} \in \mathrm{O}(V)$ is an orthogonal *twisted equivariant* map: i.e. for all $\delta \in \Delta$ we have

$$\delta Q = Q\phi(\delta), \tag{2.8}$$

where $\phi\colon \Delta \to \Delta$ is an automorphism of Δ (the *twist automorphism*) defined as

$$\phi(\delta) = \sigma^{-1}\delta\sigma.$$

In [34] it has been shown that for many groups, for example matrix groups, it is always possible to choose σ within $\sigma\Delta$ such that ϕ has finite order k. Correspondingly Q can be chosen to have order (at most) $2k$.

Important examples include algebraic groups such as compact Lie groups and Euclidean groups (which are relevant in many applications of interest). We note that if the order of the automorphism ϕ of Δ cannot be chosen to be finite then still there will exist some finite number k_V so that ϕ^{k_V} effectively acts as the identity morphism on the representation of Δ on V for an appropriate choice of σ within $\sigma\Delta$, see [34] and Section 5.3.2 below for more details. In this paper we will focus mainly on the cases when k can be chosen to be finite. In the remaining cases in which there exists no finite k, we merely need to replace k by the number k_V which depends on the representation V (whereas k only depends on the group structure).

Due to the bundle structure one can lift the differential equations on the bundle to $(\Delta \rtimes \mathbb{Z})$-equivariant differential equations on $\Gamma \times V \times \mathbb{R}$. In terms of the coordinates (γ, v, θ) on $\Gamma \times V \times \mathbb{R}$ the resulting differential equations have the form

$$\dot{\gamma} = \gamma f_\Gamma(v, \theta), \qquad \dot{v} = f_V(v, \theta), \qquad \dot{\theta} = f_\Theta(v, \theta) \tag{2.9}$$

where

$$f_\Gamma(0, \theta) = f_V(0, \theta) = 0, \qquad f_\Theta(0, \theta) = 1, \tag{2.10}$$

and the $(\Delta \rtimes \mathbb{Z})$-equivariance reads as follows ($\delta \in \Delta$):

$$\begin{gathered} f_\Gamma(\delta v, \theta) = \mathrm{Ad}_\delta\, f_\Gamma(v, \theta), \qquad f_V(\delta v, \theta) = \delta f_V(v, \theta), \\ f_\Theta(\delta v, \theta) = f_\Theta(v, \theta), \end{gathered} \tag{2.11}$$

$$\begin{gathered} f_\Gamma(Qv, \theta+1) = \mathrm{Ad}_\sigma\, f_\Gamma(v, \theta), \qquad f_V(Qv, \theta+1) = Q f_V(v, \theta), \\ f_\Theta(Qv, \theta+1) = f_\Theta(v, \theta). \end{gathered} \tag{2.12}$$

Here Ad_γ denotes the adjoint action of Γ on the Lie algebra $\mathrm{L}\Gamma$ of Γ, i.e., $\mathrm{Ad}_\gamma(\xi) = \gamma\xi\gamma^{-1}$, $\gamma \in \Gamma$.

Since Q has finite order $2k$, the (v, θ)-subsystem of (2.9) has the compact symmetry group $\Delta \rtimes \mathbb{Z}_{2k}$. Consequently a relative periodic orbit $\mathscr{P}$ corresponds to a discrete rotating wave of the $\Delta \rtimes \mathbb{Z}_{2k}$-equivariant (v, θ)-subsystem. Hence, the study of bifurcations from relative periodic orbits reduces to the study of bifurcations from discrete rotating waves which were studied by Lamb and Melbourne [21]. In fact, the latter reduces in turn to the study of bifurcations from equilibria of vector fields [22].

The above result can be improved in two ways. First, by a reparametrization of time, one may set $f_\Theta(v, \theta) \equiv 1$, so that θ becomes the natural time-coordinate and (2.9) becomes a nonautonomous differential equation on $\Gamma \times V$. Second, subject to some mild conditions on Γ, we may go to a comoving frame in which the relative periodic orbit is observed as a periodic orbit. In particular, if Γ is an algebraic group then there is a comoving frame $\xi \in LZ(\Sigma)$ (where $Z(\Sigma)$ is the centralizer of Σ in Γ) satisfying the equations $\sigma = \exp(\xi)\,\sigma'$, so that $(\sigma')^n = 1$ for some positive $n \in \mathbb{N}$ in which the whole bundle becomes periodic of period $2n$. Moreover, the order k of the twist automorphism associated to σ' divides n; see [34]. In the comoving frame the identification (2.7) and the symmetry property (2.12) hold with σ replaced by the finite order element σ'.

In this paper we show how to incorporate reversibility into the above approach towards relative periodic orbits. We obtain a parameterization of a neighborhood of the relative periodic solution as in (2.6), but with Γ replaced by Γ_ρ and with an additional identification due to reversibility of the relative periodic solution.

The paper is organized as follows. In Section 3 we first treat reversible relative equilibria as a special (simple) case. In Section 4 we discuss our results on the bundle structure near reversible relative periodic orbits in detail, including possible drifts of reversible RPOs and give some examples. Our main results on the bundle structure and the form of the differential equations on the bundle are stated in Section 4.3 and Section 4.4. Section 5 is devoted to the proofs of the main theorems. Section 6 deals with reversible maximal tori and reversible Cartan subgroups which are needed in Sections 3 and 4 to study possible drifts of reversible relative equilibria and reversible relative periodic orbits.

3. REVERSIBLE RELATIVE EQUILIBRIA

Before discussing reversible relative periodic orbits, we will first briefly treat the simpler case of reversible relative equilibria. A relative equilibrium is a flow-invariant group orbit Γu_0. Consequently, $\Phi_t(u_0) = e^{\xi t} u_0$ for some $\xi \in L\Gamma$. In a reversible system, we call a relative equilibrium Γu_0 *reversible* if $\rho\Gamma u_0 = \Gamma u_0$ for some reversing symmetry ρ and non-reversible if $\rho\Gamma u_0 \cap \Gamma u_0 = \varnothing$. The theory for relative equilibria that are not reversible is the same as when there is no reversing symmetry in the system. Hence, we focus here on the case of a ρ-reversible relative equilibrium Γu_0. Without loss of generality, we may assume that $\rho u_0 = u_0$. Namely, since $\rho\Gamma u_0 = \Gamma u_0$ there is some $\gamma \in \Gamma$ so that $\rho u_0 = \gamma u_0$ or equivalently $\gamma^{-1}\rho u_0 = u_0$, so that u_0 is in the fixed point subspace of some reversing symmetry (which we may choose to call ρ).

Let Δ_ρ be the isotropy subgroup of u_0 in Γ_ρ. Then we define $\Delta = \Delta_\rho \cap \Gamma$. Because $\rho^2 \in \Delta$ and $\rho\Delta = \Delta\rho$, we thus obtain that Δ_ρ is an index two extension of Δ. Note that the condition $\rho^2 \in \Delta$ is a restriction on the possible isotropies $\Delta \subset \Gamma$ of a reversible relative equilibrium. We call the subgroup Σ_ρ of Γ_ρ generated by Δ, ρ and the Lie algebra element ξ the *spatio-temporal symmetry group* of the relative equilibrium in Γ_ρ.

The reversibility of the relative equilibrium also leads to some natural reversibility conditions on the drift. Before stating these conditions, we introduce the notion of reversible maximal tori. Let $\Gamma = N(\Delta)/\Delta$ for a moment so that ρ is an involution, i.e., $\rho^2 = \mathrm{id}$.

DEFINITION 3.1. Let ρ be an involution. An element $\xi \in \mathrm{L}\Gamma$ is called ρ-reversible if $\mathrm{Ad}_\rho\, \xi = -\xi$. A torus T^ρ is called a *maximal ρ-reversible* torus if T^ρ is generated by a ρ-reversible $\xi \in \mathrm{L}\Gamma$, i.e., $\mathrm{Ad}_\rho(\xi) = -\xi$, and is not contained in any ρ-reversible torus of higher dimension.

In this definition, recall that Ad_γ denotes the adjoint action of Γ_ρ on $\mathrm{L}\Gamma_\rho$, i.e., $\mathrm{Ad}_\gamma(\xi) = \gamma \xi \gamma^{-1}$.

In Section 6 (Theorem 6.7) we will prove that if ρ is an involution (which holds true for $\Gamma := N(\Delta)/\Delta$) then all almost every reversible element in $\mathrm{L}\Gamma$ generates a maximal reversible torus or a line. Moreover we will show there that for symmetry groups Γ which are semidirect products of compact groups and vector spaces (like compact and Euclidean groups) all maximal reversible tori are conjugate.

The following proposition extends results of Field [10] and Ashwin and Melbourne [1] on drift of relative equilibria to reversible relative equilibria.

PROPOSITION 3.2. *Let Γu_0 be a ρ-reversible relative equilibrium, with $u_0 \in \mathrm{Fix}(\rho)$. Let $\Phi_t(u_0) = \exp(\xi t)\, u_0$. Then, without loss of generality ξ may be taken to be Δ-equivariant and ρ-reversible, i.e., $\xi \in \mathrm{L}Z(\Delta)$ and $\mathrm{Ad}_\rho(\xi) = -\xi$.*

Generically, the time orbit of each u_0 on the relative equilibrium is either diffeomorphic to a line, or, if the drift is compact, the dimension of the closure of the time orbit of u_0 is equal to $\dim(T^\rho)$, where T^ρ is the maximal ρ-reversible torus in $N(\Delta)/\Delta$ containing ξ. In other words, generically the drift velocity ξ of a reversible relative equilibrium generates a line or a maximal ρ-reversible torus T^ρ in $N(\Delta)/\Delta$.

Proof. Since $\Phi_t(u_0) = \mathrm{e}^{\eta t} u_0$ for some $\eta \in \mathrm{L}\Gamma$ we have

$$\Phi_t(\rho u_0) = \rho \Phi_{-t}(u_0) = \mathrm{e}^{-\mathrm{Ad}_\rho \eta t}(\rho u_0)$$

similarly as for reversible relative periodic orbits; cf. (2.5). Since $u_0 = \rho u_0$ we conclude that

$$\eta + \mathrm{Ad}_\rho\, \eta \in \mathrm{L}\Delta.$$

Since $e^{-\eta t}\delta e^{\eta t}u_0 = e^{-\eta t}\delta\Phi_t(u_0) = e^{-\eta t}\Phi_t(u_0) = u_0$ for $\delta \in \Delta$, $t \in \mathbb{R}$, we always have $\eta \in LN(\Delta)$.

Since Δ_ρ is compact we may choose a Δ_ρ-equivariant scalar product on $L\Gamma$ and write $\eta = \chi + \xi$, where $\chi \in L\Delta$ and $\xi \in (L\Delta)^\perp$. Now, using that the scalar product is ρ-invariant, we find that $\xi + \mathrm{Ad}_\rho(\xi) \in (L\Delta)^\perp$, but on the other hand also $\xi + \mathrm{Ad}_\rho(\xi) = \eta + \mathrm{Ad}_\rho(\eta) - \chi - \mathrm{Ad}_\rho(\chi) \in L\Delta$. Hence, $\xi + \mathrm{Ad}_\rho(\xi) = 0$. Since $\chi \in L\Delta$ we may w.l.o.g. change η to ξ. As shown in [11] we have $(L\Delta)^\perp \cap LN(\Delta) \subseteq LZ(\Delta)$. Hence also $\xi \in LZ(\Delta)$.

Thus ξ generates a reversible torus or a reversible line in $N(\Delta)/\Delta$. Since by Theorem 6.7 below almost every reversible $\xi \in L(N(\Delta)/\Delta) \simeq (L\Delta)^\perp \cap LN(\Delta)$ generates a maximal reversible torus or a line in $N(\Delta)/\Delta$ we conclude that this holds true for generic drift velocities of reversible relative equilibria with isotropy Δ. ∎

We will illustrate the importance of reversibility for generic drift with some simple examples where the generic drift in the reversible case differs from the generic drift in the nonreversible case.

EXAMPLE 3.3. Suppose that $\Gamma = SO(2)$, and $\Gamma_\rho = SO(2) \times \mathbb{Z}_2$. Then a Γ_ρ reversible relative equilibrium does not drift, because the generators of SO(2) are not ρ-reversible. However, a non-reversible equilibrium will typically drift along the SO(2) orbit (rotating wave).

EXAMPLE 3.4. Let us consider the case that $\Gamma = SO(2) \times SO(2)$ and $\Gamma_\rho = SO(2) \times O(2)$. A Γ_ρ reversible relative equilibrium will typically drift along an SO(2) orbit. The SO(2)-part of O(2) is a maximal reversible torus since the generators of SO(2) are reversible with respect to the additional involutory generator of O(2). A non-reversible relative equilibrium, however, is expected to drift along the full $SO(2) \times SO(2)$ group orbit, which is a maximal torus.

In the non-reversible case one can write a tubular neighborhood U of the relative equilibrium Γu_0 as $U = \Gamma \times_\Delta V$; see [9, 18, 26]. Here V is a Δ-invariant section, transversal to the group orbit Γu_0 in u_0.

In describing the bundle structure for reversible relative equilibria, we employ again a local section V that is transversal to the relative equilibrium. In particular we may choose this section to be Δ_ρ-invariant. Namely, since $u_0 \in \mathrm{Fix}(\Delta_\rho)$ the group Δ_ρ acts on $T_{u_0}M$, and since Δ_ρ is a compact group there exists a Δ_ρ-invariant scalar product on $T_{u_0}M$. Since $\rho\Gamma\rho^{-1} = \Gamma$ we have $\mathrm{Ad}_\rho\, L\Gamma = L\Gamma$. Therefore $\rho\xi u_0 = \mathrm{Ad}_\rho\,\xi u_0 \in L\Gamma u_0$ for $\xi \in L\Gamma$. So the space $T_{u_0}\Gamma u_0 = L\Gamma u_0$ is Δ_ρ-invariant. We hence choose $V = (T_{u_0}\Gamma u_0)^\perp$ as Δ_ρ-invariant complement of $T_{u_0}\Gamma u_0$ in $T_{u_0}M$.

The bundle structure near the relative equilibrium and the form of the corresponding differential equations are discussed in the following theorem.

THEOREM 3.5. *Let Γu_0 be a ρ-reversible relative equilibrium with $u_0 \in \mathrm{Fix}(\Delta_\rho)$. Then a neighborhood U of Γu_0 has the form*

$$U \equiv (\Gamma_\rho \times V)/\Delta_\rho = \Gamma_\rho \times_{\Delta_\rho} V. \tag{3.1}$$

The differential equations (2.1) *on U lift to $\Gamma \times V$. With coordinates $(\gamma, v) \in \Gamma \times V$ they take the form*

$$\dot{\gamma} = \gamma f_\Gamma(v), \qquad \dot{v} = f_V(v), \tag{3.2}$$

where $f_\Gamma: V \to \mathrm{L}\Gamma$ and $f_V: V \to V$. The equations (3.2) *are Δ-equivariant and reversible in the following sense:*

$$\forall \delta \in \Delta \qquad f_\Gamma(\delta v) = \mathrm{Ad}_\delta\, f_\Gamma(v), \quad f_V(\delta v) = \delta f_V(v), \tag{3.3}$$

$$f_\Gamma(\rho v) = -\mathrm{Ad}_\rho\, f_\Gamma(v), \qquad f_V(\rho v) = -\rho f_V(v). \tag{3.4}$$

Proof. We focus on the consequences of the reversibility, as the equivariant part of the result is well known; see [9, 18].

Let $\psi: T_{u_0}M \to M$ be a Δ_ρ-equivariant smooth local chart of M near u_0. Due to ρ-equivariance of ψ we have

$$\begin{aligned}(\gamma, v) &= \gamma\psi(v) = \gamma\rho^{-1}\rho\psi(v) = \gamma\rho^{-1}\psi(\rho v)\\ &= (\gamma\rho^{-1}, \rho v), \qquad v \in V.\end{aligned}$$

Moreover, Δ-equivariance implies

$$\forall \delta \in \Delta \qquad (\gamma, v) = (\gamma\delta^{-1}, \delta v). \tag{3.5}$$

By these identifications, a neighborhood U of the relative equilibrium Γu_0 is of the form $U \equiv (\Gamma_\rho \times V)/\Delta_\rho = \Gamma_\rho \times_{\Delta_\rho} V$.

In the coordinates $(\gamma, v) \in \Gamma \times V$ the differential equations on U have the form

$$\dot{\gamma} = \gamma f_\Gamma(v), \qquad \dot{v} = f_V(v).$$

Due to the identification (3.5), f_Γ and f_V are not uniquely determined if Δ is continuous. Therefore we choose $f_\Gamma(v) \in (\mathrm{L}\Delta)^\perp$ for some Δ_ρ-invariant scalar-product on $\mathrm{L}\Gamma$. Reversibility implies that whenever $u(t) = (\gamma(t), v(t)) = \gamma(t)\,\psi(v(t))$ is a solution, then so is

$$\begin{aligned}\rho u(-t) &= \rho\gamma(-t)\,\psi(v(-t)) = \rho\gamma(-t)\,\rho^{-1}\psi(\rho v(-t))\\ &= (\rho\gamma(-t)\,\rho^{-1}, \rho v(-t)).\end{aligned}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}(-t)}(\rho v) = f_V(\rho v) \Leftrightarrow -\rho f_V(v) = f_V(\rho v), \tag{3.6}$$

and similarly

$$\frac{\mathrm{d}}{\mathrm{d}(-t)}(\rho\gamma\rho^{-1}) = f_\Gamma(\rho v) \Leftrightarrow -\mathrm{Ad}_\rho\, f_\Gamma(v) = f_\Gamma(\rho v). \tag{3.7}$$

∎

In [27] we extend these results to Hamiltonian reversible relative equilibria. We conclude this section with some simple examples of reversible relative equilibria. Just as in the non-reversible case they might be rotating waves, invariant tori or travelling waves.

EXAMPLE 3.6 (Reversible Rotating Wave). Let $\Gamma_\rho = \mathrm{O}(2)$, $\Gamma = \mathrm{SO}(2)$. Then, a relative equilibrium Γu_0 with isotropy $\Delta = \mathbf{1}$ and $\Delta_\rho = \mathbb{Z}_2$ is typically a reversible rotating wave satisfying $\Phi_t(u_0) = R_{\omega t} u_0$, and thus an equilibrium in a rotating frame with angular velocity ω. By reversibility ρ reflects the rotating wave to a solution rotating with opposite orientation, i.e.,

$$\rho\Phi_t(u_0) = \Phi_{-t}(u_0) = R_{-\omega t} u_0.$$

EXAMPLE 3.7 (Reversible Invariant Torus). Let $\Gamma_\rho = \mathrm{O}(N)$, $\Gamma = \mathrm{SO}(N)$. Then, a relative equilibrium Γu_0 with $\Delta = \mathbf{1}$ and $\Delta_\rho = \mathbb{Z}_2$ is typically a reversible invariant torus satisfying $\Phi_t(u_0) = \exp(\xi t)\, u_0$, with $\xi \in \mathrm{so}(N)$. Again we have

$$\rho\Phi_t(u_0) = \Phi_{-t}(u_0) = \exp(-\xi t)\, u_0.$$

EXAMPLE 3.8 (Reversible Travelling Wave). Let $\Gamma_\rho = \mathbb{R} \rtimes Z_2 = \mathrm{E}(1)$, $\Gamma = \mathbb{R}$. Let T_a denote a translation by $a \in \mathbb{R}$ so that $\rho T_a = T_{-a}\rho$ for all $a \in \mathbb{R}$. Then, a relative equilibrium Γu_0 with $\Delta = \mathbf{1}$ and $\Delta_\rho = \mathbb{Z}_2$ is typically a reversible travelling wave satisfying

$$\Phi_t(u_0) = T_{vt} u_0, \quad \rho\Phi_t(u_0) = \Phi_{-t}(u_0) = T_{-vt} u_0.$$

4. REVERSIBLE RELATIVE PERIODIC ORBITS

Starting point for our results on reversible relative periodic orbits are the results on reversible relative equilibria presented in the previous section and the results in [28, 34] on the bundle structure near RPOs in non-reversible systems. First, in Section 4.1, we make some general statements on reversible relative periodic orbits and give some examples. In Section 4.2 we show what kinds of drifts are possible for reversible RPOs. In Section 4.3 we state our three main theorems on the bundle structure near reversible RPOs. Detailed proofs of the main theorems are deferred to Section 5. In Section 4.4 we study the differential equations in the bundle coordinates and show that reversible RPOs correspond to reversible discrete rotating waves on a slice $(V \times S^1)$.

4.1. *Some General Properties of Reversible RPOs*

Let $\mathcal{P}$ be a reversible relative periodic orbit with relative period 1, i.e., fixing $u_0 \in \mathcal{P}$ we have $\Phi_1(u_0) = \sigma u_0$ for some $\sigma \in \Gamma$ and $\rho\mathcal{P} = \mathcal{P}$.

As in the case of relative equilibria we can always choose a point $u_0 \in \mathcal{P}$ such that $u_0 \in \mathrm{Fix}(\rho)$ for some reversing symmetry ρ. Namely, since $\rho u_0 \in \mathcal{P}$ there exists a $\gamma_0 \in \Gamma$ and some $t_0 \in \mathbb{R}$ such that $\rho u_0 = \gamma_0 \Phi_{t_0}(u_0)$, and we readily verify that $\gamma_0^{-1} \rho \Phi_{t_0/2}(u_0) = \gamma_0^{-1} \Phi_{-t_0/2}(\rho u_0) = \Phi_{t_0/2}(u_0)$. So we change u_0 to $\Phi_{t_0/2}(u_0)$ and ρ to $\gamma_0^{-1}\rho$. From now on we assume that u_0 and ρ are such that $u_0 \in \mathrm{Fix}(\rho)$. We denote the isotropies of u_0 by $\Delta \subset \Gamma$ and $\Delta_\rho = \langle \Delta, \rho \rangle \subset \Gamma_\rho$. As mentioned in the case of reversible relative equilibria, Δ is a normal subgroup of Δ_ρ of index 2.

In reversible systems a point $u_0 \in \mathrm{Fix}(\rho)$ lies on a ρ-reversible T-periodic solution if and only if $\Phi_{T/2}(u_0) \in \mathrm{Fix}(\rho)$. An analogous observation applies to reversible RPOs.

LEMMA 4.1. *Let $u_0 \in \mathrm{Fix}(\rho)$. Then u_0 lies on a ρ-reversible relative periodic orbit $\mathcal{P}$ with relative (minimal) period T, if and only if $\Phi_{T/2}(u_0) \in \mathrm{Fix}(\rho')$ where ρ' is some reversing symmetry in $\rho\Gamma$.*

Proof. We readily verify that, with $\Phi_T(u_0) = \sigma u_0$ for some $\sigma \in \Gamma$, we obtain

$$\sigma\rho\Phi_{T/2}(u_0) = \sigma\Phi_{-T/2}(\rho u_0) = \sigma\Phi_{-T/2}(u_0) = \Phi_{T/2}(u_0),$$

so that indeed $\Phi_{T/2}(u_0) \in \mathrm{Fix}(\rho')$ with $\rho' = \sigma\rho$.

Vice versa, if $u_0 \in \mathrm{Fix}(\rho)$ and $\Phi_{T/2}(u_0) \in \mathrm{Fix}(\sigma\rho)$ for some $T > 0$, $\sigma \in \Gamma$ then u_0 lies on a relative periodic solution with relative period T. ∎

In the following proposition we collect some easy observations:

PROPOSITION 4.2. *Let* u_0 *be a point on a reversible relative periodic orbit* $\mathcal{P}$ *with (minimal) relative period* $T>0$, *such that* $\Phi_T(u_0)=\sigma u_0$, *and with isotropy* $\Delta_p\subset\Gamma_p$ *and let* $\Sigma=\langle\Delta,\sigma\rangle$. *Then,*

1. $\dim \mathrm{Fix}(\Delta_p)\leqslant\dim \mathrm{Fix}(\Delta)-1$.
2. *If* $\dim \mathrm{Fix}(\Delta_p)=\dim \mathrm{Fix}(\Delta)-1$ *and* Σ *is compact then* $\mathcal{P}$ *is periodic.*

Proof. The time orbit of u_0 is contained in $\mathrm{Fix}(\Delta)$, so we can restrict to the case $M=\mathrm{Fix}(\Delta)$, and $\Gamma=N(\Delta)/\Delta$. Now assume that

$$\dim \mathrm{Fix}(\rho)=\dim(M)-1.$$

In case Σ is compact, σ^ℓ can be made arbitrarily close to identity by choosing some appropriate $\ell\in\mathbb{N}$ large enough. Since

$$\Phi_\ell(u_0)=\sigma^\ell u_0\in\mathrm{Fix}(\sigma^\ell\rho\sigma^{-\ell})$$

and since $\mathrm{Fix}(\sigma^\ell\rho\sigma^{-\ell})$ is close to $\mathrm{Fix}(\rho)$ we conclude that due to the transversal crossing of $\Phi_t(u_0)$ through $\mathrm{Fix}(\sigma^\ell\rho\sigma^{-\ell})$ and since $\mathrm{Fix}(\rho)$ has codimension 1 the orbit must cross transversely through $\mathrm{Fix}(\rho)$, i.e. $\Phi_T(u_0)\in\mathrm{Fix}(\rho)$ for some $T>0$. Then u_0 lies on a periodic orbit with period $2T$.

If $\dim M=\dim \mathrm{Fix}(\rho)$ then $M=\mathrm{Fix}(\rho)$ and since $f(u)\in\mathrm{Fix}(-\rho)$ we have $f(u)=0$ on M. Hence $\mathcal{P}$ must be a steady state, which contradicts our assumptions. ∎

We conclude this Section with some examples of reversible relative periodic orbits.

EXAMPLE 4.3 (Reversible Discrete Rotating Wave). One of the simplest examples of a reversible relative periodic orbit is a reversible discrete rotating wave as studied in [19], i.e., a periodic orbit with discrete spatio-temporal symmetry. For example, let $\Gamma_\rho=\mathbb{D}_n$, $\Gamma=\mathbb{Z}_n$, let σ generate $\mathbb{Z}_n$ and let $\rho\in\Gamma_\rho\setminus\Gamma$. Then we may have a reversible periodic solution of period n passing through $u_0\in\mathrm{Fix}(\rho)$ and satisfying $\Phi_1(u_0)=\sigma u_0$.

EXAMPLE 4.4 (Reversible Modulated Travelling Wave). Another example of a reversible relative periodic orbit is a reversible modulated travelling wave. Let $\Gamma_\rho=\mathrm{E}(1)$, $\Gamma=\mathbb{R}$, and let σ generate $\mathbb{Z}\subset\mathbb{R}$. Then we may have a relative periodic orbit $\mathcal{P}$ passing through $u_0\in\mathrm{Fix}(\rho)$ satisfying $\Phi_1(u_0)=\sigma u_0$. Such a solution is called a modulated travelling wave and may have bifurcated from a reversible travelling wave; see Example 3.8.

EXAMPLE 4.5 (Reversible Modulated Rotating Wave). A reversible modulated rotating wave is a reversible relative periodic orbit that is typically nonperiodic, but has compact drift. For instance, let $\Gamma_\rho = \mathrm{O}(2)$ and $\Gamma = \mathrm{SO}(2)$ act on a two-torus $M = S^1 \times S^1$, so that Γ acts naturally on the first copy of S^1 and trivial on the second copy. Let ρ act as a reflection on both copies of S^1. Then $\dim \mathrm{Fix}(\rho) = 0$, and typically a relative periodic orbit $\mathscr{P}$ passing through $u_0 \in \mathrm{Fix}(\rho)$ satisfies $\Phi_T(u_0) = \sigma u_0$ for some $T > 0$ and some $\sigma \in \mathrm{SO}(2)$ generating SO(2). Such a solution is quasiperiodic, and a reversible modulated rotating wave.

4.2. *Drift of Reversible RPOs*

In this subsection we investigate what kind of drifts may occur for reversible RPOs. As we saw above, without loss of generality we may assume that there exists a point u_0 on the reversible RPO $\mathscr{P}$ in $\mathrm{Fix}(\rho)$.

We find that the drift element σ with $\Phi_1(u_0) = \sigma u_0$ must satisfy certain conditions due to the isotropy $\Delta_\rho \subset \Gamma_\rho$ of u_0.

LEMMA 4.6. *Let $\mathscr{P}$ be a ρ-reversible relative periodic orbit with relative period* 1, *containing a point $u_0 \in \mathrm{Fix}(\Delta_\rho)$, such that $\Phi_1(u_0) = \sigma u_0$ for some $\sigma \in \Gamma$. Then σ must be inside $N(\Delta)$ and satisfies*

$$\sigma \rho \sigma \rho^{-1} \in \Delta. \tag{4.1}$$

Proof. If $\mathscr{P}$ has isotropy Δ, then σ has to lie in the normalizer $N(\Delta)$ of the isotropy group Δ due to the fact that for all $\delta \in \Delta$

$$\sigma^{-1} \delta \sigma u_0 = \sigma^{-1} \delta \Phi_1(u_0) = u_0.$$

If $\mathscr{P}$ is ρ-reversible, with $u_0 \in \mathrm{Fix}(\rho) \cap \mathscr{P}$, we have

$$\rho \sigma \rho^{-1} u_0 = \rho \Phi_1(u_0) = \Phi_{-1}(\rho u_0) = \sigma^{-1} u_0,$$

so that $\sigma \rho \sigma \rho^{-1} \in \Delta$. ▮

In contrast to the situation of drift of reversible relative equilibria, note that in the case of reversible RPOs σ cannot always be chosen to be Δ-equivariant and ρ-reversible (here ρ-reversible means $\rho \sigma \rho^{-1} = \sigma^{-1}$), see [20, 21, 34] for counter-examples.

Let $\Sigma = \langle \Delta, \sigma \rangle$ denote the spatio-temporal symmetry of $\mathscr{P}$ with respect to u_0 in Γ. Because of Lemma 4.6 the group $\Sigma_\rho = \langle \Sigma, \rho \rangle$ is an index-2-extension of Σ. The spatial and spatio-temporal symmetry groups Δ, Δ_ρ, Σ and Σ_ρ of $\mathscr{P}$ with respect to u_0 in Γ rsp. Γ_ρ satisfy the following relations

$$\Delta \trianglelefteq \Sigma \trianglelefteq \Sigma_\rho, \qquad \Delta \trianglelefteq \Delta_\rho, \qquad \Sigma_\rho / \Sigma \cong \Delta_\rho / \Delta \cong \mathbb{Z}_2 \tag{4.2}$$

(here $A \trianglelefteq B$ denotes that A is a normal subgroup of B). Analogous statements for reversible discrete rotating waves can be found in [20].

In the following consideration let $\Gamma = N(\Delta)/\Delta$ so that ρ is an involution on Γ. We call a compact topologically cyclic subgroup C^ρ of Γ which is generated by some reversible $\sigma \in \Gamma$ and which is maximal with that property a *reversible Cartan subgroup*. Denote a connected component of the reversible group elements of Γ as reversible component. In Section 6 (Theorem 6.10) we will show that if ρ is an involution then almost every reversible $\gamma \in \Gamma$ generates a reversible Cartan subgroup or a copy of $\mathbb{Z}$. Moreover we will prove in the case of compact or Euclidean groups $\Gamma = N(\Delta)/\Delta$ that each reversible $\tilde{\sigma}$ which lies in the same reversible component of Γ as σ and generates a compact subgroup of Γ lies in a reversible Cartan subgroup $\tilde{C}^\rho$ which is conjugated to C^ρ.

Extending the results of [1, 10] to the reversible context, we obtain the following result on generic drift of reversible RPOs.

PROPOSITION 4.7. *Let $\mathscr{P}$ be a ρ-reversible relative periodic orbit, as in Lemma* 4.6. *Then generically either Σ is a reversible Cartan subgroup C^ρ of $N(\Delta)/\Delta$ or a copy of $\mathbb{Z}$ in $N(\Delta)/\Delta$. This implies that generically either the dimension of the closure of the time-orbit of u_0 is* $\dim(C^\rho)+1$ *(if the drift compact) or the time-orbit of $u_0 \in \mathscr{P}$ is a line.*

Proof. Due to the reversibility condition (4.1) and since by Theorem 6.10 almost every reversible element σ generates a reversible Cartan subgroup C^ρ or a copy of $\mathbb{Z}$ we conclude that if $\mathscr{P}$ is a generic reversible RPO then either the dimension of the closure of the time-orbit of $u_0 \in \mathscr{P}$ equals $\dim(C^\rho)+1$ (if Σ is compact), or, if the spatio-temporal symmetry Σ_ρ of the reversible RPO $\mathscr{P}$ is noncompact, for example, if $\mathscr{P}$ is a modulated travelling wave, then the time-orbit of $u_0 \in \mathscr{P}$ is always a line. ∎

In analogy to [34] we define the *reversible index m_ρ* of a reversible RPO as the least number such that

$$\sigma^{m_\rho} = \delta_0 \exp(m_\rho \zeta) \qquad \text{for some} \quad \zeta \in \mathrm{L}Z(\Sigma), \quad \mathrm{Ad}_\rho\, \zeta = -\zeta, \quad \delta_0 \in \Delta.$$

The reversible index may be infinity, e.g. if $\Sigma = \mathbb{Z}$, $\Sigma_\rho = \mathbb{D}_\infty$, $\Delta = \{\mathrm{id}\}$. If Σ is compact then m_ρ equals the number $\tilde{m}_\rho$ of connected components of the reversible Cartan subgroup $C^\rho \subseteq N(\Delta)/\Delta$ containing σ. This can be seen as follows: obviously $\tilde{m}_\rho | m_\rho$. On the other hand we know that $\sigma^{\tilde{m}_\rho} = \exp(\tilde{m}_\rho \tilde{\zeta})$ with $\mathrm{Ad}_\sigma\, \tilde{\zeta} = \tilde{\zeta}$ and $\mathrm{Ad}_\rho\, \tilde{\zeta} = -\tilde{\zeta}$ in $N(\Delta)/\Delta$. Then $\sigma^{\tilde{m}_\rho} = \delta_0 \exp(\tilde{m}_\rho \zeta)$ for some $\delta_0 \in \Delta$ and some representative ζ of $\tilde{\zeta} = \zeta + \mathrm{L}\Delta$. Choose a Σ_ρ-invariant scalar-product on $\mathrm{L}N(\Delta)$ and choose the representative $\zeta \in (\mathrm{L}\Delta)^\perp \cap \mathrm{L}N(\Delta) \subseteq \mathrm{L}Z(\Delta)$. Then $(\mathrm{Ad}_\sigma\, \zeta - \zeta) \in (\mathrm{L}\Delta)^\perp \cap \mathrm{L}\Delta$, and therefore

$\mathrm{Ad}_\sigma\,\zeta=\zeta$. Similarly, $(\mathrm{Ad}_\rho\,\zeta+\zeta)\in(\mathrm{L}\varDelta)^\perp\cap\mathrm{L}\varDelta$ and henceforth $\mathrm{Ad}_\rho\,\zeta=-\zeta$ and $\tilde m_\rho=m_\rho$.

We will see in Section 5.3.1 and Theorem 4.12 below that m_ρ is finite if Γ is an algebraic group. If m_ρ is finite denote the group generated by $\varDelta$ and the element $\tilde\sigma=\sigma\exp(-\zeta)$ by Σ^{m_ρ}, and let $\Sigma^{m_\rho}_\rho$ be the group generated by Σ^{m_ρ} and ρ. We have

$$\Sigma^{m_\rho}/\varDelta\cong\mathbb{Z}_{m_\rho},\qquad \Sigma^{m_\rho}_\rho/\varDelta\cong\mathbb{D}_{m_\rho},\qquad \Sigma^{m_\rho}_\rho/\Sigma^{m_\rho}\cong\varDelta_\rho/\varDelta\cong\mathbb{Z}_2. \tag{4.3}$$

Assume that the relative period of the reversible RPO $\mathscr{P}$ is scaled to one. In a frame moving with velocity ζ the reversible relative periodic orbit $\mathscr{P}$ is foliated by periodic orbits of period m_ρ as the following lemma shows:

Lemma 4.8. *Assume that the reversible RPO $\mathscr{P}$ has finite index m_ρ. Then*

$$\mathscr{P}=\frac{\Gamma_\rho\times S^1}{\Sigma^{m_\rho}_\rho},\qquad S^1=\mathbb{R}/m_\rho\mathbb{Z}.$$

Proof. Write $\gamma\exp(-\theta\zeta)\,\Phi_\theta(u_0)=(\gamma,\theta)$ where $\gamma\in\Gamma_\rho$, $\theta\in\mathbb{R}$. One readily verifies that

$$(\gamma\delta^{-1},\theta)=(\gamma\tilde\sigma^{-1},\theta+1)=(\gamma\rho^{-1},-\theta)$$

where $\tilde\sigma=\sigma\exp(-\zeta)$. In particular $(\gamma,\theta+m_\rho)=(\gamma\delta_0,\theta)=(\gamma,\theta)$, with $\delta_0\in\varDelta$. ∎

4.3. *Bundle Structure near Reversible RPOs*

In this section we describe the bundle structure in the neighborhood of a reversible RPO. Before we state our main theorems we need to discuss some preliminaries.

We let $\mathscr{P}$ be a reversible relative periodic solution of (2.1) with relative period 1, and let ρ be a reversing symmetry fixing a point $u_0\in\mathscr{P}$ (such a reversing symmetry ρ and point u_0 always exist, see Section 4.1). Let $\varDelta\subset\Gamma$ be the isotropy subgroup of u_0, and $\Phi_1(u_0)=\sigma u_0$. We define again $\varDelta_\rho=\langle\varDelta,\rho\rangle$.

We now extend the definition of the *twist morphism* ϕ of Section 2 to $\varDelta_\rho$ by

$$\forall\delta\in\varDelta\qquad \phi(\delta\rho)=\sigma^{-1}\delta\rho\sigma^{-1}. \tag{4.4}$$

Due to Lemma 4.6 we have $\phi(\delta\rho)\in\rho\varDelta$ for all $\delta\in\varDelta$ so that ϕ maps $\varDelta_\rho$ into itself. Note that in general ϕ is a morphism but not necessarily an automorphism or homomorphism of $\varDelta_\rho$: one readily verifies that [20]

$$\phi(\delta\rho)=\phi(\delta)\,\phi(\rho),\qquad\text{but}\qquad \phi(\rho\delta)=\phi(\rho)\,\phi^{-1}(\delta). \tag{4.5}$$

DEFINITION 4.9. Let Δ_ρ act smoothly on a manifold X. We say that a diffeomorphism $F: X \to X$ is *twisted reversible equivariant* if

$$F\delta = \phi(\delta)\, F, \qquad F\delta\rho = \phi(\delta\rho)\, F^{-1}$$

for all $\delta \in \Delta$.

Twisted (reversible) equivariant diffeomorphisms and the twist morphism ϕ were introduced in Lamb and Quispel [23] and referred to as *k-symmetric* maps where k is least such that ϕ^k is the identity morphism on Δ_ρ. In the study of RPOs linear twisted reversible equivariant maps play an important role, see below.

When Γ_ρ is algebraic, there always exists a choice of σ for which k is finite, see Theorem 4.12 below and Section 5.3 for more details. As in Section 2 we will let k_V denote the effective order of ϕ as acting on a representation of Δ_ρ on V.

In describing the bundle structure we will mention a group Ξ_ρ that is related to the spatiotemporal symmetries of the RPO. We define

$$\Xi = \Delta \rtimes \mathbb{Z} \tag{4.6}$$

as the group which was factored out in the bundle structure near non-reversible RPOs, see Section 2. We now write the elements of Ξ as (δ, i) where $\delta \in \Delta$ and $i \in \mathbb{Z}$. The group multiplication of the semidirect product is defined via the twist morphism ϕ

$$(0, i)(\delta, 0)(0, -i) = (\phi^{-i}(\delta), 0), \qquad \delta \in \Delta, \quad i \in \mathbb{Z}. \tag{4.7}$$

We define $\Xi_\rho = (\Delta \rtimes \mathbb{Z})_\rho$ as index-2-extension of Ξ by an element ρ such that

$$\rho(\delta, i)\, \rho^{-1} = (\rho\delta(\phi^i(\rho))^{-1}, -i), \delta \in \Delta, \quad i \in \mathbb{Z}, \tag{4.8}$$

and $\rho^2 \in \Delta$. Evidently,

$$\Xi \trianglelefteq \Xi_\rho, \qquad \Xi_\rho/\Xi \cong \mathbb{Z}_2, \qquad \text{and} \qquad \Delta \trianglelefteq \Xi_\rho, \qquad \Xi_\rho/\Delta \simeq \mathbb{D}_\infty. \tag{4.9}$$

Before we state our main results on the bundle, we verify the existence of a Δ_ρ-invariant section V that is locally transversal to $\mathscr{P}$.

LEMMA 4.10. *Let $\mathscr{P}$ be a ρ-reversible relative periodic orbit through u_0 with isotropy Δ_ρ. Then there exists a Δ_ρ-invariant complement V to $T_{u_0}\mathscr{P}$ in $T_{u_0}M$.*

Proof. Because $u_0 \in \mathrm{Fix}(\Delta_\rho)$, we have $\rho f(u_0) = -f(u_0)$. Therefore the space $\langle f(u_0) \rangle$ is Δ_ρ-invariant. Since $\rho\xi u_0 = \mathrm{Ad}_\rho\, \xi u_0$, $\xi \in \mathrm{L}\Gamma$, we conclude that $\mathrm{L}\Gamma u_0$ is Δ_ρ-invariant. So the tangent space $T_{u_0}\mathscr{P}$ to the relative periodic orbit in u_0 is invariant under the group Δ_ρ.

Since $u_0 \in \mathrm{Fix}(\Delta_\rho)$ and Δ_ρ is compact we can assume w.l.o.g. that Δ_ρ acts orthogonally on $T_{u_0}M$. Then the space $V = (T_{u_0}\mathscr{P})^\perp$ is a Δ_ρ-invariant complement to $T_{u_0}\mathscr{P}$. ∎

We can now state our main results. First, we formulate our most general result (replace k by k_V if k is not finite):

THEOREM 4.11. *Let $\mathscr{P}$ be a reversible RPO as discussed above. Then a neighborhood U of $\mathscr{P}$ is of the form*

$$U \equiv \frac{\Gamma_\rho \times V \times \mathbb{R}}{\Xi_\rho}.$$

Here V is a Δ_ρ-invariant section transversal to $\mathscr{P}$. Let (γ, v, θ) be coordinates on $\Gamma_\rho \times V \times \mathbb{R}$. Then the quotient by Ξ_ρ corresponds to the identifications $(\gamma \in \Gamma_\rho,\ v \in V,\ \theta \in \mathbb{R})$

$$(\gamma, v, \theta) = (\gamma\rho^{-1}, \rho v, -\theta) = (\gamma\sigma^{-1}, Qv, \theta + 1) = (\gamma\delta^{-1}, \delta v, \theta) \qquad \forall \delta \in \Delta. \tag{4.10}$$

The linear map $Q^{-1} \in \mathrm{O}(V)$ occurring in these identifications is twisted reversible equivariant (with respect to the twist morphism ϕ on Δ_ρ) and of finite order 2k.

By definition $(\Delta \rtimes \mathbb{Z}) \trianglelefteq \Xi_\rho$ and $\Xi_\rho/(\Delta \rtimes \mathbb{Z}) \cong \mathbb{Z}_2$, so that indeed the bundle structure here is a natural extension of the result obtained in [34]. The proofs of this theorem and the two following theorems can be found in Section 5.

Now we consider the case that Γ is an algebraic group, i.e., defined by algebraic equations. This assumption is often satisfied in applications: examples include compact groups, Euclidean groups and the classical Lie groups. In this case the bundle structure near a reversible relative periodic orbit becomes periodic in a comoving frame.

THEOREM 4.12. *Let $\mathscr{P}$ be a reversible RPO as in Theorem 4.11, and assume that Γ is algebraic. Then there exists a $\xi \in \mathrm{L}Z(\Sigma)$, satisfying $\mathrm{Ad}_\rho\, \xi = -\xi$, such that $\sigma^n = \exp(n\xi)$ for some $n \in \mathbb{N}$ and an appropriate choice of σ in the coset $\sigma\Delta$. As a consequence, k and m are finite and there exists a comoving frame (moving with velocity ξ) in which the neighborhood U of $\mathscr{P}$ has the form*

$$U \equiv \frac{\Gamma_\rho \times V \times \mathbb{R}/2n\mathbb{Z}}{\Xi'_\rho}.$$

Here $\Xi' = \Delta \rtimes \mathbb{Z}_{2n}$ *is defined by the group multiplication* (4.7) *where* $i \in \mathbb{Z}_{2n}$ *instead of* $i \in \mathbb{Z}$ *and* Ξ'_ρ *is defined as index-2-extension of* Ξ' *with group multiplication* (4.8) *so that* Ξ'_ρ *satisfies* $\Delta \trianglelefteq \Xi'_\rho$ *and* $\Xi'_\rho/\Delta \cong \mathbb{D}_{2n}$.

The quotient by Ξ'_ρ *corresponds to the identifications* (4.10) *with* σ *replaced by* $\sigma' = \sigma\exp(-\xi)$.

Analogously as for the group Ξ_ρ we have $(\Delta \rtimes \mathbb{Z}_{2n}) \trianglelefteq \Xi'_\rho$ and $\Xi'_\rho/(\Delta \rtimes \mathbb{Z}_{2n}) \cong \mathbb{Z}_2$.

Similarly as in the non-reversible case [34] we can obtain a $2m_\rho$-periodic bundle if the reversible index m_ρ of the reversible RPO is finite, i.e. if $\sigma^{m_\rho} = \delta_0 \exp(m_\rho \zeta)$ for some $\zeta \in \mathrm{L}Z(\Sigma)$, $\mathrm{Ad}_\rho\, \zeta = -\zeta$, $\delta_0 \in \Delta$. For the bundle construction we need the group $\tilde{\Xi}_\rho$ which is a cyclic extension of Δ_ρ of order $2m_\rho$. The group $\tilde{\Xi}_\rho$ is generated by Δ, ρ and some element R satisfying

$$R^{-1}\delta R = \phi(\delta), \qquad R^{-1}\delta\rho R^{-1} = \phi(\delta\rho) \qquad \text{for} \qquad \delta \in \Delta, \qquad R^{2m_\rho} = \delta_0^2.$$

THEOREM 4.13. *Let* $\mathscr{P}$ *be are reversible RPO of index* m_ρ *and write* $\sigma^{m_\rho} = \delta_0 \exp(m_\rho\, \zeta)$ *where* $\zeta \in \mathrm{L}Z(\Sigma)$, $\mathrm{Ad}_\rho\, \zeta = -\zeta$ *and* $\delta_0 \in \Delta$. *Form the group* $\tilde{\Xi}_\rho$ *as described above and denote* $\tilde{\sigma} = \sigma\exp(-\zeta)$. *Then, there is a neighborhood* U *of the RPO* $\mathscr{P}$ *such that, in a comoving frame, moving with velocity* ζ

$$U \equiv \frac{\Gamma_\rho \times V \times S^1}{\tilde{\Xi}_\rho}, \qquad S^1 = \mathbb{R}/(2m_\rho\mathbb{Z}).$$

Here V *is an orthogonal representation of the group* $\tilde{\Xi}_\rho$. *The action of* $\tilde{\Xi}_\rho$ *on* $\Gamma_\rho \times V \times S^1$ *is given as*

$$\delta \cdot (\gamma, v, \theta) = (\gamma\delta^{-1}, \delta v, \theta),$$
$$R \cdot (\gamma, v, \theta) = (\gamma\tilde{\sigma}^{-1}, Rv, \theta + 1),$$
$$\rho \cdot (\gamma, v, \theta) = (\gamma\rho^{-1}, \rho v, -\theta).$$

4.4. *Differential Equations on the Bundle*

In the following theorem we state the form of the differential equations in the bundle coordinates on $\Gamma_\rho \times V \times \mathbb{R}$. We will see that the analysis of the bundle structure reduces the study of bifurcations from reversible RPOs to the study of bifurcations from reversible discrete rotating waves in the slice $V \times S^1$. Whereas the overall symmetry group Γ may be noncompact the symmetry group of the slice which determines bifurcations is compact.

THEOREM 4.14. *In terms of the bundle coordinates* $(\gamma, v, \theta) \in \Gamma_\rho \times V \times \mathbb{R}$, *the differential equations on the lifted bundle have the form* (2.9)*:*

$$\dot{\gamma} = \gamma f_\Gamma(v, \theta), \qquad \dot{v} = f_V(v, \theta), \qquad \dot{\theta} = f_\Theta(v, \theta).$$

They are ρ-reversible, i.e.,

$$\begin{gathered} f_\Gamma(\rho v, -\theta) = -\mathrm{Ad}_\rho\, f_\Gamma(v, \theta),\, f_V(\rho v, -\theta) = -\rho f_V(v, \theta), \\ f_\Theta(\rho v, -\theta) = f_\Theta(v, \theta) \end{gathered} \tag{4.11}$$

and $(\Delta \rtimes \mathbb{Z})$*-equivariant in the sense of* (2.11)*,* (2.12)*. Further* $f_\Gamma(0, \theta) = 0$*,* $f_V(0, \theta) = 0$*,* $f_\Theta(0, \theta) = 1$.

If there exists a comoving frame $\xi \in \mathrm{LZ}(\Sigma)$ *in which the bundle becomes periodic as in Theorem* 4.12*, then* (2.12) *holds with* σ *replaced by* σ'*, and* $f_\Gamma(0, \theta) = \xi$.

Proof. Due to reversibility, whenever $u(t) = (\gamma(t), v(t), \theta(t))$ is a solution of (2.9), then also

$$\rho u(-t) = (\rho\gamma(-t), v(-t), \theta(-t)) = (\rho\gamma(-t)\, \rho^{-1}, \rho v(-t), -\theta(-t))$$

is a solution of (2.9). Here we used identification (4.10). In analogy to the case of reversible relative equilibria, see Section 3, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}(-t)}(\rho v) = f_V(\rho v, -\theta) \Leftrightarrow -\rho f_V(v, \theta) = f_V(\rho v, -\theta)$$

and that

$$\frac{\mathrm{d}}{\mathrm{d}(-t)}(\rho\gamma\rho^{-1}) = \rho\gamma\rho^{-1} f_\Gamma(\rho v, -\theta) \Leftrightarrow -\mathrm{Ad}_\rho\, f_\Gamma(v, \theta) = f_\Gamma(\rho v, -\theta)$$

and finally, that

$$\frac{\mathrm{d}}{\mathrm{d}(-t)}(-\theta) = f_\Theta(\rho v, -\theta) \Leftrightarrow f_\Theta(v, \theta) = f_\Theta(\rho v, -\theta)$$

which yields the conditions (4.11). The equivariance conditions (2.11), (2.12) are known from [34]. In Section 5.4 below we will show how the vector fields f_Γ, f_V and f_Θ are related to the vector field f from (2.1). ∎

Again, we may reparametrize time to set $f_\Theta(v, \theta) = 1$. In particular, whenever Γ_ρ is algebraic, we can proceed to a comoving frame in which the relative periodic orbit is periodic, and the equations take the form of a periodically driven reversible equivariant system on $\Gamma \times V$.

Since Q has finite order $2k$ (rsp. $2k_V$) we note that relative periodic solutions correspond to reversible discrete rotating waves (periodic solutions with discrete spatiotemporal symmetry) of the (v, θ)-subsystem. Hence, modulo drifts, the problem of bifurcation from reversible relative periodic orbits reduces to the problem of bifurcation from reversible discrete rotating waves. More precisely, reversible RPOs become fixed points of the twisted reversible equivariant diffeomorphism $F: V \to V$ defined as $F = Q^{-1}g^{(1)}$ where $g^{(1)}$ is the time-one map of the nonautonomous v-equation in (2.9).

There is no general bifurcation theory available yet for reversible discrete rotating waves, although it is clear that the techniques of [21] for discrete rotating waves in equivariant systems extend to the context of reversible equivariant systems: By employing Birkhoff normal form techniques [19] the problem of bifurcation from discrete rotating waves may be reduced to a problem of fixed point bifurcation of a $(\Delta \rtimes \mathbb{Z}_{2k})$-equivariant and ρ-reversible diffeomorphism (up to higher order terms), see, e.g., [20] for some examples.

As mentioned before, a general local steady-state bifurcation theory for reversible equivariant systems is being developed in [5]; for the linear theory, see [16, 25].

5. PROOFS OF THE MAIN THEOREMS

In this section we prove our main Theorems 4.11, 4.12, 4.13 on the bundle structure near a reversible relative periodic solution.

First, in Section 5.1, we briefly review the bundle construction of [28, 34] and sketch what modifications need to be made in the reversible case. Section 5.2 deals with decompositions of twisted reversible equivariant linear maps which occur as linearizations of relative periodic solutions. In Section 5.3 we show that the order of the twist morphism ϕ is finite for appropriate choices of σ in the coset of $\sigma\Delta$ if Γ is algebraic. Moreover we show that for any twisted reversible equivariant linear map $L \in \mathrm{Mat}(V)$ there is some k_V and $\delta_0 \in \Delta$ such that $(\delta_0 L)^{k_V}$ is equivariant and reversible. Finally, in Section 5.4 we prove Theorems 4.11, 4.12 and 4.13 and discuss the relationship between the equations (2.9) in bundle coordinates and the original vector field (2.1) in more detail.

5.1. *The Bundle Construction of* [28, 34]

Let $\mathscr{P}$ be a relative periodic orbit of (2.1) of relative period 1. Fixing $u_0 \in \mathscr{P}$ there is a $\sigma \in \Gamma$ such that $\sigma^{-1}\Phi_1(u_0) = u_0$. We let $\Delta \subset \Gamma$ denote the isotropy of u_0 and $\Sigma \subset \Gamma$ denote the group generated by σ and Δ. As mentioned in Section 2, in [28, 34] it is shown that a neighborhood U of $\mathscr{P}$ can be parametrized as (2.6) with identifications (2.7).

To prove this result, in [28] a family of cross-sections V_θ around the relative periodic solution is constructed and the linearized flow $(\mathrm{D}\Phi_\theta)_{u_0}$ is used to define coordinates on V_θ. First, write

$$T_{\Phi_\theta(u_0)}M = T_{\Phi_\theta(u_0)}\mathscr{P} \oplus V_\theta, \tag{5.1}$$

where $V_0 = V$ and the cross-sections V_θ are Δ-invariant and depend smoothly on θ. Let $P_\theta\colon T_{\Phi_\theta(u_0)}M \to V_\theta$ denote the associated family of Δ-equivariant projections. The spaces V_θ are chosen such that the linearized flow $(\mathrm{D}\Phi_\theta)_{u_0}$ restricts to a Δ-equivariant map $P_\theta(\mathrm{D}\Phi_\theta)_{u_0}\colon V \to V_\theta$ and that

$$P_{\theta+1} = \sigma P_\theta \sigma^{-1}. \tag{5.2}$$

The neighborhood U of the relative periodic solution is parametrized by the submersion $\tau\colon U \to \Gamma \times V \times \mathbb{R}$ where

$$u = \tau(\gamma, v, \theta) = \gamma\psi(\Phi_\theta(u_0), P_\theta \mathrm{D}\Phi_\theta(u_0)\, J_\theta v). \tag{5.3}$$

Here $\psi(u, w) \in M$, $u \in M$, $w \in T_u M$, is a local smooth Γ-equivariant chart for M near u and $J_\theta \in \mathrm{GL}(V)$ is a Δ-equivariant homotopy such that

$$J_0 = \mathrm{id},\ P_0\sigma^{-1}\mathrm{D}\Phi_1(u_0)\, J_{\theta+1} = J_\theta Q^{-1} \tag{5.4}$$

and $Q^{-1} \in \mathrm{O}(V)$ is an orthogonal twisted equivariant matrix of finite order $2k_V$ (rsp. $2k$ if k is finite, e.g. in the case of algebraic groups Γ). Using the parameterization (5.3) one verifies the identifications (2.7).

In the following two Sections we show how to adapt this approach to reversible systems.

5.2. *Twisted Reversible Equivariant Linear Maps*

In this Section we show that twisted reversible equivariant linear maps can be decomposed into a product of two commuting matrices one of them being reversibly and equivariantly homotopic to identity and the other one being orthogonal and of twice the order k of the twist morphism ϕ. We need this result to find reversible homotopies J_θ which are used later in defining the parametrization of a neighborhood of a reversible RPO, cf. (5.3), (5.4).

LEMMA 5.1. *Let Δ_ρ be a compact Lie group acting orthogonally on some finite-dimensional vector space V and suppose that ϕ is a morphism of finite order k, such that $\phi|_\Delta \in \mathrm{Aut}(\Delta)$ and for all $\delta \in \Delta$ $\phi(\delta\rho) = \phi(\delta)\,\phi(\rho)$ and $\phi(\rho\delta) = \phi(\rho)\,\phi^{-1}(\delta)$. Let $L\colon V \to V$ be a twisted reversible equivariant nonsingular linear map. Then there exists a twisted reversible equivariant* *orthogonal map $A: V \to V$ such that $A^{2k} = I$ and $A^{-1}L$ is homotopic to the identity with a homotopy $J_\theta = \exp(\theta\eta)$ which is Δ-equivariant, commutes with A and L and is reversible*

$$\rho J_\theta \rho^{-1} = J_\theta^{-1} = J_{-\theta}.$$

Proof. We decompose $L = L_s L_n$ where L_s is the semisimple part of L, L_n is unipotent and $[L_s, L_n] = 0$. As shown in [19, Lemma 2.2] the map L_s is twisted reversible equivariant as L. Therefore the matrix L_n is Δ-equivariant and ρ-reversible. Note that $L_n - \mathrm{id}$ is nilpotent. Therefore $\ln(L_n)$ is a polynomial in $L_n - \mathrm{id}$ and consequently Δ-equivariant. Moreover

$$\begin{aligned} \rho \ln(L_n)\, \rho^{-1} &= \rho \sum_i \frac{(-1)^i}{i!} (L_n - \mathrm{id})^i\, \rho^{-1} = \sum_i \frac{(-1)^i}{i!} (L_n^{-1} - \mathrm{id})^i \\ &= \ln(L_n^{-1}) = -\ln(L_n). \end{aligned}$$

Therefore the unipotent part L_n of L is Δ-equivariantly and reversibly homotopic to id and the homotopy $\mathrm{e}^{\ln(L_n)\,\theta}$ commutes with L and L_s. From now on we can assume without loss of generality that L is semisimple.

The map L^k is equivariant in the usual sense. Let μ denote an eigenvalue of L^k and denote the corresponding eigenspace by E_μ. Note that E_μ is Δ-invariant and is also invariant under L. However, E_μ is not necessarily invariant under ρ: If μ is an eigenvector of L^k with eigenvector v_μ then μ^{-1} is also an eigenvalue of L^k, with eigenvector $v_{\mu^{-1}} = \rho v_\mu$. Hence a Δ_ρ-invariant subspace for ρ is given by $\hat{E}_\mu = E_\mu + E_{\mu^{-1}}$. To prove the lemma, it is sufficient to restrict to $\hat{E}_\mu$, or, if $\mu \in \mathbb{C}$, to $\hat{E}_\mu + \hat{E}_{\bar{\mu}}$.

If $\mu \in \mathbb{R}^+$ then let λ be the positive real kth root of μ and define

$$\eta|_{E_\mu} := \ln(\lambda), \qquad \eta|_{E_{\mu^{-1}}} := -\ln(\lambda).$$

Then $J_\theta = \mathrm{e}^{\eta\theta}$ defines a Δ-equivariant reversible homotopy commuting with L such that $J_0 = \mathrm{id}$ and $J_k = L^k$.

If L^k has an eigenvalue $\mu \in \mathbb{R}^-$, then we homotopy $L^k|_{\hat{E}_\mu}$ to $-\mathrm{id}|_{\hat{E}_\mu}$ using the matrix η defined above.

If the eigenvalue $\mu \in \mathbb{C} \setminus \mathbb{R}$ of L^k lies on the unit circle then again we choose a kth root λ of μ and define

$$\eta|_{E_\mu} = \mathrm{i} \arg(\lambda), \qquad \eta|_{E_{\bar{\mu}}} = -\mathrm{i} \arg(\lambda)$$

and obtain a real-valued reversible homotopy $\mathrm{e}^{\eta\theta}$ between id and L^k on the real subspace corresponding to $\hat{E}_\mu$.

If the eigenvalue $\mu \in \mathbb{C} \setminus \mathbb{R}$ of L^k has module different from 1 then we get a quadruple of eigenvalues

$$\{\mu, \bar{\mu}, \mu^{-1}, \bar{\mu}^{-1}\}.$$

Again we choose a kth root λ of μ and define the homotopy $J_\theta = e^{\eta\theta}$ where

$$\eta|_{E_\mu} = \ln(|\lambda|) + \mathrm{i}\,\ln(\arg(\lambda)), \qquad \eta|_{E_{\mu^{-1}}} = -\ln(|\lambda|) - \mathrm{i}\,\ln(\arg(\lambda))$$

and the complex conjugated equations hold on $E_{\bar\mu}$, $E_{\bar\mu^{-1}}$. Again the matrix η anticommutes with ρ and is Δ-equivariant.

In all cases we obtain a reversible Δ-equivariant path J_θ with

$$\rho J_\theta \rho^{-1} = J_{-\theta} = J_\theta^{-1}, \qquad \delta J_\theta = J_\theta \delta, \quad \delta \in \Delta, \quad \theta \in \mathbb{R}$$

which is smooth in θ, commutes with L and satisfies

$$J_0 = \mathrm{id}, \qquad J_{\theta+1} = A^{-1} L J_\theta, \quad A^{2k} = \mathrm{id}.$$

The matrix A is orthogonal, twisted reversible equivariant and commutes with J_θ and L. ∎

Remark 5.2. The order of A in the above proof is not optimally chosen concerning orientability issues. In fact it is often possible to choose A of order k even if L^k has eigenvalues in $\mathbb{R}^-$. Whether or not this is possible depends on the type of representation of Ξ_ρ on V.

5.3. *Finiteness of k and k_V*

In this subsection we study the order of the twist morphism $\phi\colon \Delta_\rho \to \Delta_\rho$ and the corresponding twisted reversible equivariant matrices L defined on some Δ_ρ-invariant finite-dimensional vector space V. Remember that the morphism ϕ is defined by some $\sigma \in N(\Delta)$ with $\rho\sigma\rho^{-1}\sigma \in \Delta$. As before, Δ is a compact subgroup of Γ with $\rho^2 \in \Delta$ and Δ_ρ is the index-2-extension of Δ by ρ. In Section 5.3.1 we show that the order k of the twist morphism ϕ is finite for an appropriate choice of σ in the coset of $\sigma\Delta$ if the overall symmetry group Γ is algebraic. In Section 5.3.2 we show that there is always a $k_V \in \mathbb{N}$ and a $\delta_0 \in \Delta$ such that $(\delta_0 L)^{k_V}$ is reversible and Δ-equivariant for any twisted reversible equivariant matrix $L \in \mathrm{GL}(V)$ even if Γ is not algebraic.

We will need these results in Section 5.4 below to prove that the matrix Q occurring in the identification (4.10) can always be chosen to have finite order $2k$ or $2k_V$.

5.3.1. *Algebraic groups.* We now assume that Γ is an algebraic group. Compact and Euclidean groups are algebraic groups whereas $\Gamma = \mathbb{Z}$ is an example of a non-algebraic group.

LEMMA 5.3. *Let Γ be an algebraic group, let $\Delta \subseteq \Gamma$ be compact with $\rho^2 \in \Delta$ and assume that $\sigma \in N(\Delta)$ satisfies $\sigma\rho\sigma\rho^{-1} \in \Delta$. Further let Σ be the* *group generated by σ and Δ. Then there is some $n \in \mathbb{Z}$ and some choice of σ in the coset $\sigma\Delta$ such that $\sigma^n = \mathrm{e}^{n\xi}$ where $\xi \in \mathrm{L}Z(\Sigma)$ and $\mathrm{Ad}_\rho\, \xi = -\xi$. In particular, the morphism ϕ of Δ_ρ defined by σ has finite order k.*

Proof. Let

$$C = c(Z(\sigma)) \cap N(\Delta),$$

where $c(G) \subset G$ denotes the center of the group G. We have $\sigma \in C$. The group C is Abelian and so the set

$$C^- = \{\gamma \in C \mid \gamma\rho\gamma\rho^{-1} \in \Delta\}$$

is a group: let $\gamma_1, \gamma_2 \in C^-$. Then there are $\delta_1, \delta_2 \in \Delta$ such that

$$\gamma_i = \delta_i \rho \gamma_i^{-1} \rho^{-1}, \qquad i = 1, 2.$$

Therefore

$$\gamma_1\gamma_2 = \delta_1\rho\gamma_1^{-1}\rho^{-1}\delta_2\rho\gamma_2^{-1}\rho^{-1} = \delta_1\rho\gamma_1^{-1}\tilde{\delta}_2\gamma_2^{-1}\rho^{-1} = \delta_1\hat{\delta}_2\rho(\gamma_1\gamma_2)^{-1}\rho^{-1},$$

where $\tilde{\delta}_2, \hat{\delta}_2 \in \Delta$. Here we used that ρ and γ_1 are normal in Δ and that C^- is Abelian.

The group C^- is algebraic and $\sigma \in C^-$. Hence by [14] there is some $n \in \mathbb{N}$ and some $\eta \in \mathrm{L}C^-$ such that $\sigma^n = \mathrm{e}^{n\eta}$.

Now we first treat the case that Σ is compact. Then Σ_ρ is compact. So we can choose a Σ_ρ-invariant scalar-product on $\mathrm{L}\Gamma$. As in [34, Lemmata 6.1, 6.2] we decompose orthogonally

$$\eta = \xi + \chi,$$

where $\chi \in \mathrm{L}\Delta$ and $\xi \in (\mathrm{L}\Delta)^\perp$. As shown in [34, Lemmata 6.1, 6.2] we have $\xi \in \mathrm{L}Z(\Sigma)$. The argument there is as follows: since the scalar product on $\mathrm{L}\Gamma$ is Σ-invariant we know that $\mathrm{Ad}_\gamma\, \xi - \xi \in (\mathrm{L}\Delta)^\perp$ for $\gamma \in \Sigma$. Further $\mathrm{Ad}_\gamma\, \xi - \xi = \mathrm{Ad}_\gamma(\eta - \chi) - (\eta - \chi) \in \mathrm{L}\Delta$ for $\gamma \in \Sigma$ since $\sigma \in N(\Delta)$ and $\eta \in \mathrm{L}N(\Delta) \cap \mathrm{L}Z(\sigma)$. Consequently $\mathrm{Ad}_\gamma\, \xi - \xi \in \mathrm{L}\Delta \cap (\mathrm{L}\Delta)^\perp = \{0\}$. In particular letting $\sigma = \mathrm{id}$ so that $\Sigma = \Delta$ the above argument shows that for any compact group Δ and any Δ-invariant scalar-product on $\mathrm{L}\Gamma$ the relation $\mathrm{L}N(\Delta) \cap (\mathrm{L}\Delta)^\perp \subseteq \mathrm{L}Z(\Delta)$ holds; cf. [11].

With a similar argument we can show that $\mathrm{Ad}_\rho\, \xi = -\xi$: Since the scalar product on $\mathrm{L}\Gamma$ is ρ-invariant we conclude that $\mathrm{Ad}_\rho\, \xi + \xi \in (\mathrm{L}\Delta)^\perp$. Further, since $\eta \in \mathrm{L}C^-$ we have $\mathrm{Ad}_\rho\, \eta + \eta \in \mathrm{L}\Delta$. Therefore $\mathrm{Ad}_\rho\, \xi = \mathrm{Ad}_\rho(\eta - \chi) \in -\xi + \mathrm{L}\Delta$ and consequently $\mathrm{Ad}_\rho\, \xi + \xi = 0$. Changing σ to $\sigma \exp(-\chi)$ we see that $k < \infty$.

If Σ is non-compact then the relative periodic orbit has to be a modulated travelling wave. We proceed as in Lemma 6.2 of [34]: We choose a Δ_ρ-invariant scalar-product $(\cdot,\cdot)$ on $\mathrm{L}\Gamma$. We decompose $\eta = \tilde{\xi} + \tilde{\chi}$ orthogonally with respect to this scalar product, i.e., $\tilde{\xi} \in \mathrm{L}N(\Delta) \cap (\mathrm{L}\Delta)^{\perp} \subseteq \mathrm{L}Z(\Delta)$, $\tilde{\chi} \in \mathrm{L}\Delta$.

Now we define a new scalar product on $X := \mathbb{R}\eta + \mathrm{L}\Delta = \mathbb{R}\tilde{\xi} \oplus \mathrm{L}\Delta$ as

$$\langle u, v\rangle = \sum_{i=0}^{n-1} (\mathrm{Ad}_{\sigma^i} u, \mathrm{Ad}_{\sigma^i} v), \qquad u, v \in X.$$

Since $\sigma \in N(\Delta)$ this new scalar product is Δ-invariant. Since $\tilde{\xi} \in \mathrm{L}Z(\Delta)$ we have $\mathrm{Ad}_{\sigma^n} = \mathrm{Ad}_{\exp(n\tilde{\chi})}$ on $\mathrm{L}\Delta$ and $\mathrm{Ad}_{\sigma^n} \tilde{\xi} = \mathrm{Ad}_{\exp(n\tilde{\chi})\exp(n\tilde{\xi})} \tilde{\xi} = \mathrm{Ad}_{\exp(n\tilde{\chi})} \tilde{\xi} = \tilde{\xi}$. Therefore $(\mathrm{Ad}_{\sigma^n} u, \mathrm{Ad}_{\sigma^n} v) = (u, v)$ for $u, v \in X$ and the new scalar product $\langle\cdot,\cdot\rangle$ is σ-invariant. Finally $(\cdot,\cdot)$ is Δ_ρ-invariant and $\sigma\rho = \phi^{-1}(\rho)\,\sigma^{-1} \in \Delta\rho\sigma^{-1}$. Therefore

$$\begin{aligned}
\langle \mathrm{Ad}_\rho u, \mathrm{Ad}_\rho v\rangle &= \sum_{i=0}^{n-1} (\mathrm{Ad}_{\sigma^i} \mathrm{Ad}_\rho u, \mathrm{Ad}_{\sigma^i} \mathrm{Ad}_\rho v)\\
&= \sum_{i=0}^{n-1} (\mathrm{Ad}_{\phi^{-i}(\rho)} \mathrm{Ad}_{\sigma^{-i}} u, \mathrm{Ad}_{\phi^{-i}(\rho)} \mathrm{Ad}_{\sigma^{-i}} v)\\
&= \sum_{i=0}^{n-1} (\mathrm{Ad}_{\sigma^{-i}} u, \mathrm{Ad}_{\sigma^{-i}} v)\\
&= \langle u, v\rangle.
\end{aligned}$$

Now we decompose $\eta = \xi + \chi$ with respect to this new Σ_ρ-invariant scalar product on X. Proceeding as in the case of compact spatio-temporal symmetry Σ_ρ we see that again $\mathrm{Ad}_\rho \xi = -\xi$. Changing σ to $\sigma \exp(-\chi)$ we achieve that k becomes finite. ∎

If the spatiotemporal symmetry group Σ is algebraic then we can define $\Gamma := \Sigma$ and the above lemma applies. This means that in particular for discrete rotating waves and modulated rotating waves the order of the automorphism ϕ is always finite for appropriate choices of σ.

5.3.2. *Non-algebraic groups.* In the case of discrete travelling waves k may be ∞ for any choice of σ in the coset $\sigma\Delta$, see [34, Example 6.6]. Let V be a finite-dimensional representation of Δ_ρ and let $L\colon V \to V$ be a Δ_ρ twisted reversible equivariant matrix wrt. the twist morphism induced by σ. In [34], for the twisted equivariant case, we proved that there is still a finite k_V such that $(\delta_0 L)^{k_V}$ is Δ-equivariant for an appropriate $\delta_0 \in \Delta$. Similarly in the twisted reversible equivariant case there is some finite k_V and some $\delta_0 \in \Delta$ such that $(\delta_0 L)^{k_V}$ is Δ-equivariant and ρ-reversible as we will now show. This means that the effective order k_V of the twist morphism induced by $\sigma\delta_0^{-1}$ on V is finite. We will apply this to the linearization along reversible relative periodic orbits of non-algebraic group actions in Section 5.4 below.

First we prove that it suffices to treat the case that L is orthogonal.

PROPOSITION 5.4. *Assume that the compact group Δ_ρ acts orthogonally on the finite-dimensional vector space V. Suppose that $L: V \to V$ is a twisted reversible equivariant nonsingular linear map. Then there exists a twisted reversible equivariant orthogonal map $A: V \to V$ such that $A^{-1}L$ is Δ-equivariant and (ρA)-reversible.*

Proof. By polar decomposition, we can write L uniquely as $L = AB$ where A is orthogonal and B is symmetric and positive definite. One easily computes, see [34, Proposition 1.1], that B is equivariant and A is twisted equivariant.

Recall that B is defined as the unique symmetric positive definite square root of $B^2 = L^T L$. We show that B satisfies $(\rho A)\, B = B^{-1}(\rho A)$. Using orthogonality of the Δ_ρ-action on V, we find

$$\begin{aligned} B^2(\rho A) &= L^T L \rho A = L^T \phi(\rho)\, L^{-1} A = ((\phi(\rho))^{-1}\, L)^T\, L^{-1} A = (L^{-1} \rho^{-1})^T\, L^{-1} A \\ &= \rho (L^{-1})^T\, L^{-1} A = \rho (A^{-1})^T\, (B^{-1})^T\, B^{-1} A^{-1} A = \rho A B^{-2}. \end{aligned}$$

Thus, the positive definite matrix $B' = (\rho A)^{-1}\, B(\rho A)$ satisfies

$$(B')^2 = (\rho A)^{-1}\, L^T L(\rho A) = B^{-2}.$$

Since B is the unique positive definite square root of $L^T L$, it follows that $B^{-1} = B' = (\rho A)^{-1}\, B(\rho A)$. The matrix A is twisted reversible equivariant. Namely,

$$A\rho = LB^{-1}\rho = LB^{-1}\rho A A^{-1} = L\rho A B A^{-1} = \phi(\rho)\, L^{-1} L A^{-1} = \phi(\rho)\, A^{-1}. \quad \blacksquare$$

Now consider the subgroup G of $\mathrm{O}(V)$ generated by the representation Δ_V of Δ on V and the orthogonal, twisted reversible equivariant matrix A. The group Δ_V is normal in G. Let G_ρ denote the index two extension of G by the orthogonal representation ρ_V of ρ on V and let ϕ_V be the morphism of Δ_V defined by $\phi_V(\delta) = A\delta A^{-1}$, $\delta \in \Delta_V$, and $\phi_V(\delta\rho_V) = A\delta\rho_V A$. Since G is compact and therefore algebraic we can apply Lemma 5.3 and conclude that there is some $\delta_0 \in \Delta$ and some $k_V \in \mathbb{N}$ such that $(\delta_0 A)^{k_V}$ is Δ-equivariant and ρ-reversible. Changing A to $A_{\text{new}} = \delta_0 A$ we see that now $A_{\text{new}}^{k_V}$ is Δ-equivariant and ρ-reversible. Since A_{new} and $L_{\text{new}} = \delta_0 L$ define the same morphism ϕ_V of Δ_V by Proposition 5.4 we conclude that also $L_{\text{new}}^{k_V}$ is Δ-equivariant and reversible. We summarize this result in the following lemma.

Lemma 5.5. *Suppose that $L: V \to V$ is a twisted reversible equivariant nonsingular linear map. Then there exists a finite number $k_V \in \mathbb{N}$ and some $\delta_0 \in \Delta$ such that $(\delta_0 L)^{k_V}$ is reversible and Δ-equivariant.*

5.4. *Parametrization and Equations on the Slice*

In this section we come towards the proofs of Theorems 4.11, 4.12, 4.13. Let $\mathscr{P}$ be a reversible relative periodic solution with minimal relative period one. Choose some $u_0 \in \mathscr{P}$. As we saw in Section 4.1 we may assume without loss of generality that $u_0 \in \mathrm{Fix}(\rho)$. We have $\Phi_1(u_0) = \sigma u_0$ for some $\sigma \in \Gamma$ and $\Phi_t(u_0) \notin \Gamma u_0$ for $0 < t < 1$. Let Δ and Σ denote the spatial and spatio-temporal symmetry groups of $\mathscr{P}$ in Γ. Remember that the isotropy subgroup $\Delta \subset \Gamma$ is compact and that Σ is the closed subgroup of Γ generated by Δ and σ.

Since Φ_1 is ρ-reversible and Δ-equivariant, i.e., $\Phi_1 \rho = \rho \Phi_{-1}$ and $\delta \Phi_1 = \Phi_1 \delta$ for all $\delta \in \Delta$, it follows that the diffeomorphism $\sigma^{-1}\Phi_1$ is twisted reversible equivariant: we obtain [20]

$$(\sigma^{-1}\Phi_1)\,\delta\rho = \sigma^{-1}\delta\rho\sigma^{-1}\sigma\Phi_{-1} = \phi(\delta\rho)\,\sigma\Phi_{-1} = \phi(\delta\rho)\,\Phi_1^{-1}\sigma, \qquad (5.5)$$

where ϕ is the twist morphism as defined before. Accordingly, the linearization $\sigma^{-1}\mathrm{D}\Phi_1$ is also twisted reversible equivariant.

In [28], it was first shown how the flow in a neighborhood U of the relative periodic solution $\mathscr{P}$ can be written as a skew product flow on a space of the form $\Gamma \times V \times \mathbb{R}$, and we shortly described this approach in Section 2 and Section 5.1. We will use the parametrization (5.3) from [28], but need to take reversibility into account. First of all we choose V as a Δ_ρ-invariant cross-section to $\mathscr{P}$. In the following lemma we show how the projections a P_θ and the homotopy J_θ mentioned in Section 5.1 can be chosen to be ρ-reversible.

Lemma 5.6. *Let $L = P_0\sigma^{-1}(\mathrm{D}\Phi_1)_{u_0}: V \to V$.*

(a) *There is an orthogonal twisted reversible equivariant map $A: V \to V$ of finite order $2k_V$ and a smooth family of Δ-equivariant nonsingular linear maps $J_\theta: V \to V$, $\theta \in \mathbb{R}$ commuting with A, such that*

$$J_0 = I, \qquad \rho J_\theta \rho^{-1} = J_{-\theta} = J_\theta^{-1}, \qquad LJ_{\theta+1} = J_\theta A, \quad \theta \in \mathbb{R}. \qquad (5.6)$$

(b) *The Δ-equivariant projections P_θ can be chosen so that*

$$P_\theta \rho = \rho P_{-\theta}, \qquad P_{\theta+1}\sigma = \sigma P_\theta, \quad \theta \in \mathbb{R}.$$

Proof. Let $\langle\, ,\rangle$ be a Δ_ρ-invariant inner product on $T_{u_0}M$ and define P_0 to be the orthogonal projection onto V such that $T_{u_0}\mathscr{P} = (\mathrm{id} - P_0)\,T_{u_0}M$ and that P_0 is Δ_ρ-equivariant. Since $\sigma^{-1}(\mathrm{D}\Phi_1)_{u_0}$ is twisted reversible equivariant as we saw above and $\sigma^{-1}(\mathrm{D}\Phi_1)_{u_0}$ maps $T_{u_0}\mathscr{P}$ into itself also $L = P_0\sigma^{-1}(\mathrm{D}\Phi_1)_{u_0} \in \mathrm{GL}(V)$ is twisted reversible equivariant, i.e., $L\delta = \phi(\delta)\,L$ and $L\delta\rho = \phi(\delta\rho)\,L^{-1}$ for all $\delta \in \Delta$, where ϕ is the twist morphism on Δ_ρ. By Lemma 5.5 there is some $k_V \in \mathbb{N}$ and some $\sigma \in \sigma\Delta$ such that L^{k_V} is Δ-equivariant and reversible. Now we can apply Lemma 5.1 from Section 5.2 with ϕ replaced by ϕ_V and conclude that there is some orthogonal twisted reversible matrix A of order $2k_V$ such that $A^{-1}L$ is homotopic to identity by a Δ-equivariant reversible homotopy J_θ which commutes with A.

Next, we prove part (b). We repeat the argument above, but applied to the twisted equivariant map $\tilde{L} = \sigma^{-1}(\mathrm{D}\Phi_1)_{u_0}: T_{u_0}M \to T_{u_0}M$ to obtain an orthogonal map $\tilde{A}: T_{u_0}M \to T_{u_0}M$ and a smooth family of Δ-equivariant nonsingular linear maps $\tilde{J}_\theta: T_{u_0}M \to T_{u_0}M$ such that

$$\tilde{J}_0 = I, \qquad \rho\tilde{J}_\theta = \tilde{J}_{-\theta}\rho, \qquad \tilde{L}\tilde{J}_{\theta+1} = \tilde{J}_\theta\tilde{A}, \quad \theta \in \mathbb{R}.$$

Let $\langle\ ,\ \rangle_\theta$ be the Δ-invariant inner product on $T_{\Phi_\theta(u_0)}M$ defined by

$$\langle v, w\rangle_\theta = \langle \tilde{J}_\theta^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}\, v, \tilde{J}_\theta^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}\, w\rangle, \qquad v, w \in T_{\Phi_\theta(u_0)}M.$$

Following [34], V_θ is defined to be the orthogonal complement to $T_{\Phi_\theta(u_0)}\mathscr{P}$ in $T_{\Phi_\theta(u_0)}M$ with respect to the inner product $\langle\ ,\ \rangle_\theta$. We let $P_\theta: T_{\Phi_\theta(u_0)}M \to V_\theta$ be the orthogonal projection with respect to this inner product. One easily computes, see [34], that $\langle v, w\rangle_{\theta+1} = \langle \sigma^{-1}v, \sigma^{-1}w\rangle_\theta$. Therefore we see that indeed $P_{\theta+1}\sigma = \sigma P_\theta$ as required. It remains to be shown that also $P_\theta\rho = \rho P_{-\theta}$. We compute that

$$(\mathrm{D}\Phi_{-\theta})_{u_0} = \mathrm{D}(\rho^{-1}\Phi_\theta\rho)_{u_0} = \rho^{-1}(\mathrm{D}\Phi_\theta)_{\rho u_0}\,\rho = \rho^{-1}(\mathrm{D}\Phi_\theta)_{u_0}\,\rho. \tag{5.7}$$

Here we used that $\rho u_0 = u_0$. Hence

$$\tilde{J}_{-\theta}^{-1}(\mathrm{D}\Phi_{-\theta})_{u_0}^{-1} = \tilde{J}_{-\theta}^{-1}\rho^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}\,\rho = \rho^{-1}\tilde{J}_\theta^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}\rho.$$

Using the orthogonality of ρ on $T_{u_0}M$, it follows that

$$\begin{aligned}\langle v, w\rangle_{-\theta} &= \langle \tilde{J}_{-\theta}^{-1}(\mathrm{D}\Phi_{-\theta})_{u_0}^{-1}\, v, \tilde{J}_{-\theta}^{-1}(\mathrm{D}\Phi_{-\theta})_{u_0}^{-1}\, w\rangle \\ &= \langle \rho^{-1}\tilde{J}_\theta^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}\,\rho v, \rho^{-1}\tilde{J}_\theta^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}\,\rho w\rangle \\ &= \langle \tilde{J}_\theta^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}\,\rho v, \tilde{J}_\theta^{-1}(\mathrm{D}\Phi_\theta)_{u_0}^{-1}\,\rho w\rangle \\ &= \langle \rho v, \rho w\rangle_\theta.\end{aligned}$$

Since $P_\theta: T_{\Phi_\theta(u_0)}M \to V_\theta$ is the orthogonal projection with respect to the inner product $\langle\ ,\ \rangle_\theta$ we see that $P_\theta\rho = \rho P_{-\theta}$ as required. ∎

Now, consider the Γ-equivariant submersion $\tau: \Gamma_\rho \times V \times \mathbb{R} \to M$ defined by

$$\tau(\gamma, v, \theta) = \gamma\psi(\Phi_\theta(u_0)\,,\, P_\theta(\mathrm{D}\Phi_\theta)_{u_0}\, J_\theta v). \tag{5.8}$$

Here $\psi(u, w) \in M$, $u \in M$, $w \in T_u M$, is a local smooth Γ_ρ-equivariant chart for M near u. It follows from Δ-equivariance of P_θ, $(\mathrm{D}\Phi_\theta)_{u_0}$ and J_θ, and Δ-invariance of the points $\Phi_\theta(u_0) \in \mathscr{P}$, that $\tau(\gamma\delta, v, \theta) = \tau(\gamma, \delta v, \theta)$ for all $\delta \in \Delta$. Also as in [34] we compute that

$$\tau(\gamma, v, \theta+1) = \tau(\gamma\sigma, Q^{-1}v, \theta), \tag{5.9}$$

where $Q = A^{-1}$. In the presence of reversibility, we moreover have

$$\begin{aligned}\tau(\gamma, v, -\theta) &= \gamma\psi(\Phi_{-\theta}(u_0), P_{-\theta}(\mathrm{D}\Phi_{-\theta})_{u_0} J_{-\theta} v)\\ &= \gamma\psi(\rho^{-1}\Phi_\theta(u_0), \rho^{-1}P_\theta\rho(\mathrm{D}\Phi_{-\theta})_{u_0} J_{-\theta} v)\\ &= \gamma\psi(\rho^{-1}\Phi_\theta(u_0), \rho^{-1}P_\theta(\mathrm{D}\Phi_\theta)_{u_0} J_\theta \rho v)\\ &= \tau(\gamma\rho^{-1}, \rho v, \theta).\end{aligned}$$

Here we used the relations

$$\rho\Phi_\theta(u_0) = \Phi_{-\theta}(u_0), \; \mathrm{D}\Phi_\theta(u_0) = \rho \mathrm{D}\Phi_{-\theta}(u_0)\, \rho^{-1}.$$

This proves Theorem 4.11.

Proof of Theorem 4.12. If Γ is algebraic then we can simplify the bundle structure by passing to a convenient comoving frame. Due to Lemma 5.3 we can write $\sigma^n = \exp(n\xi)$ where $\xi \in \mathrm{L}Z(\Sigma)$ and $\mathrm{Ad}_\rho\, \xi = -\xi$. Let $\sigma' = \exp(-\xi)\,\sigma$, so that σ' has order n. We define the new submersion

$$\tau_{\text{new}}(\gamma, v, \theta) = \tau(\gamma \exp(-\theta\xi), v, \theta). \tag{5.10}$$

Note that τ_{new} remains Γ-equivariant, since Γ acts on the left. Since $\tau(\gamma\delta, v, \theta) = \tau(\gamma, \delta v, \theta)$ for all $\delta \in \Delta$, and $\xi \in \mathrm{L}Z(\Delta)$, it is immediate that $\tau_{\text{new}}(\gamma\delta, v, \theta) = \tau_{\text{new}}(\gamma, \delta v, \theta)$ for all $\delta \in \Delta$. Similarly, see [34], we compute that

$$\begin{aligned}\tau_{\text{new}}(\gamma, v, \theta+1) &= \tau(\gamma \exp(-\xi) \exp(-\theta\xi), v, \theta+1)\\ &= \tau(\gamma \exp(-\xi) \exp(-\theta\xi)\, \sigma, Q^{-1}v, \theta)\\ &= \tau(\gamma\sigma' \exp(-\theta\xi), Q^{-1}v, \theta)\\ &= \tau_{\text{new}}(\gamma\sigma', Q^{-1}v, \theta).\end{aligned}$$

Now we need to check the reversibility condition in (4.10). We get

$$\begin{aligned}\tau_{\text{new}}(\gamma\rho^{-1}, \rho v, -\theta) &= \tau(\gamma\rho^{-1} \exp(-\theta\xi), \rho v, -\theta)\\ &= \tau(\gamma \exp(\theta\xi)\, \rho^{-1}, \rho v, -\theta)\\ &= \tau(\gamma \exp(\theta\xi), v, \theta) = \tau_{\text{new}}(\gamma, v, \theta).\end{aligned} \tag{5.11}$$

Here we used that $\mathrm{Ad}_\rho\, \xi = -\xi$.

Since σ' has order n and $Q^{2n} = \mathrm{id}$, it follows that $\tau_{\text{new}}(\gamma, v, \theta + 2n) = \tau_{\text{new}}(\gamma, v, \theta)$. Hence τ_{new} induces a Γ-equivariant and $(\Delta \rtimes \mathbb{Z}_{2n})$-equivariant submersion

$$\tau_{\text{new}}: \Gamma_\rho \times V \times S^1 \to M, \qquad S^1 = \mathbb{R}/2n\mathbb{Z},$$

onto a uniform neighborhood U of the reversible RPO $\mathscr{P}$ which respects reversibility in the sense of (5.11). This proves Theorem 4.12.

Proof of Theorem 4.13. Now let $\mathscr{P}$ be a reversible RPO with finite reversible index m_ρ. Write $\sigma^{m_\rho} = \delta_0 \exp(m_\rho \zeta)$ where $\zeta \in LZ(\Sigma)$, $\mathrm{Ad}_\rho\, \zeta = -\zeta$ and $\delta_0 \in \Delta$ and let $\tilde{\sigma} = \sigma \exp(-\zeta)$. The matrix $\delta_0^{-1} Q^{m_\rho}$ with Q as before is Δ-equivariant and ρ-reversible. Moreover, since $\mathrm{Ad}_\sigma\, \delta_0 = \delta_0$ also $Q\delta_0 Q^{-1} = \delta_0$. Let $G \in \mathrm{O}(V)$ be the compact group generated by Q and the representation Δ_V of Δ on V. Since Q^{-1} is twisted reversible equivariant $\rho Q \rho^{-1} \in Q^{-1} \Delta_V \subset G$ and therefore $G_\rho = \langle G, \rho_V \rangle$ is an index-2-extension of G (here ρ_V is the representation of ρ on V). We saw in Lemma 5.1 that the square of a G-equivariant and ρ-reversible matrix is G-equivariantly and ρ-reversibly homotopic to identity by a homotopy $J_\theta = \exp(\eta\theta)$. This implies that $(\delta_0^{-1} Q^{m_\rho})^2 = \exp(-2m_\rho\, \eta)$ where $\eta \in \mathrm{so}(V)$ is Δ-equivariant, commutes with Q and satisfies $\rho_V \eta \rho_V^{-1} = -\eta$. Set $R = Q \exp(\eta) \in \mathrm{O}(V)$ and define similarly as in the proof of Theorem 4.12

$$\tilde{\tau}(\gamma, v, \theta) = \tau(\gamma \exp(-\theta\zeta), \exp(-\theta\eta)\, v, \theta). \tag{5.12}$$

One easily verifies (see [34]) that

$$\tilde{\tau}(\gamma\tilde{\sigma}^{-1}, Rv, \theta + 1) = \tilde{\tau}(\gamma\delta^{-1}, \delta v, \theta) = \tilde{\tau}(\gamma, v, \theta), \qquad \delta \in \Delta.$$

Since $\mathrm{Ad}_\rho\, \eta = -\eta$ and $\mathrm{Ad}_\rho\, \zeta = -\zeta$ we moreover see that

$$\tilde{\tau}(\gamma\rho^{-1}, \rho v, -\theta) = \tilde{\tau}(\gamma, v, \theta). \tag{5.13}$$

Since $\tilde{\sigma}^{m_\rho} = \delta_0$ and $R^{2m_\rho} = \delta_0^2$ it follows that

$$\tilde{\tau}(\gamma, v, \theta + 2m_\rho) = \tilde{\tau}(\gamma\delta_0^2, \delta_0^{-2} v, \theta) = \tilde{\tau}(\gamma, v, \theta)$$

so that the bundle is indeed $2m_\rho$-periodic in θ. Hence $\tilde{\tau}$ induces a Γ-equivariant diffeomorphism $\tilde{\tau}$: $(\Gamma_\rho \times V \times S^1)/\tilde{\Xi}_\rho \cong U$ where $S^1 = \mathbb{R}/(2m_\rho\mathbb{Z})$, $\tilde{\Xi}_\rho$ is as in the theorem and U is a uniform neighborhood of the reversible RPO. This proves the theorem.

Appendix on the Proof of Theorem 4.14. The differential equations (2.9) are defined by the equation

$$D\tau(\gamma, v, \theta)(\gamma f_\Gamma(v, \theta), f_V(v, \theta), f_\Theta(v, \theta)) = f(\tau(\gamma, v, \theta)).$$

If Δ is continuous then $D\tau$ has a kernel due to the identification (2.11). Therefore f_Γ is only determined within $f_\Gamma + L\Delta$. In the following we will show how to choose f_Γ within $f_\Gamma + L\Delta$ in such a way that $f_\Gamma(v, \theta)$ is smooth in v, θ and that the equivariance and reversibility conditions (2.11), (2.12), and (4.11) are satisfied. We choose $f_\Gamma(v, \theta) \in \mathbf{m}_\theta$ where $\mathbf{m}_\theta \subset L\Gamma$ is a complement to $L\Delta$ which is invariant under the adjoint action of Δ_ρ on $L\Gamma$. More precisely we set $\mathbf{m}_\theta = \exp(\eta\theta)\,\mathbf{m}_0$ where $\eta \in \mathrm{Mat}(L\Gamma)$ is defined as follows.

We first deal with the setting of Theorem 4.11. We apply Lemma 5.5 on the space $X = L\Gamma \times V$ and the twisted reversible equivariant matrix

$$L = \begin{pmatrix} \mathrm{Ad}_\sigma^{-1} & 0 \\ 0 & P_0\sigma^{-1}\, D\Phi_1(u_0) \end{pmatrix}$$

and conclude that there is some finite k_X such that L^{k_X} is Δ-equivariant and ρ-reversible for an appropriate choice of $\sigma \in \sigma\Delta$ (this may enlarge the original k_V a bit). Applying Lemma 5.1 on the space $L\Gamma$ we see that there is some twisted reversible equivariant matrix A of finite order $2k_X$ and a reversible Δ-equivariant homotopy $\exp(\eta\theta)$ commuting with A such that $\mathrm{Ad}_\sigma^{-1} = A\exp(-\eta)$. We choose an inner product on $L\Gamma$ which is invariant under the compact matrix group generated by $A \in \mathrm{Mat}(L\Gamma)$, $\mathrm{Ad}_\delta \in \mathrm{Mat}(L\Gamma)$, $\delta \in \Delta$, and $\mathrm{Ad}_\rho \in \mathrm{Mat}(L\Gamma)$ and choose $\mathbf{m}_0$ as orthogonal complement to $L\Delta$ with respect to this scalar product on $L\Gamma$. Since $\mathbf{m}_\theta = \exp(\eta\theta)\,\mathbf{m}_0$ we get $\mathbf{m}_{\theta+1} = \mathrm{Ad}_\sigma\,\mathbf{m}_\theta$ so that $f_\Gamma(v, \theta)$ is smooth in v, θ and so that the equivariance and reversibility conditions (2.11), (2.12) and (4.11) for the equations on the group make sense.

In the case of Theorems 4.12 and 4.13 we set $\mathbf{m}_\theta \equiv \mathbf{m} := (L\Delta)^\perp \subset L\Gamma$ where in the setting of algebraic groups (Theorem 4.12) the scalar product on $L\Gamma$ is chosen to be invariant under the compact group generated by Δ_ρ and σ' and in the setting of Theorem 4.13 it is chosen to be $\Sigma_\rho^{m_\rho}$-invariant.

6. MAXIMAL REVERSIBLE TORI AND REVERSIBLE CARTAN SUBGROUPS

In this section we prove results on maximal reversible tori and reversible Cartan subgroups. These results are used in Sections 3 and 4.2 to study the drift of reversible relative equilibria and reversible relative periodic orbits.

For an account on maximal tori and Cartan subgroups see, e.g., [4, 6, 15, 33]. We will extend the techniques used in these references to treat reversible maximal tori and reversible Cartan subgroups in this section. Most results are restricted to either compact or Euclidean groups as these are the most relevant groups showing up in applications.

6.1. *Maximal Reversible Tori of Compact Groups*

This section deals with the proof of the following result:

THEOREM 6.1. *Let Γ_ρ be a finite-dimensional compact Lie group such that $\Gamma_\rho = \Gamma \cup \rho\Gamma$ where Γ is a normal connected subgroup of Γ_ρ and $\rho \in \Gamma_\rho$ has order* 2. *Then*

(a) *all maximal reversible tori T^ρ are conjugated by elements in $(Z(\rho))^0$;*

(b) *almost every reversible $\xi \in \mathrm{L}\Gamma$ generates a maximal reversible torus.*

This theorem can be applied to study generic drifts of reversible relative equilibria of compact symmetry groups Γ with isotropy Δ by defining $\Gamma := N(\Delta)/\Delta$; see Section 3. Of course, the theorem applies to non-connected groups Γ as well since tori are subgroups of Γ^0 anyway.

We know that Ad_ρ is an involution on $\mathrm{L}\Gamma = \mathbf{g}$. So we can decompose $\mathbf{g} = \mathbf{g}^+ \oplus \mathbf{g}^-$ where Ad_ρ acts trivially on $\mathbf{g}^+ = \mathrm{L}Z(\rho)$ and Ad_ρ acts as $-\mathrm{id}$ on $\mathbf{g}^-$. Before we prove the theorem we give an example.

EXAMPLE 6.2. Let $\Gamma = \mathrm{SO}(3)$ and let $\rho = \kappa_3$ be a reflection along the (x_1, x_2)-plane. Let ξ_i, $i = 1, 2, 3$, denote the generators of the rotations around the x_i-axis. Then $\mathbf{g}^- = \mathrm{span}(\xi_1, \xi_2)$ and $\mathbf{g}^+ = \mathrm{span}(\xi_3)$. In this case maximal tori are copies of SO(2), some of the maximal tori of SO(3) are reversible, namely those generated by elements $\xi \in \mathrm{span}(\xi_1, \xi_2)$, the maximal torus generated by ξ_3 lies in $Z(\rho)$, and all other maximal tori are neither reversible nor commute with ρ.

For $\xi \in \mathrm{L}\Gamma$ let $T(\xi) = \{e^{t\xi}; t \in \mathbb{R}\}$. For the proof of Theorem 6.1 we need the following lemma:

LEMMA 6.3. *Let G be a finite-dimensional Lie group with Lie algebra $\mathbf{g}$, let $\xi \in \mathbf{g}$ with $T(\xi)$ compact and let $\mathbf{z} = \mathrm{L}Z(\xi)$ be the Lie algebra of the centralizer of ξ. Then the map $F\colon \mathbf{z} \times \mathbf{z}^\perp \to \mathbf{g}$, defined as*

$$F(\chi, \eta) = \mathrm{Ad}_{\exp(\eta)}(\chi + \xi), \qquad \chi \in \mathbf{z}, \quad \eta \in \mathbf{z}^\perp$$

is locally bijective near $(\chi, \eta) = 0$. Here we choose the inner product on $\mathbf{g}$ to be $T(\xi)$-invariant.

Proof. The derivative of F at $\chi = 0$, $\eta = 0$ is given by

$$\mathrm{D}_\chi F(0, 0) = \mathrm{id}|_{\mathbf{z}}, \qquad \mathrm{D}_\eta F(0, 0) = [\,\cdot\,, \xi]|_{\mathbf{z}^\perp}.$$

The kernel of the map $[\cdot, \xi]$ is $\mathbf{z}$. Since $[\cdot, \xi]$ is a skew-symmetric matrix the image of $[\cdot, \xi]$ is $\mathbf{z}^{\perp}$ and so $DF(0,0)$ is bijective. By the implicit function theorem, F is locally bijective near 0. ▮

Proof of Theorem 6.1. (a) Let $S = \mathrm{Ad}_{(Z(\rho))^0} \, \mathrm{L}T^{\rho}$ where T^{ρ} is a maximal reversible torus. We will show that S is closed and $S \setminus \{0\}$ is open in $\mathbf{g}^-$ and deduce from that that $S = \mathbf{g}^-$.

Let $(\mathrm{Ad}_{\gamma_n} \chi_n)_{n \in \mathbb{N}}$, where $\chi_n \in \mathrm{L}T^{\rho}$, $\gamma_n \in (Z(\rho))^0$, be a sequence converging to $\xi \in \mathbf{g}^-$. Since Γ is compact there is some subsequence $(n_j)_{j \in \mathbb{N}}$ such that γ_{n_j} converges to $\gamma \in (Z(\rho))^0$ as $j \to \infty$. Consequently χ_{n_j} converges to some $\chi \in \mathrm{L}T^{\rho}$. Therefore $\xi = \mathrm{Ad}_{\gamma} \chi$ and S is closed in $\mathbf{g}^-$.

To prove openness of $S \setminus \{0\}$ we assume that the identity component $(c(\Gamma))^0$ of the center $c(\Gamma)$ of Γ is trivial. If this is not the case then we choose some Γ_ρ-invariant complement $\hat{\mathbf{g}}$ to the Γ_ρ-invariant Lie algebra $\mathrm{L}c(\Gamma)$ of $c(\Gamma)$ in $\mathbf{g}$ and denote the group generated by $\hat{\mathbf{g}}$ as $\hat{\Gamma}$; then $\hat{\Gamma}$ is semisimple. Taking any maximal reversible torus T^{ρ} of Γ define $\hat{T}^- := T^{\rho} \cap \hat{\Gamma}$ and $T^- := T^{\rho} \cap c(\Gamma)$. Then T^- is the unique maximal reversible torus of $c(\Gamma)$ and $\hat{T}^-$ is a maximal reversible torus of $\hat{\Gamma}$. Therefore we are only left with proving the conjugacy statement for the semisimple group $\hat{\Gamma}$, so we can assume wlog. that Γ is semisimple. To prove openness of $S \setminus \{0\}$ we proceed by induction over $\dim(\Gamma)$. For $n = 1$ the statement of Theorem 6.1(a) is clear. Assume the statement of Theorem 6.1 a) is proved for all groups K of dimension $\dim(K) < \dim(\Gamma)$ satisfying the assumptions of Theorem 6.1.

Let $\xi_0 \in S$, ie, $\xi_0 = \mathrm{Ad}_{\gamma_0} \chi_0$ for some $\chi_0 \in \mathrm{L}T^{\rho}$, $\gamma_0 \in (Z(\rho))^0$. Choose an Ad_ρ-invariant scalar product on $\mathbf{g}$. Every $\xi \approx \xi_0$ can be written as

$$\xi = \mathrm{Ad}_{\exp(\eta)}(\xi_0 + \chi), \qquad \chi \in \mathbf{z} = \mathrm{L}Z(\xi_0), \quad \eta \in \mathbf{z}^{\perp}$$

by Lemma 6.3 and this decomposition is locally unique. Let $\xi \in \mathbf{g}^-$. Then also

$$\xi = \mathrm{Ad}_{\exp(\mathrm{Ad}_\rho \eta)}(\xi_0 - \mathrm{Ad}_\rho \chi)$$

and since $\mathbf{z}$ and $\mathbf{z}^{\perp}$ are invariant under conjugation by ρ we conclude that $\mathrm{Ad}_\rho \chi \in \mathbf{z}$, $\mathrm{Ad}_\rho \eta \in \mathbf{z}^{\perp}$ and that by local uniqueness $\mathrm{Ad}_\rho \chi = -\chi$, $\mathrm{Ad}_\rho \eta = \eta$. Note that $\mathrm{Ad}_{\gamma_0} \mathrm{L}T^{\rho} \subseteq \mathrm{L}Z(\xi_0) \cap \mathbf{g}^-$ and if ξ_0 generates a maximal reversible torus then $\mathrm{Ad}_{\gamma_0} \mathrm{L}T^{\rho} = \mathrm{L}Z(\xi_0) \cap \mathbf{g}^-$. In the latter case it is immediate that a small ball around ξ_0 in $\mathbf{g}^-$ still belongs to S. If $\xi_0 \neq 0$ then, since we assume that Γ is semisimple, $\mathrm{L}Z(\xi_0) \neq \mathbf{g}$. As $(Z(\xi_0))^0$ is invariant under conjugation by ρ and has smaller dimension than Γ we can apply our induction hypothesis on the group $(Z(\xi_0))^0_\rho$ to conclude that $\xi_0 + \chi \in (\mathrm{L}Z(\xi_0))^-$ is conjugate to some element in the Lie algebra of the maximal reversible torus $\gamma_0 T^{\rho} \gamma_0^{-1}$ of $Z(\xi_0)$ by some $\gamma \in Z(\xi_0) \cap Z(\rho)$, and altogether that $\xi \in \mathrm{Ad}_{\exp(\eta)\gamma\gamma_0} \mathrm{L}T^{\rho} \subset S$. This proves that $S \setminus \{0\}$ is open in $\mathbf{g}^-$.

Finally the set $\mathbf{g}^- \setminus \{0\}$ is connected unless $\dim(\mathbf{g}^-) = 1$. In the first case we conclude that $\mathbf{g}^- \setminus \{0\} \subset S$ and hence that $\mathbf{g}^- = S$. In the latter case the theorem holds anyway.

(b) Let T^ρ be a maximal reversible torus of Γ. We saw in the proof of part (a) that $\Psi\colon Z(\rho)/(Z(T^\rho) \cap Z(\rho)) \times \mathrm{L}T^\rho \to \mathbf{g}^-$, $\Psi(g, \chi) = \mathrm{Ad}_g\, \chi$, is surjective and that $\dim(\mathbf{g}^-) = \dim(T^\rho) + \dim(Z(\rho)/(Z(T^\rho) \cap Z(\rho))$. Since this reduces integration in $\mathbf{g}^-$ to integration over $Z(\rho)/(Z(T^\rho) \cap Z(\rho))$ and integration over $\mathrm{L}T^\rho$ and clearly almost every element χ of $\mathrm{L}T^\rho$ generates T^ρ we conclude that almost every reversible ξ generates a maximal reversible torus. ∎

6.2. *Reversible Cartan Subgroups of Compact Groups*

In this section we prove similar statements as in Section 6.1 for reversible Cartan subgroups. Let $\Gamma^- = \{\gamma \in \Gamma \mid \rho\gamma\rho = \gamma^{-1}\}$ denote the reversible elements of Γ and let $\Gamma^-_{\sigma_0}$ be the connected component of Γ^- containing $\sigma_0 \in \Gamma^-$. We have the following theorem:

THEOREM 6.4. *Let Γ_ρ be a compact finite-dimensional Lie group, let Γ be a normal subgroup of Γ_ρ and let $\rho \in \Gamma_\rho$ be an involution such that $\Gamma_\rho = \Gamma \cup \rho\Gamma$ as before.*

(a) *Let C^ρ be a reversible Cartan subgroup generated by some reversible $\sigma_0 \in \Gamma$. Then each reversible σ which is in the same connected component $\Gamma^-_{\sigma_0}$ of Γ^- as σ_0 lies in a reversible Cartan subgroup which is conjugated to C^ρ by some element in $(Z(\rho))^0$.*

(b) *Almost every reversible $\sigma \in \Gamma$ generates a reversible Cartan subgroup.*

This theorem can be applied to study generic drifts of reversible relative periodic orbits with isotropy Δ by defining $\Gamma = N(\Delta)/\Delta$, cf. Section 4.2, provided that Γ is compact.

The following example shows that even if Γ is connected Γ^- need not be connected so that two reversible elements σ, $\hat{\sigma}$ lying in conjugated Cartan subgroups may not lie in conjugated reversible Cartan subgroups.

EXAMPLE 6.5. Consider Example 6.2 again, i.e., let $\Gamma = \mathrm{SO}(3)$ and let $\rho = \kappa_3$ be a reflection along the (x_1, x_2)-plane. Let ξ_i, $i = 1, 2, 3$, denote the generators of the rotations around the x_i-axis. Then $\mathbf{g}^- = \mathrm{span}(\xi_1, \xi_2)$ and $\mathbf{g}^+ = \mathrm{span}(\xi_3)$. The set Γ^- contains rotations about vectors $\xi \in \mathbf{g}^-$, and a rotation by π about the x_3-axis which we denote by σ:

$$\sigma = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}.$$

The element σ lies in $\Gamma^0 = \Gamma = \mathrm{SO}(3)$, but it is isolated within the set Γ^-: this follows from the fact that $\gamma\sigma \in \Gamma^-$ for $\gamma \in \Gamma$ means that $\rho(\gamma\sigma)\rho = (\gamma\sigma)^{-1}$ which is equivalent to $\gamma = \gamma^{-1}$ because $\sigma = -\rho$. The reversible element $\hat{\sigma} = \mathrm{id}$ lies in the reversible Cartan subgroup $\hat{C}^\rho = T^\rho \sim S^1$ and the Cartan subgroups C rsp. $\hat{C}$ containing σ rsp. $\hat{\sigma}$ are conjugated copies of S^1. But since $C \subseteq Z(\rho)$ the element σ lies in the reversible Cartan subgroup $C^\rho \sim \mathbb{Z}_2$ which is not conjugated to $\hat{C}^\rho \sim S^1$.

For $\sigma \in \Gamma$ let $C(\sigma)$ denote the group generated by σ. To prove Theorem 6.4 we employ the following lemma:

LEMMA 6.6. *Let G be a finite-dimensional Lie group and let $\sigma_0 \in G$ with $C(\sigma_0)$ compact. Choose a $C(\sigma_0)$-invariant inner product on $\mathbf{g}$. Then the map $F\colon \mathbf{z} \times \mathbf{z}^\perp \to G$ where $\mathbf{z} = \mathrm{L}Z(\sigma_0)$, defined as*

$$F(\chi, \eta) = \exp(\mathrm{Ad}_{\sigma_0}^{-1}\, \eta) \exp(\chi) \exp(-\eta)$$

is locally bijective from a neighborhood of 0 *in $\mathbf{z} \times \mathbf{z}^\perp$ to a neighborhood of* id *in G.*

Proof. We have

$$\mathrm{D}F(0) = (\mathrm{id}|_{\mathbf{z}}, (\mathrm{Ad}_{\sigma_0}^{-1} - \mathrm{id})|_{\mathbf{z}^\perp}).$$

Since $\mathrm{Im}(\mathrm{Ad}_{\sigma_0}^{-1} - \mathrm{id}) = \mathbf{z}^\perp$ the matrix $\mathrm{D}F(0)$ is bijective, and therefore, by the implicit function theorem, F is locally bijective near 0. ∎

Proof of Theorem 6.4. (a) Let C^ρ be a reversible Cartan subgroup generated by $\sigma_0 \in \Gamma$. We show that the set

$$S = \{\gamma\sigma_0 \exp(\chi)\, \gamma^{-1} \mid \gamma \in (Z(\rho))^0, \chi \in \mathrm{L}C^\rho\}$$

is open and closed in the set $\Gamma^-_{\sigma_0}$ of reversible $\gamma \in \Gamma$ which lie in the same connected component of Γ^- as the reversible element σ_0. Closedness follows from compactness of Γ.

To show that S is open let $s_0 \in S$, i.e., $s_0 \in \gamma_0 C^\rho \gamma_0^{-1}$, $\gamma_0 \in (Z(\rho))^0$. Let $\sigma \in \Gamma^-$, $\sigma \approx s_0$. Write $\sigma = s_0 \exp(\xi)$ with $\xi \approx 0$ and choose the scalar product on $\mathbf{g}$ to be Ad_ρ-invariant. By Lemma 6.6 we have $\exp(\xi) = \exp(\mathrm{Ad}_{s_0}^{-1}\, \eta) \exp(\chi) \exp(-\eta)$ where $\chi \in \mathbf{z} = \mathrm{L}Z(s_0)$, $\eta \in \mathbf{z}^\perp$ which implies that $\sigma = \exp(\eta)\, s_0 \exp(\chi) \exp(-\eta)$ where η, χ are locally unique. As $\sigma \in \Gamma^-$ we have $\sigma = \rho\sigma^{-1}\rho$ so that

$$\sigma = \exp(\mathrm{Ad}_\rho\, \eta)\, s_0 \exp(-\mathrm{Ad}_\rho\, \chi) \exp(-\mathrm{Ad}_\rho\, \eta).$$

Since $\mathbf{z}$ and $\mathbf{z}^\perp$ are invariant under conjugation by ρ we conclude that $\eta \in \mathbf{g}^+$, $\chi \in (\mathrm{L}Z(s_0))^-$. Since $T^\rho_{s_0} := \gamma_0 (C^\rho)^0\, \gamma_0^{-1}$ is a maximal reversible torus in $(Z(s_0))^0$, by Theorem 6.1 there is some element $\gamma \in (Z(s_0) \cap Z(\rho))^0$ and some

$\zeta \in \mathrm{L}T^{\rho}_{s_0}$ such that $\chi = \mathrm{Ad}_{\gamma}\,\zeta$. Hence σ is conjugated to the element $s_0 \exp(\zeta)$ of $\gamma_0 C^{\rho} \gamma_0^{-1}$ by the element $\exp(\eta)\,\gamma \in (Z(\rho))^0$ and consequently s_0 is an interior point of S and S is open in $\Gamma^{-}_{\sigma_0}$.

(b) Now let σ_0 be any reversible element in Γ. As in part (a) we see that every $\sigma \in \Gamma^{-}$ close to σ_0 is of the form $\sigma = \exp(\eta)\,\sigma_0 \exp(\chi) \exp(-\eta)$ with $\chi \in \mathbf{z}^{-}$ and $\eta \in \mathbf{z}^{\perp} \cap \mathbf{g}^{+}$ where $\mathbf{z} = \mathrm{L}Z(\sigma_0)$. Let $C^{\rho} = \mathbb{Z}_{m_\rho} \times T^{\rho}_{\sigma_0}$ be a reversible Cartan subgroup containing σ_0. Then $T^{\rho}_{\sigma_0}$ is a maximal reversible torus in $Z(\sigma_0)$. Write $\sigma_0 = \hat{\sigma}_0 \exp(\chi_0)$ where $\hat{\sigma}_0 \in C^{\rho}$ has order m_ρ and $\chi_0 \in \mathrm{L}C^{\rho}$. Since $\sigma_0 \exp(\chi)$ generates a reversible Cartan subgroup iff $\chi_0 + \chi$ generates a maximal reversible torus in $Z(\sigma_0)$ we can employ Theorem 6.1(b) to conclude that almost every element close to σ_0 in $Z(\sigma_0)^{-}$ generates a reversible Cartan subgroup in $Z(\sigma_0)$. Moreover we saw above that integration over a small neighborhood of σ_0 in Γ^{-} amounts to integration over a neighborhood of σ_0 in $Z(\sigma_0)^{-}$ and over a neighborhood of 0 in $\mathbf{z}^{\perp} \cap \mathbf{g}^{+}$. Therefore almost every $\sigma \in \Gamma^{-}$ close to σ_0 generates a reversible Cartan subgroup. ∎

6.3. *Extensions to Noncompact Groups*

In this subsection we deal with extensions of the above results to noncompact groups. First we treat reversible maximal tori of noncompact groups.

THEOREM 6.7. *If Γ is noncompact, but meets the other assumptions of Theorem* 6.1 *then*

(a) *almost every reversible $\xi_0 \in \mathbf{g}^{-}$ generates a maximal reversible torus or a line.*

(b) *If $\xi_0 \in \mathbf{g}^{-}$ generates a maximal reversible torus T^{ρ} then every element $\xi \approx \xi_0$ with $\xi \in \mathbf{g}^{-}$ and $T(\xi)$ compact is conjugate to some element in* $\mathrm{L}T^{\rho}$ *by some close to identity element in* $(Z(\rho))^0$.

(c) *If Γ is a semidirect product of a compact group and a finite-dimensional vector space then all maximal reversible tori are conjugate.*

Recall that the group action for a semidirect product $\Gamma = K \ltimes V$, $V = \mathbb{R}^N$, where K is a compact group acting on the vectorspace V linearly is given as

$$(R_1, a_1)(R_2, a_2) = (R_1 R_2, a_1 + R_1 a_2),\ R_i \in K,\ a_i \in \mathbb{R}^N,\ i = 1, 2.$$

The symmetry groups we have in mind in part (c) of Theorem 6.7 are the Euclidean symmetry groups E(2) and E(3) of the plane and of the three-space as these are the most common non-commutative noncompact symmetry groups showing up in applications.

For the proof of part (c) of Theorem 6.7 we need the following lemma:

LEMMA 6.8. *Let $\Gamma = K \ltimes V$ be a semidirect product of a compact group K with a finite-dimensional vectorspace V. Let ρ be an involution such that $\rho\Gamma\rho = \Gamma$ and that $\Gamma_\rho = \Gamma \cup \rho\Gamma$ is an index-two extension of Γ. Then*

(a) *We can choose K and V to be invariant under conjugation by ρ, and make this assumption from now on.*

(b) *$C((R, a))$ is compact iff $a \in \mathrm{im}(R-\mathrm{id})$. In this case (R, a) is conjugate to $(R, 0)$ by the element (id, b) with $b = (R-\mathrm{id})^{-1} a$.*

(c) *$\Gamma^- = \{(R, a) \mid \rho R\rho = R^{-1}, -a = R\rho a\}$*

(d) *If $(R, a) \in \Gamma^-$ and $C((R, a))$ is compact then (id, b) from (b) is in $Z(\rho)$.*

Proof. Since Γ_ρ has finitely many connected components for any compact subgroup L of $G := \Gamma_\rho$ we can choose a maximal compact subgroup H of G such that $L \subset H$, see Theorem 3.1 of [15, p. 180]. So there is a maximal compact subgroup H of G containing $L = \{\mathrm{id}, \rho\}$ and a maximal compact subgroup H_K of G containing K. Since maximal compact subgroups are unique up to conjugacy [6, 15], there is some $g \in G$ such that $H_K = gHg^{-1}$. As H is invariant under conjugation by ρ we can choose $g \in \Gamma$. Therefore $g^{-1}Kg \subset \Gamma \cap H$ is a maximal compact subgroup of Γ and we can assume wlog that $g = \mathrm{id}$. The group $H \cap \Gamma$ is a compact subgroup of Γ containing K and as K is maximal compact we have $K = H \cap \Gamma$. Since Γ and H are invariant under conjugation by ρ so is K. We write $\Gamma = K \ltimes V_K$ where V_K is a vectorspace; the unique radical of Γ (i.e. the unique maximal solvable ideal of Γ, see, e.g., [33]) is given by $\mathbf{r} = \mathrm{L}c(K^0) \oplus \mathrm{L}V_K$. Since $\mathbf{r}$ is unique it is K_ρ-invariant and so is the abelian Lie algebra generated by $[\mathbf{r}, \mathbf{r}] = [\mathrm{L}c(K^0), \mathbf{r}] \subset \mathrm{L}V_K$. A K_ρ-invariant complement to $[\mathbf{r}, \mathbf{r}]$ in $\mathbf{r}$ which contains $\mathrm{L}c(K^0)$ is the abelian Lie algebra $\mathbf{a} = \mathrm{L}z(c(K^0)) \cap \mathbf{r}$. Choose a K_ρ-invariant complement $\mathbf{w}$ to $\mathrm{L}c(K^0)$ in $\mathbf{a}$, and let $\mathrm{L}V := [\mathbf{r}, \mathbf{r}] + \mathbf{w}$. Then V invariant under conjugation by ρ as well. This proves (a). Parts (b)–(d) are elementary computations. ▮

Proof of Theorem 6.7. (a) It is immediate that the elements $\xi \in \mathbf{g}^-$ generating a maximal reversible torus or a line are dense in $\mathbf{g}^-$. We use induction over $n = \dim(\Gamma)$ to prove that the elements of $\mathbf{g}^-$ generating a maximal reversible torus or a line have full measure. For $n = 1$ the theorem holds true trivially. Assume that (a) holds true for all groups $\hat{\Gamma}$ of dimension $\dim(\hat{\Gamma}) < n$ which satisfy the assumptions of Theorem 6.7(a) and let $n = \dim(\Gamma)$.

Let $\xi_0 \in \mathbf{g}^-$ be such that $T(\xi_0)$ is a torus and choose an Ad_ρ-invariant and $T(\xi_0)$-invariant inner product on $\mathbf{g}$. Then, as in the proof of Theorem 6.1 every element $\xi \in \mathbf{g}^-$ close to ξ_0 can be written as uniquely

$$\xi = \mathrm{Ad}_{\exp(\eta)}(\xi_0 + \chi), \qquad \chi \in \mathbf{z}^-, \quad \eta \in \mathbf{z}^\perp \cap \mathbf{g}^+, \quad \chi, \eta \approx 0, \qquad (6.1)$$

where $\mathbf{z} = \mathrm{L}Z(\xi_0)$. So integration over a small ball $B^-(\xi_0)$ around ξ_0 in $\mathbf{g}^-$ corresponds to integration over a small ball $B^+(0)$ around 0 in $\mathbf{g}^+ \cap \mathbf{z}^\perp$ and integration over a small ball around ξ_0 in $\mathbf{z}^-$. Assume that ξ_0 does not lie in the Lie algebra of the center $c(\Gamma)$ of Γ; then $\dim(Z(\xi_0)) < \dim(\Gamma)$. Since $\hat{\Gamma} := Z(\xi_0)$ is invariant under conjugation by ρ we can apply the induction hypothesis to conclude that almost every $\xi \in \mathrm{L}Z(\xi_0)$ generates a maximal reversible torus or a line. As conjugation by elements in $Z(\rho)$ maps maximal reversible tori onto maximal reversible tori and reversible lines onto reversible lines we then conclude that almost every $\xi \in B^-(\xi_0)$ generates a maximal reversible torus or a reversible line. Since the set $\mathbf{g}^- \setminus \mathrm{L}c(\Gamma)$ has full measure or $\mathbf{g}^- \subset \mathrm{L}c(\Gamma)$ (in which case the theorem is obvious) this proves that almost every $\xi \in \mathbf{g}^-$ generates a maximal reversible torus or a line.

(b) Let $\xi_0 \in \mathbf{g}^-$ generate a maximal reversible torus $T(\xi_0)$ and let $\xi \in \mathbf{z}^-$, $\xi \approx \xi_0$, generate a reversible torus. Then in the above decomposition (6.1), $\chi \in \mathrm{L}T^\rho$ because $T^\rho \subseteq Z(\xi_0)$ is a maximal reversible torus. This proves the local conjugacy argument as well.

(c) Let $\Gamma = K \ltimes V$ be a semidirect product of a compact group and a vectorspace, and choose K and V to be invariant under conjugation by ρ as in Lemma 6.8. Let T^ρ, $\hat{T}^\rho$ be maximal reversible tori in Γ generated by ξ and $\hat{\xi}$. Then by Lemma 6.8(d) we can conjugate ξ and $\hat{\xi}$ to elements η and $\hat{\eta}$ in the Lie algebra $\mathrm{L}K$ of K by elements in $(Z(\rho))^0$. By Theorem 6.1 the maximal reversible tori $T(\hat{\eta}) \subseteq K$ and $T(\eta) \subseteq K$ are conjugate by an element in $(Z(\rho))^0$ as well which proves the conjugacy statement. ▮

Remark 6.9. The problem with extending the local conjugacy result of Theorem 6.7(b) to a global conjugacy result for general noncompact groups with the techniques of the proof of Theorem 6.1 is that it is not clear whether the set the element $\xi \in \mathbf{g}^- \setminus \{0\}$ with $T(\xi)$ compact form a connected set for $\dim(\mathbf{g}^-) > 1$.

Now we come to reversible Cartan subgroups of noncompact groups:

THEOREM 6.10. *Let Γ be a finite-dimensional Lie group, let ρ be an involution with $\rho\Gamma\rho = \Gamma$ and let $\Gamma_\rho = \Gamma \cup \rho\Gamma$.*

(a) *Almost every reversible σ generates a reversible Cartan subgroup or a copy of $\mathbb{Z}$;*

(b) *Let C^ρ be a reversible Cartan subgroup generated by some reversible $\sigma_0 \in \Gamma$. Then every $\sigma \in \Gamma^-_{\sigma_0}$ with $\sigma \approx \sigma_0$ and $C(\sigma)$ compact lies in a reversible Cartan subgroup which is conjugate to C^ρ by some close to identity element in $(Z(\rho))^0$.*

(c) *Let* C^ρ *be a reversible Cartan subgroup generated by some reversible* $\sigma_0 \in \Gamma$. *If* Γ *is a semidirect product of a compact group K with a finite-dimensional vectorspace V then each reversible* σ *with* $C(\sigma)$ *compact which is in the same connected component of* Γ^- *as* σ_0 *lies in a reversible Cartan subgroup which is conjugated to* C^ρ *by some element in* $(Z(\rho))^0$.

Proof. To show (a) observe that the proof of Theorem 6.4(b) basically works for noncompact groups as well if we replace the statement that almost every element in Γ^- generates a reversible Cartan subgroup by the corresponding statement for noncompact groups: let $\sigma_0 \in \Gamma^-$ with $C(\sigma_0)$ compact; then, as in the proof of Theorem 6.4, every $\sigma \in \Gamma^-_{\sigma_0}$ close to σ_0 can be written uniquely as

$$\sigma = \exp(\eta)\, \sigma_0 \exp(\chi) \exp(-\eta) \qquad \text{with} \quad \eta \in \mathrm{L}Z(\rho) \cap \mathbf{z}^\perp, \quad \chi \in \mathbf{z}^-, \quad \chi, \eta \approx 0 \tag{6.2}$$

where $\mathbf{z} = \mathrm{L}Z(\sigma_0)$. Depending on whether $T(\chi)$ is compact or not, $C(\sigma)$ is compact or not, and, as in the proof of Theorem 6.4(b), the statement follows from the fact that almost every $\chi \in (\mathrm{L}Z(\sigma_0))^-$ generates a maximal reversible torus or a line by Theorem 6.7(a).

To prove (b) let $\sigma_0 \in \Gamma^-$ generates a reversible Cartan subgroup. If $C(\sigma)$ is compact then $T(\chi)$ with χ from (6.2) is a torus which implies that $\chi \in \mathrm{L}C^\rho$ because $(C^\rho)^0$ is the unique maximal reversible torus in $Z(\sigma_0)$. This proves the conjugacy statement (b).

To prove (c) let $\Gamma = K \ltimes V$ be a semidirect product of a compact group K and a vectorspace V, both of which are chosen to be invariant under conjugation by ρ by virtue of Lemma 6.8(a), let σ_0 generate the reversible Cartan subgroup C^ρ and let $\sigma \in \Gamma^-_{\sigma_0}$ be such that $C(\sigma)$ is compact. Because of Lemma 6.8(d) we can conjugate $\sigma_0 = (R_0, a_0)$ and $\sigma = (R, a)$ to the elements $(R_0, 0)$ and $(R, 0)$ in K by $(\mathrm{id}, b_0) \in (Z(\rho))^0$ resp. $(\mathrm{id}, b) \in (Z(\rho))^0$, where $b_0 = (R_0 - \mathrm{id})^{-1}\, a$, $b = (R - \mathrm{id})^{-1}\, a$. From Lemma 6.8(c) we further see that $(R, a) \in \Gamma^-_{(R_0, a_0)}$ iff $(R, a) \in \Gamma^-$ and $R \in K^-_{R_0}$. The conjugacy result now follows from Theorem 6.4(a) applied onto K. ∎

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