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The Weak Series Reduction Property Implies Pseudomodularity

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1. INTRODUCTION

Pseudomodular matroids have been introduced by Björner and Lovász in [1]. Hochstättler and Kern [3] have proven that pseudomodular matroids have the matching property defined by Dress and Lovász in [2].

In [2], the authors have also defined the strong and weak series reduction properties. They proved that the weak series reduction property implies the matching property.

In this paper, we fill in the gap by showing that in fact the weak series reduction property implies pseudomodularity. In the process, we have also obtained a somewhat weaker characterisation of pseudomodularity than that in [3].

In [2], it is shown that full algebraic matroids, full graphic matroids and full transversal matroids have the weak series reduction property. So the result in this paper shows that these matroids are pseudomodular, a fact also mentioned in [1].

2. DEFINITIONS

2.1. *Basic definitions on matroids.* We recall some basic definitions and results. We also introduce some new notation convenient for our purposes. For a fundamental treatment of matroids, the reader is refered to [4].

We find it convenient to define matroids using the rank function. Other equivalent definitions of matroids are found in [4].

DEFINITION 2.1.1. A *matroid* is a pair (E, r) where E is a set (which may be finite or infinite) and r is a non-negative integer function over the subsets of E satisfying the following:

(i) r is finitely generated, i.e.

for every $A \subseteq E$, $r(A)$ is finite and there is a finite set $A_0 \subseteq A$ such that $r(A) = r(A_0)$;

(ii) $r(\phi) = 0$;

(iii) r is non-decreasing, i.e.

$$A \subseteq B \subseteq E \text{ implies } r(A) \leqslant r(B);$$

(iv) r is submodular, i.e.

$$r(A \cup B) + r(A \cap B) \leqslant r(A) + r(B) \text{ for every } A, B \subseteq E;$$

(v) for every finite subset $A \subseteq E$, $r(A) \leqslant |A|$.

Normally, a matroid is defined on a finite set. Here, we have extended the definition to infinite sets. This is to include the 'full' infinite matroids introduced by Dress and Lovász in [2]. The rank is finite and finitely generated to preserve the essential properties of finite matroids. Possibly, the only interesting property lost is that of the dual matroid.

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For $e \in E$, we shall write e both for the element e itself and the singleton $\{e\}$ unless confusion arises. So, for example, $A \cup e$ would mean $A \cup \{e\}$.

Let (E, r) be a matroid. A *circuit* is a finite subset $C \subseteq E$ such that $r(C) = r(C - e) = |C| - 1$ for every $e \in C$. An *independent set* is a finite subset $X \subseteq E$ such that $r(X) = |X|$. A *basis* of a set $A \subseteq E$ is an independent set X such that $X \subseteq A$ and $r(A) = r(X) = |X|$.

Very often, we would like to say things like $\{\beta, \beta\}$ is a circuit where $r(\beta) = 1$. For this purpose, we shall use the following device.

Definition 2.1.2. Let (E, r) be a matroid. Let $A, B \subseteq E$. We say that $A \uplus B$ is independent if $r(A \cup B) = |A| + |B|$. We say that $A \uplus B$ is a circuit if, for any $e \in A$, $r(A \cup B) = r((A - e) \cup B) = |A| + |B| - 1$ and, for any $e \in B$, $r(A \cup B) = r(A \cup (B - e)) = |A| + |B| - 1$.

Some of the proofs in this paper will involve duplicating or 'doubling' of elements of a matroid. We will use the symbol $\uplus$ whenever we want to take a union with 'doubling' of the elements in the intersection. This is more or less an abuse of concept. We leave it to the reader to work out the proper details.

Definition 2.1.3. Let $M = (E, r)$ be a matroid and let $A \subseteq E$. The matroid $M/A = (E, r_A)$ obtained by *contracting* the set A is defined on the same set E by $r_A(X) = r(X \cup A) - r(A)$.

This is slightly different from the usual definition of contraction. Here, the contracted elements are left as loops. The rank function r_A is identical to that of the usual definition when restricted to $E - A$. Our definition is more convenient for tackling matroid matching problems, and in particular for proving the results in this paper.

The *closure* of a set $A \subseteq E$ is the set $\bar{A} = \{x \in E : r(A \cup x) = r(A)\}$. We have $\bar{A}^{M/V} = \overline{V \cup A}$, where $\bar{A}^{M/V}$ denotes the closure of A in the contracted matroid M/V. A set S is said to be the *disjoint union* of A and B if $S = A \cup B$ and $A \cap B = \phi$. We denote this by $S = A + B$. We say that S is the *direct sum* of A and B if S is the disjoint union of A and B and $r(S) = r(A) + r(B)$. We denote this by $S = A \oplus B$. If S cannot be expressed as a direct sum of two non-empty sets, we say that S is *connected.*

2.2. *The series reduction property.* In [2], Dress and Lovász gave the definition of the strong and weak series reduction properties. To define these terms, one has first to define what is meant by 'in series'. Due to our approach, our definition of 'in series' is slightly different from theirs. However, there is no change to the series reduction properties.

Definition 2.2.1. Let $M = (E, r)$ be a matroid. Let $S, V \subseteq E$. We say that S is *in series* with V if S and V are both finite and S is a circuit in M/V.

Note that when we use the phrase 'S is in series with V', we mean specifically that V is finite. Otherwise, we would simply say 'S is a circuit in M/V'.

Definition 2.2.2. Let $M = (E, r)$ be a matroid. We say that M has the *strong series reduction property* if for every $S, V \subseteq E$ such that S is in series with V, the following property holds:

There exists $\beta \in E$ such that, for any $T \subseteq V$, $T \uplus S$ is independent iff $T \uplus \beta$ is independent (or, equivalently, $T \uplus S$ is a circuit iff $T \uplus \beta$ is a circuit).

We say that M has the *weak series reduction property* if the above property holds for all $S, V \subseteq E$ such that S is in series with V and, in addition, V is connected.

The equivalence mentioned in Definition 2.2.2 can be proved using Lemma 3.1.1, stated later in this paper.

2.3. *Pseudomodularity.* The definition of pseudomodularity we adopt is that of [3]. We then derive other equivalent definitions. For even more equivalent definitions, see [1].

DEFINITION 2.3.1. A matroid $M = (E, r)$ is said to be *pseudomodular* if the following holds:

If $A, B, C \subseteq E$ are such that $r_A(C) = r_B(C) = r_{A \cup B}(C)$, then $r(\overline{A \cup C} \cap \overline{B \cup C}) - r(\bar{A} \cap \bar{B}) = r(A \cup C) - r(A)$.

PROPOSITION 2.3.2. *Let $M = (E, r)$ be a matroid. The following properties are equivalent*:
(1) *M is pseudomodular.*
(2) *If $S \subseteq E$ is a circuit in each of the three contracted matroids M/A, M/B and $M/A \cup B$, then S is also a circuit of the contracted matroid $M/\bar{A} \cap \bar{B}$.*
(3) *If $W, X, Y, S \subseteq E$ are such that $W + X$ and $W + Y$ are independent sets, with $\overline{W + X} \cap \overline{W + Y} = \bar{W}$, and if S is a circuit in each of the three contracted matroids $M/(W + X)$, $M/(W + Y)$, $M/(W + X + Y)$, then S is also a circuit of the contracted matroid M/W.*
(4) *If Z is independent in $M/A \cup B$, then $\overline{A \cup Z} \cap \overline{B \cup Z} = \overline{(\bar{A} \cap \bar{B}) \cup Z}$.*

PROOF. (2)$\Rightarrow$(3) Put $A = W + X$, $B = W + Y$.
(3)$\Rightarrow$(2) Let W be a basis of $\bar{A} \cap \bar{B}$. Complete this to a basis $W + X$ of $\bar{A}$ and a basis $W + Y$ of $\bar{B}$.
(1)$\Rightarrow$(2) Let $e \in S$. By definition of S, we have

$$r_A(S) = r_A(S - e) = r_B(S) = r_B(S - e) = r_{A \cup B}(S) = r_{A \cup B}(S - e) = |S| - 1.$$

By pseudomodularity, we have

$$r(\overline{A \cup S} \cap \overline{B \cup S}) - r(\bar{A} \cap \bar{B}) = r(\overline{A \cup (S - e)} \cap \overline{B \cup (S - e)}) - r(\bar{A} \cap \bar{B}) = |S| - 1.$$

But $(\bar{A} \cap \bar{B}) \cup S \subseteq \overline{A \cup S} \cap \overline{B \cup S}$ and $(\bar{A} \cap \bar{B}) \cup (S - e) \subseteq \overline{A \cup (S - e)} \cap \overline{B \cup (S - e)}$. Therefore $r_{\bar{A} \cap \bar{B}}(S) \leqslant |S| - 1$. But $r_{\bar{A} \cap \bar{B}}(S) \geqslant r_A(S) = |S| - 1$. Therefore $r_{\bar{A} \cap \bar{B}}(S) = |S| - 1$. Similarly, $r_{\bar{A} \cap \bar{B}}(S - e) = |S| - 1$.
(2)$\Rightarrow$(4) Let $e \in \overline{A \cup Z} \cap \overline{B \cup Z} - Z$. Let S be the circuit of M/A such that $e \in S \subseteq Z + e$. But $S - e$ is independent in $M/A \cup B$, so $r_{A \cup B}(S) = |S| - 1$. By Lemma 3.1.2 (stated later in the paper) S is also the circuit in $M/A \cup B$ with $e \in S \subseteq Z + e$. By the symmetrical argument, S is also the circuit in M/B with $e \in S \subseteq Z + e$. Propery (2) tells us that S is a circuit in $M/\bar{A} \cap \bar{B}$. Thus $e \in \bar{Z}^{M/\bar{A} \cap \bar{B}} = \overline{(\bar{A} \cap \bar{B}) \cup Z}$.
(4)$\Rightarrow$(1) Let Z be a basis of C in $M/A \cup B$. Z is also a basis of C in M/A and M/B. Thus

$$\overline{A \cup C} \cap \overline{B \cup C} = \bar{C}^{M/A} \cap \bar{C}^{M/B} = \bar{Z}^{M/A} \cap \bar{Z}^{M/B} = \overline{A \cup Z} \cap \overline{B \cup Z} = \overline{(\bar{A} \cap \bar{B}) \cup Z}.$$

Since Z is independent in $M/_{\bar{A}\cap\bar{B}}$, we have $r((\bar{A}\cap\bar{B})\cup Z)=r(\bar{A}\cap\bar{B})+|Z|=r(\bar{A}\cap\bar{B})+r(\bar{A}\cup\bar{Z})-r(A)=r(\bar{A}\cap\bar{B})+r(A\cup C)-r(A)$ and 1 follows. □

3. Weak Characterisation of Pseudomodularity

In fact, a slightly weaker form of property 3 of Proposition 2.3.2 characterises pseudomodularity.

Definition 3.0.1. A matroid $M=(E,r)$ is said to satisfy condition WCP (Weak Characterisation of Pseudomodularity) if the following holds.

Let W, X and Y be finite pairwise disjoint subsets of E such that $W+X$ and $W+Y$ are independent sets with $\overline{W+X}\cap\overline{W+Y}=\bar{W}$.

Put $V=W+X+Y$. Suppose that V is *connected.*

Let S be a circuit in each of the three contracted matroids $M/(W+X)$, $M/(W+Y)$ and $M/(W+X+Y)$.

Then S is also a circuit of the contracted matroid M/W.

Theorem 3.0.2. *A matroid $M=(E,r)$ is pseudomodular iff it satisfies condition* WCP.

The rest of this section is devoted to proving this theorem.

3.1. *Preliminary propositions*

Lemma 3.1.1. *Let $M=(E,r)$ be a matroid. Let $S,V\subseteq E$ such that $S\cap V=\varnothing$. Suppose that S is a circuit of M/V. Then every circuit C of M such that $C\subseteq S\cup V$ either is disjoint from S or contains S entirely.*

There exists at least one circuit C_0 of M such that $S\subseteq C_0\subseteq S\cup V$. If, in addition, V is an independent set, then C_0 is unique.

Proof. Let $C\subseteq S\cup V$ be a circuit of M. Suppose that $C\cap S\neq\varnothing$. We have $|C\cap V|<|C|$ and $r(C\cap V)=|C\cap V|$ since C is a circuit of M. We thus have $r_V(C\cap S)=r(C\cup V)-r(V)\leqslant r(C)-r(C\cap V)=|C|-1-|C\cap V|=|C\cap S|-1$. But S is a circuit of M/V and so $C\supseteq S$.

Now suppose that $S-V\neq\phi$. Let $x\in S-V$. $r(V\cup S)=r_V(S)+r(V)=r_V(S-x)+r(V)=r(V\cup S-x)$. Therefore there exists a circuit C of M with $x\in C\subseteq V\cup S$. Since $C\cap S\neq\phi$, the first part of the proof tells us that $C\supseteq S$. If there are two distinct circuits C and C' such that $S\subseteq C\subseteq S+V$ and $S\subseteq C'\subseteq S+V$, then there exists a circuit $C_0\subseteq C\cup C'-s$ for some $s\in S$. But from what is proved above, $C_0\subseteq V$, so V cannot be independent. □

Lemma 3.1.2. *Let $M=(E,r)$ be a matroid and let $V\subseteq E$. Let $S\subseteq E-V$ and let C be a circuit of M such that $S\subseteq C\subseteq S+V$. Suppose that $r_V(S)\geqslant|S|-1$. Then S is a circuit of M/V.*

Proof. Let $e\in S$. *We have* $e\in\overline{C-e}\subseteq\overline{V\cup(S-e)}$ and so $r(V\cup S)=r(V\cup(S-e))$. Therefore, $r_V(S-e)=r(V\cup(S-e))-r(V)=r(V\cup S)-r(V)\geqslant|S|-1$. On the other hand, $r_V(S-e)\leqslant|S|-1$. □

PROPOSITION 3.1.3. *Let (E, r) be a matroid and let W, X and Y be pairwise disjoint subsets of E. Suppose that $W + X$ and $W + Y$ are independent sets and that*

$$\overline{W + X} \cap \overline{W + Y} = \bar{W}.$$

Then, for any $W' \subseteq W$, $X' \subseteq X$ and $Y' \subseteq Y$, we have

$$\overline{W' + X'} \cap \overline{W' + Y'} = \bar{W}'.$$

PROOF. Let $A = \overline{W' + X'} \cap \overline{W' + Y'}$. We have $A \subseteq \overline{W + X} \cap \overline{W + Y} = \bar{W}$. Hence

$$\begin{aligned} r(A) &= r(\overline{W' + X'} \cap \overline{W' + Y'} \cap \bar{W}) \leqslant r(\overline{W' + X} \cap \bar{W}) \\ &\leqslant r(\overline{W' + X}) + r(\bar{W}) - r(\overline{W + X}) \\ &= r(W') + r(X) + r(W) - r(W) - r(X) = r(W'). \end{aligned}$$

On the other hand, obviously $\bar{W}' \subseteq A$. □

PROPOSITION 3.1.4. *Let $M = (E, r)$ be a matroid. Let $S, V \subseteq E$ such that $S \neq \phi$, $S \cap V = \phi$ and S is a circuit of M/V. Suppose that $V = V_1 \oplus V_2$. Let C_1 and C_2 be two circuits such that $S \subseteq C_1 \subseteq S + V$ and $S \subseteq C_2 \subseteq S + V$. Then $C = S + (C_1 \cap V_1) + (C_2 \cap V_2)$ is a circuit.*

We first prove the following weaker form.

LEMMA 3.1.5. *Under the hypothesis of the above proposition, there exists a circuit C of M such that $S \subseteq C \subseteq S + (C_1 \cap V_1) + (C_2 \cap V_2)$.*

PROOF. Let C be a circuit such that $S \subseteq C \subseteq S + V$, $C \cap V_1 \subseteq C_1 \cap V_1$ and such that $|(C \cap V_2) - C_2|$ is a minimum with these properties. We want to show that $(C \cap V_2) - C_2 = \phi$. Suppose that we have the contrary and let $a \in (C \cap V_2) - C_2$. Now $a \in C$, $a \notin C_2$ and $S \subseteq C \cap C_2$. So there exists a circuit C_3 such that $a \in C_3 \subseteq C \cup C_2 - s$ for some $s \in S$. But from Lemma 3.1.1, $C_3 \cap S = \phi$, so $a \in C_3 \subseteq (C \cup C_2) - S \subseteq V$. Note that $C_3 \subseteq V_2$ because $a \in V_2$, and $V = V_1 \oplus V_2$. Now we have $a \in C \cap C_3$, $S \subseteq C - C_3$. Thus there exists a circuit C' such that $s \in C' \subseteq C \cup C_3 - a$, where $s \in S$ but, again from Lemma 3.1.1, $C' \supseteq S$. We now have $C' \cap V_1 \subseteq C \cap V_1 \subseteq C_1 \cap V_1$ and $(C' \cap V_2) - C_2 \subseteq (C \cap V_2) - a - C_2$ contradicting the assumption that $|(C \cap V_2) - C_2|$ is minimal. □

PROOF OF PROPOSITION 3.1.4. From Lemma 3.1.5, let C be a circuit such that $S \subseteq C \subseteq S + (C_1 \cap V_1) + (C_2 \cap V_2)$. Using two more times the Lemma 3.1.5 there exists a circuit C_1' such that $S \subseteq C_1' \subseteq S + (C \cap V_1) + (C_1 \cap V_2) \subseteq C_1$ and a circuit C_2' such that $S \subseteq C_2' \subseteq S + (C_2 \cap V_1) + (C \cap V_2) \subseteq C_2$. We thus have $C_1' = C_1$ and $C_2' = C_2$ by definition of circuits and thus $C \cup V_1 = C_1 \cap V_1$ and $C \cap V_2 = C_2 \cap V_2$. □

PROPOSITION 3.1.6. *Let $M = (E, r)$ be a matroid. Let $S, V \subseteq E$ such that $S \neq \phi$, $S \cap V = \phi$ and S is a circuit of M/V. Suppose that $V = V_1 \oplus V_2$. Let C be a circuit such that $S \subseteq C \subseteq S + V$. Then $S' = S + (C \cap V_1) = C \cap (S + V_1) = C - V_2$ is a circuit of M/V_2.*

PROOF. We have $|S| - 1 = r_{C \cap V}(S) = r_{(C \cap V_1) \cup (C \cap V_2)}(S) \geqslant r_{(C \cap V_1) \cup V_2}(S) \geqslant r_V(S) = |S| - 1$, and so $r_{(C \cap V_1) \cup V_2}(S) = |S| - 1$. Also $r((C \cap V_1) \cup V_2) = r(C \cap V_1) + r(V_2)$ because

$V = V_1 \oplus V_2$. Thus $r_{V_2}(S') = r(S' \cup V_2) - r(V_2) = r(S \cup (C \cap V_1) \cup V_2) - r(V_2) = |S| - | +$ $r((C \cap V_1) \cup V_2) - r(V_2) \simeq |S| - 1 + |C \cap V_1| = |S'| - 1$ and therefore, from proposition 3.1.2, S' is a circuit in M/V_2. □

3.2. *Proof of the weak characterisation theorem*

PROOF OF THEOREM 3.0.2. We shall prove property (3) of Proposition 2.3.2. Let $W, X, Y \subseteq E$ be such that $W + X$ and $W + Y$ are independent and $\overline{W + X} \cap \overline{W + Y} = \bar{W}$. Let $S \subseteq E$ such that S is a circuit in each of the three contracted matroids $M/W + X$, $M/W + Y$, $M/W + X + Y$. We want to prove that S is a circuit in M/W.

If S is a singleton, the result is obvious, so assume that S is not a singleton. This means that $S \cap (W + X + Y) = \phi$.

By Proposition 3.1.1, there exists a unique circuit C_1 of M such that $S \subseteq C_1 \subseteq S + W + X$ and a unique circuit C_2 of M such that $S \subseteq C_2 \subseteq S + W + Y$.

Put $V = W + X + Y$. Let V' be a connected component of V. So, $V = V' \oplus V''$, where $V'' = V - V'$. Write $W' = W \cap V'$, $X' = X \cap V'$ and $Y' = Y \cap V'$.

We will now prove that $C_1 \cap V' = C_2 \cap V'$.

By Proposition 3.1.4, $C_3 = S + (C_2 \cap V') + (C_1 \cap V'')$ is a circuit of M.

Put $S' = S + (C_1 \cap V'') = C_1 \cap (S + V'') = C_3 \cap (S + V'')$. Now, by Proposition 3.1.6, S' is a circuit in $M/W' + W'$, $M/W' + Y'$ and $M/W' + X' + Y'$. Thus, from the hypothesis and Proposition 3.1.3, S' is a circuit in M/W'. So, by Proposition 3.1.1, there exists a circuit C_4 such that $S' \subseteq C_4 \subseteq S' + W'$. Again by 3.1.1, $C_4 = C_1 = C_3$ because $W + X$ and $W + Y$ are independent. This shows that $C_1 \cap V' = C_2 \cap V'$.

Since this is true for every connected component V' of V, we have in fact $C_1 = C_2 \subseteq S + W$ and thus S is a circuit of M/W. □

4. THE MAIN THEOREM

MAIN THEOREM. *If a matroid has the weak series reduction property, then it is pseudomodular.*

PROOF. Let W, X, Y, V and S be as in the 'hypothesis part' of condition WCP.

By the weak series reduction property, there exists $\beta \in E$ such that for any $T \subseteq V$, $T + S$ is a circuit if and only if $T \uplus \beta$ is a circuit.

S is a circuit of $M/(W + X)$, so by Lemma 3.1.1, there exists $T_1 \subseteq W + X$ such that $S \uplus T_1$ is a circuit. Thus, $\beta \uplus T_1$ is a circuit and so $\beta \in \overline{W + X}$. Similarly, $\beta \in \overline{W + Y}$. Therefore $\beta \in \bar{W}$. Thus, there exists $T_0 \subseteq W$ such that $\beta \uplus T_0$ is a circuit. This means that $S \uplus T_0$ is a circuit. By Lemma 3.1.2, S is a circuit in M/W. □

5. A COUNTEREXAMPLE TO THE CONVERSE OF THE MAIN THEOREM

The converse to the main theorem is not true. Here is a counterexample.

We shall consider a subset E of the affine plane $\mathbf{R}^2$. Remember that the dimension of a point is 1 and that of a line is 2.

Put $a_1 = (1, 0)$, $a_2(2, 0)$, $a_3 = (3, 0)$ and $b_1 = (1, 1)$, and $b_2 = (2, 2)$.

Let $E = \{a_1, a_2, a_3, b_1, b_2\}$. Let r be the dimension function.

One verifies easily that the circuits are $\{a_1, a_2, a_3\}$ and $\{a_i, a_j, b_1, b_2\}$, with $i, j \in \{1, 2, 3\}$, $i \neq j$.

The connected sets are the circuits and E itself.

(E, r) is pseudomodular. This is easy to verify using the weak characterisation.

Consider $V = \{a_1, a_2, a_3\}$, $S = \{b_1, b_2\}$. β with the property in Definition 2.3.1 does not exist.

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