ABSTRACT

This paper is a continuation of an earlier one by the author. We obtain a new lower bound for the Perron root $r(A)$ of a nonnegative $n \times n$ matrix $A$. We show that for $A$ with at least one principal submatrix of order two, with different diagonal entries, and with at least one positive off-diagonal entry, the bound is at least as good as the one from [7], and we determine a class of nonnegative matrices for which it is essentially better. We also show that not every nonnegative square matrix is similar to a nonnegative square matrix with identical diagonal entries.

RESULTS

Since in this paper we will use a result from [7], we begin by recalling it.

THEOREM 1 [7]. Let $A = (a_{ij})$ be an $n \times n$ nonnegative real matrix, $n \geq 2$, and define

$$a = \min_i \{a_{ii}\}, \quad K^2 = \frac{1}{n} \text{Tr}((A - aI)^2),$$

where $\text{Tr}(A)$ denotes the trace of $A$, and $I$ is the identity matrix. Then for the Perron root $r(A)$ of $A$, the following inequality holds:

$$r(A) \geq a + \begin{cases} \frac{K}{\sqrt{n-1} \sum_{1 \leq i < j \leq n} a_{ij}a_{ji}} & \text{for } n \text{ even,} \\ \max \left\{ \frac{1}{K}, \sqrt{\frac{2}{n-1} \sum_{1 \leq i < j \leq n} a_{ij}a_{ji}} \right\} & \text{for } n \text{ odd} \end{cases}$$

(1)
It is noticed in [7] that the bound (1) is invariant under diagonal similarity transformations of \( A \). So, in order to improve it, we should look for other similarity transformations of \( A \).

**Theorem 2.** Let \( A = (a_{ij}) \) be an \( n \times n \) nonnegative real matrix with \( n \geq 2 \). Assume that there exists a pair \((k, s)\) of indices such that

\[
(a_{ss} - a_{kk}) a_{ks} > 0,
\]

and define

\[
M = \min \left\{ \min_{i \in (j: a_{kj} > 0)} \left\{ \frac{a_{si}}{a_{ki}}, \frac{a_{ss} - a_{kk}}{2a_{ks}} \right\} \right\},
\]

\[
\tilde{a} = \min_{i \in (1, \ldots, n) \setminus \{k, s\}} \{ a_{ii}, a_{kk} + Ma_{ks} \},
\]

\[
\tilde{K}^2 = \frac{1}{n} \text{Tr}\left((A - \tilde{a}I)^2\right).
\]

Then for the Perron root \( r(A) \) of \( A \), the following inequality holds:

\[
r(A) \geq \bar{a} + \begin{cases} 
\tilde{K} & \text{for } n \text{ even} \\
\max \left\{ \tilde{K}, \sqrt{\frac{2}{n-1} \left( \sum_{1 \leq i < j \leq n} a_{ij} a_{ji} + Ma_{ks}(a_{ss} - a_{kk}) - (Ma_{ks})^2 \right)} \right\} & \text{for } n \text{ odd}.
\end{cases}
\]

Moreover, for a matrix with the property (2), the bound (3) is always at least as good as (1).

**Proof.** The proof will be completed in two steps:

**Step 1.** We prove the inequality (3). Starting with \( A \), we define the nonsingular matrix \( B = (b_{ij}) \) by

\[
b_{ij} = \begin{cases} 
1 & \text{for } i = j, \\
M & \text{for } (i, j) = (s, k), \\
0 & \text{otherwise}.
\end{cases}
\]

Then \( B \) transforms \( A \) into a similar matrix \( A' = (a'_{ij}) = B^{-1}AB \). Notice that
A' differs from A only in the sth row and kth column, and

\[
\begin{cases}
  a'_{jk} = a_{jk} + Ma_{js} & \text{for } j \in \{1, \ldots, n\} \setminus \{s\}, \\
  a'_{sj} = a_{sj} - Ma_{kj} & \text{for } j \in \{1, \ldots, n\} \setminus \{k\}, \\
  a'_{sk} = a_{sk} - Ma_{kk} + M(a_{ss} - Ma_{ks}).
\end{cases}
\]

(4)

By (2) and the definition of M, the above formulas ensure the nonnegativity of A'. Therefore, we can adopt the bound (1) to the matrix A', which gives

\[
r(A) = r(A') \geq \bar{a} + \begin{cases}
  \tilde{K} & \text{for } n \text{ even} \\
  \max \left \{ \tilde{K}, \sqrt{\frac{2}{n-1} \sum_{1 \leq i < j \leq n} a'_{ij}a'_{ji}} \right \} & \text{for } n \text{ odd},
\end{cases}
\]

(5)

where \(\bar{a} = \min_i a'_{ii}\) and \(\tilde{K}^2 = (1/n)\text{Tr}((A' - \bar{a}I)^2)\). By (2), (4), and the definition of M, it follows that \(\bar{a} = \tilde{a}\), and (5) takes the form

\[
r(A) = r(A') \geq \bar{a} + \begin{cases}
  \tilde{K} & \text{for } n \text{ even}, \\
  \max \left \{ \tilde{K}, \sqrt{\frac{2}{n-1} \sum_{1 \leq i < j \leq n} a'_{ij}a'_{ji}} \right \} & \text{for } n \text{ odd},
\end{cases}
\]

(6)

where \(\tilde{K}^2 = (1/n)\text{Tr}((A' - \bar{a}I)^2)\). In view of the similarity of A and A', we get

\[
\sum_{1 \leq i < j \leq n} (a'_{ii}a'_{jj} - a'_{ij}a'_{ji}) = \sum_{1 \leq i < j \leq n} (a_{ii}a_{jj} - a_{ij}a_{ji})
\]

(7)

and

\[
\text{Tr}^2(A) = \text{Tr}^2(A').
\]

(8)

Taking (7) and (8) into account, we obtain

\[
\sum_{1 \leq i < j \leq n} a'_{ij}a'_{ji} = \sum_{1 \leq i < j \leq n} a_{ij}a_{ji} + \frac{1}{2} \left( \sum_{i=1}^{n} a_{ii}^2 - \sum_{i=1}^{n} (a'_{ii})^2 \right)
\]
which, by the form of diagonal entries of $A'$, leads us to

$$\sum_{1 \leq i < j \leq n} a_{ij}a'_{ji} = \sum_{1 \leq i < j \leq n} a_{ij}a_{ji} + Ma_{ks}(a_{ss} - a_{kk}) - (Ma_{ks})^2. \quad (9)$$

Using again the similarity of $A$ and $A'$, we have

$$\text{Tr}((A' - \bar{a}I)^2) = \text{Tr}((A - \bar{a}I)^2),$$

which implies

$$\hat{K} = \overline{K}. \quad (10)$$

Combining (6) with (9) and (10), we obtain (3), which finishes step 1.

**Step 2.** We show that for a matrix with the property (2) the bound (3) is always at least as good as (1). Notice that, according to the assumptions,

$$\bar{a} \geq a \quad (11)$$

and

$$Ma_{ks}(a_{ss} - a_{kk}) - (Ma_{ks})^2 \geq 0.$$

Hence to prove the assertion it suffices to show that

$$\bar{a} + \overline{K} = \bar{a} + \sqrt{\frac{1}{n} \text{Tr}((A - \bar{a}I)^2)} \geq a + K = a + \sqrt{\frac{1}{n} \text{Tr}((A - aI)^2)}. \quad (12)$$

To verify the last inequality we introduce the real function

$$\varphi(t) = t + \sqrt{\frac{1}{n} \text{Tr}((A - tI)^2)}.$$

By differentiating $\varphi$ it is immediately seen that $\varphi$ is monotone increasing, which, together with (11), implies (12).

**Corollary 1.** The bound (3) is essentially better than (1) for a nonnegative matrix with the property (2) for which, in the notation of
Theorem 2,

\[ a_{kk} < a_{ii} \text{ and } \text{sign}(a_{st}) \geq \text{sign}(a_{ki}) \]

for \( i \in \{1, \ldots, n\} \setminus \{k\} \) and \( i \in \{1, \ldots, n\} \setminus \{s\} \), respectively.

Example 1. Let

\[
A = \begin{bmatrix}
2 & 10 & 1 \\
7 & 1 & 11 \\
2 & 11 & 1
\end{bmatrix}.
\]

In this case the bound (3), for \( k = 3 \) and \( s = 1 \), yields

\[ r(A) > 1 + \sqrt{193.25}. \]

This result is better than that obtained by methods from the papers [1–7].

Remark 1. It should be noticed that the bound given by Theorem 2 is not always better than the one we could get from the methods that are known. For example, considering the matrix

\[
A = \begin{bmatrix}
1 & 1 & 2 \\
2 & 1 & 3 \\
2 & 3 & 5
\end{bmatrix},
\]

our Theorem 2, for \( k = 1 \) and \( s = 3 \), gives

\[ r(A) > 1 + \sqrt{19}. \]  \hspace{1cm} (13)

This result is better than that obtained by the classical inequality of Frobenius and by methods from the papers [4–7], and is worse than that obtained by methods from the papers [1–3].

Remark 2. According to Theorem 2 we observe that for some matrices, starting with \( A_1 = A \), we can iteratively form a sequence \( \{ A_p = (a_{ij}^{(p)}) \} \), \( p = 1, 2, \ldots \), of similar matrices with terms given by

\[ A_{p+1} = B_p^{-1}A_pB_p, \]
where \( B_p \) differs from the identity matrix in at most one off-diagonal entry with coordinates and value defined by a pair \((k, s)\) of indices for which \( A_p \) possesses the property (2) [if, for some \( p \), \( A_p \) does not possess the property (2), we set \( B_p = I \)]. Observing that

\[
\sum_{i=1}^{n} \left( a_{ii}^{(p)} - \min_j a_{jj}^{(p)} \right)^2 \geq \sum_{i=1}^{n} \left( a_{ii}^{(p+1)} - \min_j a_{jj}^{(p+1)} \right)^2,
\]

we stop the process when the following inequality holds:

\[
\sqrt{\sum_{i=1}^{n} \left( a_{ii}^{(p)} - \min_j a_{jj}^{(p)} \right)^2} - \sum_{i=1}^{n} \left( a_{ii}^{(p+1)} - \min_j a_{jj}^{(p+1)} \right)^2 < \varepsilon,
\]

where \( \varepsilon \) is a given sufficiently small positive number. As a lower bound for \( r(A) \) we accept the bound given for \( A_{p+1} \) by (1).

**Example 2.** According to Remark 2, for the matrix

\[
A = \begin{bmatrix}
1 & 1 & 2 \\
2 & 1 & 3 \\
2 & 3 & 5
\end{bmatrix},
\]

we can form the following sequence of similar matrices:

\[
A_1 = \begin{bmatrix}
1 & 1 & 2 \\
2 & 1 & 3 \\
2 & 3 & 5
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
3 & 1 & 2 \\
5 & 1 & 3 \\
4 & 2 & 3
\end{bmatrix}, \quad A_3 = \begin{bmatrix}
3 & 5 & 3 & 2 \\
5 & 2 & 3
\end{bmatrix},
\]

\[
A_4 = \begin{bmatrix}
2.5 & 103/60 & 1.7 \\
5 & 2.5 & 3 \\
7/3 & 77/30 & 2
\end{bmatrix}, \quad A_5 = \begin{bmatrix}
2.5 & 103/60 & 5751/3090 \\
105/22 & 2.25 & \sqrt{1952} \\
7/3 & 77/30 & 2.25
\end{bmatrix},
\]

\[
A_6 = \begin{bmatrix}
2.375 & 5772/3560 & 154197/86240 \\
105/22 & 2.375 & 1861/616 \\
7/3 & 473/180 & 2.25
\end{bmatrix}.
\]
Choosing $\epsilon = 0.2$ and applying our stop criterion, we obtain

$$r(A) = r(A_6) \geq 2.25 + \sqrt{20.328125}.$$  

This result is essentially better than (13).

Completing our example, we observe that $A_6$ possesses the property (2). This observation, together with the form of $a^{(6)}_i$, enables us to infer that, up to an ordering,

$$\text{diag}(A_7) = (2.375, 2.3125, 2.3125),$$

where diag($A_7$) denotes the diagonal of $A_7$. Applying our stop criterion to the diagonal entries of $A_6$ and $A_7$, we obtain

$$r(A) = r(A_7) \geq 2.3125 + \sqrt{20.332031} > 6.8216,$$

which improves the bound given above. Notice that, by the similarity of $A$ and $A_7$, it is unnecessary to determine the off-diagonal entries of $A_7$ explicitly.

CONCLUDING REMARKS

From our considerations it follows that the best possible result derived from the bound (3) can be achieved when a matrix $A$ is similar to a nonnegative square matrix with identical diagonal entries. One can give examples of matrices which, after transformations from Remark 2, form sequences convergent to the matrices with required properties (see Example 2). In other cases, to achieve the same goal one would have to use transformations different from those which we have proposed (perhaps more complicated from the computational point of view). Therefore it is natural to ask whether for every nonnegative square matrix there exists a nonnegative matrix, similar to it, with identical diagonal entries. The negative answer to this question provides the following corollary of Theorem 1.

**Corollary 2.** Not every nonnegative square matrix is similar to a nonnegative matrix with identical diagonal entries.
Proof. Let

\[
A = \begin{bmatrix}
5 & 6 & 1 \\
7 & 5 & 0 \\
0 & 1 & 11
\end{bmatrix}.
\]

Suppose that there exists a nonsingular matrix \(B\) which transforms \(A\) into a similar matrix \(A' = BAB^{-1} = (a'_{ij})\) such that \(a'_{11} = a'_{22} = a'_{33} = 7\). Since

\[
\sum_{1 \leq i < j \leq 3} (a_{ij}a_{jj} - a_{ij}a_{ji}) = \sum_{1 \leq i < j \leq 3} (a'_{ij}a'_{jj} - a'_{ij}a'_{ji}),
\]

therefore

\[
\sum_{1 \leq i < j \leq 3} a'_{ij}a'_{ji} = 54.
\]

Then the bound (1) yields

\[
r(A) = r(A') \geq 7 + \sqrt{54}.
\]  

(14)

On the other hand it is easy to verify that \(r(A) = 12\), which contradicts (14).

We shall conclude the paper with the following open problem.

**Problem.** Determine as wide as possible a class of nonnegative matrices the elements of which are similar to nonnegative matrices with identical diagonal entries.

The author would like to express his gratitude to Professor Ludwig Elsner for his comments and remarks on the earlier version of this paper.

REFERENCES


*Received 18 December 1987; final manuscript accepted 19 April 1988*