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Minimal periods of periodic solutions of some Lipschitzian differential equations

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ABSTRACT

A problem of finding lower bounds for periods of periodic solutions of a Lipschitzian differential equation, expressed in the supremum Lipschitz constant, is considered. Such known results are obtained for systems with inner product norms. However, utilizing the supremum norm requires development of a new technique, which is presented in this paper. Consequently, sharp bounds for equations of even order, both without delay and with arbitrary time-varying delay, are found. For both classes of system, the obtained bounds are attained in linear differential equations.

1. Introduction

This paper is devoted to the following problem: for a given class of autonomous Lipschitzian systems, find a lower bound for the periods of the set of non-constant periodic solutions $x(t) = x(t+T)$, expressed in the Lipschitz constant L. The first such result (Yorke [1]),

$$T \geq 2\pi/L, \tag{1.1}$$

has been established for the equation

$$\dot{x} = f(x), \quad x \in \mathrm{R}^n \tag{1.2}$$

where the function $f(x)$ satisfies the Lipschitz condition

$$\left\| f(x') - f(x'') \right\| \leq L \left\| x' - x'' \right\| \tag{1.3}$$

with the Euclidean norm. Bound (1.1) is attained in the system $\dot{x}_1 = Lx_2, \dot{x}_2 = -Lx_1, \dot{x}_i = 0$ for $i > 2$.

Inequality (1.1) was extended (Lasota, Yorke [2]) to Eq. (1.2), which is defined on any Hilbert space and could contain a constant time delay. It remains sharp for Eq. (1.2) in the space with the supremum norm (Zevin [3]). For the case of a general Banach space, Busenberg, Fisher and Martelli [4] proved that $T \geq 6/L$ and presented an infinite-dimensional system which reaches this bound.

For periodic solutions of the equation

$$x^{(2k)} = f(x), \tag{1.4}$$

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($x^{(2k)}$ defines a $2k$ derivative of x), Mawhin and Walter [5] established the inequality

$$T \geq 2\pi / L^{1/2k}. \tag{1.5}$$

This bound is also precise, it is attained in the equation $(-1)^k x^{(2k)} = Lx$.

In the first part of this paper, we consider Eq. (1.4), where the function $f(x)$ satisfies the Lipschitz condition

$$\left| f(x') - f(x'') \right| \leq L \left| x' - x'' \right| \tag{1.6}$$

with the supremum norm ($|x| = \max |x_i|$). Theorem 1 shows that, in this case, the constant 2π in inequality (1.5) remains precise. Note that in paper [4] the following question is posed: "if the period scale of a normed space **E** is 2π, is **E** necessarily an inner product space?" Theorem 1 gives a negative answer to this question for Eq. (1.4).

In the second part of the paper, we consider the delay equation

$$x^{(2k)}(t) = f(x(\tau(t))), \tag{1.7}$$

where $\tau(t)$ is an arbitrary piece-wise continuous function and $f(x)$ satisfies condition (1.6). Theorem 2 provides a sharp lower bound for the periods of solutions of system (1.6) and (1.7). Note that such known bounds relate to the first order delay equation

$$\dot{x}(t) = f(x(\tau(t))) \tag{1.8}$$

for which rough lower bounds were first obtained by Slomczynsky [6] and Medved [7]. The precise bound, $T = 4/L$ (valid for the both Euclidean and supremum Lipschitz constants L), was found by Zevin [8]. This value is attained in the equation $\dot{x} = Lx(\tau(t))$ with the piece-wise constant function $\tau(t) = kT$ for $t \in [kT/2, (k+1)T/2)$, $k = 0, 1, \ldots$.

2. Bounds for the periods of solutions of Eq. (1.4)

First we briefly illustrate that the technique, used in [5] to derive bound (1.5) for system (1.4) and (1.3), is not applicable in the case of the supremum norm. Really, condition (1.3) implies the inequality

$$L^2 \left(\frac{T}{2\pi} \right)^{2k} \int_0^{2\pi} \|v(t)\|^2 \, dt \geq \int_0^{2\pi} \left\| v^{(k)}(t) \right\|^2 dt \tag{2.1}$$

where $v(t)$ is a function with a zero mean value. Then, required bound (1.5) follows from the known Wirtinger's inequality [9],

$$\int_0^{2\pi} \|v(t)\|^2 \, dt \leq \int_0^{2\pi} \left\| v^{(k)}(t) \right\|^2 dt. \tag{2.2}$$

However, in the case of the supremum norm, one has in (2.1), instead of $\|v(t)\|$ and $\left\| v^{(k)}(t) \right\|$,

$$|v(t)| = \max_i |v_i(t)|, \qquad \left| v^{(k)}(t) \right| = \max_i \left| v_i^{(k)}(t) \right|. \tag{2.3}$$

Since $\left| v^{(k)}(t) \right| \neq d^k |v(t)| / dt^k$ for $n > 1$, then inequality (2.2) is not applicable. Thus, in the case of the supremum norm, a new approach is required.

First let us establish some preliminary results. Consider the eigenvalue problem

$$(-1)^k \lambda u^{(2k)} = u, \tag{2.4}$$
$$u^{(2i)}(0) = u^{(2i)}(T/2) = 0, \quad i = 0, 1, \ldots, k-1,$$

where $u \in \mathrm{R}$.

Lemma. *The largest eigenvalue of problem* (2.4),

$$\lambda_0 = (T/2\pi)^{1/2k}. \tag{2.5}$$

Proof. Problem (2.4) is equivalent to the integral equation

$$\lambda u = Au, \tag{2.6}$$
$$Au(t) = \int_0^{T/2} g(t, s, k) u(s) ds,$$

where $g(t, s, k)$ is the Green function. This function satisfies the equation $(-1)^k u^{(2k)} = 0$ for $t \neq s$, boundary conditions (2.4) and the equality

$$(-1)^k \left[g^{(2k-1)}(s+0, s, k) - g^{(2k-1)}(s-0, s, k) \right] = 1.$$

Let us show that, for any k,

$$g(t, s, k) > 0 \quad \text{for } t, s \in (0, T/2). \tag{2.7}$$

Setting $u_i = u^{(2(k-i))}$, we reduce the equation $(-1)^k u^{(2k)} = 0$ to the form

$$\begin{aligned} &-\ddot{u}_1 = 0, \\ &-\ddot{u}_{i+1} = u_i, \quad i = 1, \ldots, k-1. \end{aligned} \tag{2.8}$$

One can readily check that

$$g(t, s, 1) = (1 - 2s/T)t > 0 \quad \text{for } t < s, \qquad g(t, s, 1) = (1 - 2s/T)t - t + s > 0 \quad \text{for } t > s. \tag{2.9}$$

From (2.8) it is clear that

$$g(t, s, k) = \int_0^{T/2} g(t, w, 1) g(w, s, k-1) \mathrm{d}w, \tag{2.10}$$

so, setting $k = 2, 3, \ldots$, we obtain inequality (2.7).

Problem (2.7) and, therefore, (2.8) admit the solution

$$u_0(t) = \sin \lambda_0 t, \quad \lambda_0 = (T/2\pi)^{1/2k}. \tag{2.11}$$

By (2.7), the operator A is positive ($Au(t) > 0$ on $(0, T/2)$ for any nonzero $u(t) \geq 0$). Since $u_0(t) > 0$ for $t \in (0, T/2)$, then λ_0 is the largest eigenvalue of problem (2.8) and, therefore, (2.4) [10, Theorem 2.13]. □

The following theorem provides a lower bound for the periods of solutions of Eq. (1.4).

Theorem 1. *The period of a solution $x(t) = x(t + T) \neq const$ of system (1.4), (1.6) satisfies inequality (1.5).*

Proof. Put

$$y(t) = x(t) - x(-t). \tag{2.12}$$

Clearly, $y(t) = -y(-t) = y(t + T)$, so

$$y^{(2i)}(0) = y^{(2i)}(T/2) = 0, \quad i = 0, 1, \ldots, k-1. \tag{2.13}$$

In view of (1.4) and (2.7), $y(t)$ satisfies the equation

$$y^{(2k)}(t) = p(t), \tag{2.14}$$

where

$$p(t) = f(x(t)) - f(x(-t)). \tag{2.15}$$

The solution of boundary problem (2.13), (2.14) can be represented in the form

$$y(t) = \int_0^{T/2} G(t, s) p(s) \mathrm{d}s \tag{2.16}$$

where the Green function $G(t, s, k) = (-1)^k \mathrm{diag}\,[g(t, s, k)]$.

From (2.15), (1.6) and (2.16)we have

$$|p(t)| \leq L\,|y(t)| \leq L \int_0^{T/2} |G(t, s, k)|\,|p(s)|\,\mathrm{d}s \leq LA\,|p(t)| \tag{2.17}$$

and, therefore,

$$L^{-1}\,|p(t)| = A\,|p(t)| + \delta(t) \tag{2.18}$$

for some $\delta(t) \leq 0$.

As is shown above that the operator A is positive. So, the solution $x(t)$ of the equation $\lambda x = Ax + \delta(t)$ with $\lambda > \lambda_0$ and $\delta(t) \geq 0$ is positive [10, Theorem 2.16]. Therefore, $x(t) < 0$ when $\lambda > \lambda_0$ and $\delta(t) \leq 0$. Meanwhile, Eq. (2.18) with $\delta(t) \leq 0$ has the positive solution $|p(t)|$. The obtained contradiction shows that $L^{-1} \leq \lambda_0 = (T/2\pi)^{1/2k}$ which implies inequality (1.5). □

3. Bounds for the periods of solutions of Eq. (1.7)

Consider, now, delay differential equation (1.7) where $f(x)$ satisfies condition (1.6) and $\tau(t)$ is an arbitrary piece-wise continuous function. Note that, here, the known technique, based on inequality (2.2), is not applicable, even in the case of an inner product norm, because here the function $v^{(k)}$ depends on t while the function v on $\tau(t)$.

Put

$$\alpha(k) = \frac{1}{y(0.25, k)^{1/2k}} \tag{3.1}$$

where $y(t, k)$ is the solution of the boundary problem

$$\begin{aligned} &(-1)^k y^{(2k)}(t) = 1, \\ &y^{(2i)}(0) = y^{(2i)}(0.5) = 0, \quad i = 0, 1, \dots, k-1. \end{aligned} \tag{3.2}$$

Theorem 2. *The period of any periodic solution $x(t) = x(t+T) \neq const$ of system (1.7), (1.6) satisfies the inequality*

$$T \geq \frac{\alpha(k)}{L^{1/2k}}. \tag{3.3}$$

Proof. Putting $p_* = |p(t_*)| = \max|p(t)|$, from (2.17) we have

$$p_* \leq L p_* z(t_*) \leq L p_* \max|z(t, k)|, \tag{3.4}$$

where

$$z(t, k) = \int_0^{T/2} g(t, s, k) \mathrm{d}s. \tag{3.5}$$

From (3.5) it follows that $z(t, k)$ is the solution of the problem

$$\begin{aligned} &(-1)^k z^{(2k)}(t) = 1, \\ &z^{(2i)}(0) = z^{(2i)}(T/2) = 0, \quad i = 0, 1, \dots, k-1. \end{aligned} \tag{3.6}$$

From (3.2) and (3.6) we have

$$z(t, k) = (T)^{2k} y(t/T, k). \tag{3.7}$$

Since

$$\ddot{z}(t, k) = -\int_0^{T/2} g(t, s, k-1) \mathrm{d}s < 0,$$

then the function $z(t, k)$ is concave. Obviously, $z(t, k) = z((T/2) - t, k)$, so

$$\max|z(t, k)| = z(T/4, k) = T^{2k} y(1/4, k) = [T/\alpha(k)]^{2k}. \tag{3.8}$$

Putting (3.8) in (3.4), we obtain the required inequality (3.3). □

4. Discussion

First, let us show that the obtained bounds are precise. Really, bound (1.5) is reached for the equation

$$x^{(2k)} = (-1)^k L x \tag{4.1}$$

which admits the periodic solution $x(t) = \sin(2\pi t/T)$ with the period $T = 2\pi/L^{1/2k}$.

In turn, bound (3.3) is reached for the equation

$$x^{(2k)}(t) = (-1)^k L x(\tau(t)), \tag{4.2}$$

where

$$\tau(t) = T/4 \quad \text{for } t \in [0, T/2), \qquad \tau(t + kT/2) = \tau(t) + kT/2, \quad k = 1, 2, \dots \tag{4.3}$$

Really, putting

$$x(t) = -x(-t) = x(t+T), \qquad x(t) = y(t/T) \quad \text{for } t \in [0, T/2], \tag{4.4}$$

we find

$$(-1)^k x^{(2k)}(t) = \frac{(-1)^k y^{(2k)}(t/T)}{T^{2k}}, \qquad Lx(\tau(t)) = Ly(1/4) = L/\alpha^{2k}(k). \tag{4.5}$$

Therefore, for $T = \alpha(k)/L^{1/2k}$, Eq. (4.2) has T-periodic solution (4.4).

As is seen from the proof of Theorem 1, bound (1.5) holds true for the delay equation

$$x^{(2k)}(t) = f(x(t-\tau)). \tag{4.6}$$

This bound is reached for the equation

$$x^{(2k)}(t) = (-1)^k Lx(t-\tau_q),$$
$$\tau_q = 2\pi q/L^{1/2k}, \quad q = 1, 2, \dots$$

which admits the T-periodic solution $x(t) = \sin(2\pi t/T)$ with $T = 2\pi/L^{1/2k}$.

Bound (3.3) can be extended to the equation

$$x^{(2k)}(t) = \sum_{i=1}^{m} f_i(x(\tau_i(t))) \tag{4.7}$$

where $\tau_i(t)$ are arbitrary piece-wise continuous functions, while the functions $f_i(x)$ satisfy the Lipschitz condition

$$\left| f_i(x') - f_i(x'') \right| \le L_i \left| x' - x'' \right|. \tag{4.8}$$

Really, analogously to the proof of Theorem 2, we find that the periods of non-constant periodic solutions of system (4.7), (4.8) satisfy inequality (3.3) with $L = \sum_{i=1}^{m} L_i$.

Note that for a specific system (1.4), the Euclidean and supremum Lipschitz constants have, in general, different values. As a result, the corresponding bounds for the period, provided by inequality (1.5), are different (clearly, the better bound is offered by the smaller constant).

The value $\alpha(k)$ in (3.1) can be easily determined. Taking into account conditions (3.2) for $t = 0$, we write the solution of problem (3.2) in the form

$$y(t,k) = \sum_{i=1,3,\dots}^{2k-1} y^i(0)\frac{t^i}{i!} + \frac{t^{2k}}{(2k)!}. \tag{4.9}$$

The values $y^{(i)}(0)$, $i = 2k-1, 2k-3, \dots, 1$ are successively determined by conditions (3.2) for $t = 0.5$: $y^{(2i)}(0.5) = 0$, $i = k-1, k-2, \dots, 0$. Then the required value $y(0.25, k)$ is found from (4.9). In particular, the corresponding calculations give $\alpha(1) = 5.667$, $\alpha(2) = 5.921$, $\alpha(3) = 6.033$, $\alpha(4) = 5.146$.

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