Minimal periods of periodic solutions of some Lipschitzian differential equations

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1. Introduction

This paper is devoted to the following problem: for a given class of autonomous Lipschitzian systems, find a lower bound for the periods of the set of non-constant periodic solutions $x(t) = x(t + T)$, expressed in the Lipschitz constant $L$. The first such result (Yorke \cite{1}),

$$T \geq \frac{2\pi}{L}, \quad (1.1)$$

has been established for the equation

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \quad (1.2)$$

where the function $f(x)$ satisfies the Lipschitz condition

$$\|f(x') - f(x'')\| \leq L \|x' - x''\| \quad (1.3)$$

with the Euclidean norm. Bound (1.1) is attained in the system $\dot{x}_1 = Lx_2, \dot{x}_2 = -Lx_1, \dot{x}_i = 0$ for $i > 2$.

Inequality (1.1) was extended (Lasota, Yorke \cite{2}) to Eq. (1.2), which is defined on any Hilbert space and could contain a constant time delay. It remains sharp for Eq. (1.2) in the space with the supremum norm (Zevin \cite{3}). For the case of a general Banach space, Busenberg, Fisher and Martelli \cite{4} proved that $T \geq 6/L$ and presented an infinite-dimensional system which reaches this bound.

For periodic solutions of the equation

$$x^{2k} = f(x), \quad (1.4)$$

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(\chi^{(2k)} defines a 2k derivative of \chi), Mawhin and Walter [5] established the inequality
\[ T \geq 2\pi / L^{1/2k}. \] (1.5)
This bound is also precise, it is attained in the equation \((-1)^k \chi^{(2k)} = Lx.

In the first part of this paper, we consider Eq. (1.4), where the function \( f(x) \) satisfies the Lipschitz condition
\[ |f(x') - f(x'')| \leq L |x' - x''| \] (1.6)
with the supremum norm \(|x| = \max |x_j|\). Theorem 1 shows that, in this case, the constant \( 2\pi \) in inequality (1.5) remains precise. Note that in paper [4] the following question is posed: “if the period scale of a normed space \( E \) is \( 2\pi \), is \( E \) necessarily an inner product space?” Theorem 1 gives a negative answer to this question for Eq. (1.4).

In the second part of the paper, we consider the delay equation
\[ \chi^{(2k)}(t) = f(x(t)), \] (1.7)
where \( t \) is an arbitrary piece-wise continuous function and \( f(x) \) satisfies condition (1.6). Theorem 2 provides a sharp lower bound for the periods of solutions of system (1.6) and (1.7). Note that such known bounds relate to the first order delay equation
\[ \dot{x}(t) = f(x(t)), \] (1.8)
for which rough lower bounds were first obtained by Slomczynski [6] and Medved [7]. The precise bound, \( T = 4/L \) (valid for the both Euclidean and supremum Lipschitz constants \( L \)), was found by Zevin [8]. This value is attained in the equation \( \dot{x} = Lx(t) \) with the piece-wise constant function \( t = kT \) for \( t \in [kT/2, (k + 1)T/2], k = 0, 1, \ldots \).

2. Bounds for the periods of solutions of Eq. (1.4)

First we briefly illustrate that the technique, used in [5] to derive bound (1.5) for system (1.4) and (1.3), is not applicable in the case of the supremum norm. Really, condition (1.3) implies the inequality
\[ L^2 \left( \frac{T}{2\pi} \right)^{2k} \int_0^{2\pi} \| v(t) \|^2 dt \geq \int_0^{2\pi} \| v^{(k)}(t) \|^2 dt \] (2.1)
where \( v(t) \) is a function with a zero mean value. Then, required bound (1.5) follows from the known Wirtinger’s inequality [9],
\[ \int_0^{2\pi} \| v(t) \|^2 dt \leq \int_0^{2\pi} \| v^{(k)}(t) \|^2 dt. \] (2.2)
However, in the case of the supremum norm, one has in (2.1), instead of \( \| v(t) \| \) and \( \| v^{(k)}(t) \| \),
\[ |v(t)| = \max_i |v_i(t)|, \quad |v^{(k)}(t)| = \max_i |v^{(k)}_i(t)|. \] (2.3)
Since \( |v^{(k)}(t)| \neq d^k |v(t)| / dt^k \) for \( n > 1 \), then inequality (2.2) is not applicable. Thus, in the case of the supremum norm, a new approach is required.

First let us establish some preliminary results. Consider the eigenvalue problem
\[ (-1)^k \lambda \chi^{(2k)} = u, \] (2.4)
\[ u^{(2i)}(0) = u^{(2i)}(T/2) = 0, \quad i = 0, 1, \ldots, k - 1, \]
where \( u \in \mathbb{R} \).

Lemma. The largest eigenvalue of problem (2.4),
\[ \lambda_0 = (T/2\pi)^{1/2k}. \] (2.5)

Proof. Problem (2.4) is equivalent to the integral equation
\[ \lambda u = Au, \] (2.6)
\[ Au(t) = \int_0^{T/2} g(t, s, k)u(s)ds, \]
where \( g(t, s, k) \) is the Green function. This function satisfies the equation \((-1)^k u^{(2k)} = 0 \) for \( t \neq s \), boundary conditions (2.4) and the equality
\[ (-1)^k \left[ g^{(2k-1)}(s + 0, s, k) - g^{(2k-1)}(s - 0, s, k) \right] = 1. \]
Let us show that, for any $k$, 
\[ g(t, s, k) > 0 \quad \text{for } t, s \in (0, T/2). \] (2.7)

Setting $u_i = u^{(2(k-i))}$, we reduce the equation $(-1)^k u^{(2k)} = 0$ to the form 
\[
-\ddot{u}_1 = 0, \\
-\ddot{u}_{i+1} = u_i, \quad i = 1, \ldots, k - 1.
\] (2.8)

One can readily check that 
\[
g(t, s, 1) = (1 - 2s/T)t > 0 \quad \text{for } t < s, \quad g(t, s, 1) = (1 - 2s/T)t - t + s > 0 \quad \text{for } t > s.
\] (2.9)

From (2.8) it is clear that 
\[
g(t, s, k) = \int_0^{T/2} g(t, w, 1)g(w, s, k - 1)dw,
\] (2.10)

so, setting $k = 2, 3, \ldots$, we obtain inequality (2.7).

Problem (2.7) and, therefore, (2.8) admit the solution 
\[
u_0(t) = \sin \lambda_0 t, \quad \lambda_0 = (T/2\pi)^{1/2k}.
\] (2.11)

By (2.7), the operator $A$ is positive ($Au > 0$ on $(0, T/2)$ for any nonzero $u(t) \geq 0$). Since $u_0(t) > 0$ for $t \in (0, T/2)$, then $\lambda_0$ is the largest eigenvalue of problem (2.8) and, therefore, (2.4) [10, Theorem 2.13].

The following theorem provides a lower bound for the periods of solutions of Eq. (1.4).

**Theorem 1.** The period of a solution $x(t) = x(t + T) \neq \text{const of system (1.4), (1.6) satisfies inequality (1.5).}

**Proof.** Put 
\[ y(t) = x(t) - x(-t). \] (2.12)

Clearly, $y(t) = -y(-t) = y(t + T)$, so 
\[
y^{(2i)}(0) = y^{(2i)}(T/2) = 0, \quad i = 0, 1, \ldots, k - 1.
\] (2.13)

In view of (1.4) and (2.7), $y(t)$ satisfies the equation 
\[
y^{(2k)}(t) = p(t),
\] (2.14)

where 
\[ p(t) = f(x(t)) - f(x(-t)). \] (2.15)

The solution of boundary problem (2.13), (2.14) can be represented in the form 
\[
y(t) = \int_0^{T/2} G(t, s)p(s)ds
\] (2.16)

where the Green function $G(t, s, k) = (-1)^k \text{diag} [g(t, s, k)]$.

From (2.15), (1.6) and (2.16) we have 
\[
|p(t)| \leq L|y(t)| \leq L \int_0^{T/2} |G(t, s, k)| |p(s)| ds \leq LA|p(t)|
\] (2.17)

and, therefore, 
\[
L^{-1}|p(t)| = A|p(t)| + \delta(t)
\] (2.18)

for some $\delta(t) \leq 0$.

As is shown above that the operator $A$ is positive. So, the solution $x(t)$ of the equation $\lambda x = Ax + \delta(t)$ with $\lambda > \lambda_0$ and $\delta(t) \geq 0$ is positive [10, Theorem 2.16]. Therefore, $x(t) < 0$ when $\lambda > \lambda_0$ and $\delta(t) \leq 0$. Meanwhile, Eq. (2.18) with $\delta(t) \leq 0$ has the positive solution $|p(t)|$. The obtained contradiction shows that $L^{-1} \leq \lambda_0 = (T/2\pi)^{1/2k}$ which implies inequality (1.5).
3. Bounds for the periods of solutions of Eq. (1.7)

Consider, now, delay differential equation (1.7) where \( f(x) \) satisfies condition (1.6) and \( \tau(t) \) is an arbitrary piece-wise continuous function. Note that, here, the known technique, based on inequality (2.2), is not applicable, even in the case of an inner product norm, because here the function \( v_k \) depends on \( t \) while the function \( v \) on \( \tau(t) \).

Put
\[
\alpha(k) = \frac{1}{y(0.25, k)^{1/2k}}
\]
where \( y(t, k) \) is the solution of the boundary problem
\[
\begin{align*}
(-1)^k y^{(2k)}(t) &= 1, \\
y^{(2)}(0) &= y^{(2)}(0.5) = 0, \quad i = 0, 1, \ldots, k - 1.
\end{align*}
\]

**Theorem 2.** The period of any periodic solution \( x(t) = x(t + T) \neq \text{const of system (1.7), (1.6)} \) satisfies the inequality
\[
T \geq \frac{\alpha(k)}{L^{1/2k}}.
\]

**Proof.** Putting \( p_* = |p(t_*)| = \max |p(t)| \), from (2.17) we have
\[
p_* \leq Lp_* z(t_*) \leq Lp_* \max |z(t, k)|,
\]
where
\[
z(t, k) = \int_0^{T/2} g(t, s, k) ds.
\]
From (3.5) it follows that \( z(t, k) \) is the solution of the problem
\[
\begin{align*}
(-1)^k z^{(2k)}(t) &= 1, \\
z^{(2)}(0) &= z^{(2)}(T/2) = 0, \quad i = 0, 1, \ldots, k - 1.
\end{align*}
\]
From (3.2) and (3.6) we have
\[
z(t, k) = (T)^{2k} y(t/T, k).
\]
Since
\[
\tilde{z}(t, k) = -\int_0^{T/2} g(t, s, k - 1) ds < 0,
\]
then the function \( z(t, k) \) is concave. Obviously, \( z(t, k) = z((T/2) - t, k) \), so
\[
\max |z(t, k)| = z(T/4, k) = T^{2k} y(1/4, k) = [T/\alpha(k)]^{2k}.
\]
Putting (3.8) in (3.4), we obtain the required inequality (3.3). \( \square \)

4. Discussion

First, let us show that the obtained bounds are precise. Really, bound (1.5) is reached for the equation
\[
x^{(2k)} = (-1)^k L^t x
\]
which admits the periodic solution \( x(t) = \sin(2\pi t/T) \) with the period \( T = 2\pi / L^{1/2k} \).

In turn, bound (3.3) is reached for the equation
\[
x^{(2k)}(t) = (-1)^k L x(\tau(t)),
\]
where
\[
\tau(t) = T/4 \quad \text{for } t \in [0, T/2), \quad \tau(t + kT/2) = \tau(t) + kT/2, \quad k = 1, 2, \ldots
\]\(\text{and } \tau(t) = T/4 - t \text{ for } t \in [T/2, T), \quad \tau(t + kT/2) = \tau(t) + kT/2, \quad k = 1, 2, \ldots \)

Really, putting
\[
x(t) = -x(-t) = x(t + T), \quad x(t) = y(t/T) \quad \text{for } t \in [0, T/2],
\]

we find
\[( -1)^k x^{(2k)}(t) = \frac{(-1)^k y^{(2k)}(t/T)}{T^{2k}}, \quad Lx(\tau(t)) = Ly(1/4) = L/\alpha^{2k}(k). \] (4.5)

Therefore, for \( T = \alpha(k)/L^{1/2k} \), Eq. (4.2) has \( T \)-periodic solution (4.4).

As is seen from the proof of Theorem 1, bound (1.5) holds true for the delay equation
\[ x^{(2k)}(t) = f(x(t - \tau)). \] (4.6)

This bound is reached for the equation
\[ x^{(2k)}(t) = (-1)^k Lx(t - \tau), \]
\[ \tau = 2\pi q/L^{1/2k}, \quad q = 1, 2, \ldots \]

which admits the \( T \)-periodic solution \( x(t) = \sin(2\pi t/T) \) with \( T = 2\pi/L^{1/2k} \).

Bound (3.3) can be extended to the equation
\[ x^{(2k)}(t) = \sum_{i=1}^{m} f_i(x(\tau_i(t))) \] (4.7)
where \( \tau_i(t) \) are arbitrary piece-wise continuous functions, while the functions \( f_i(x) \) satisfy the Lipschitz condition
\[ |f_i(x') - f_i(x'')| \leq L_i |x' - x''| \] (4.8)

Really, analogously to the proof of Theorem 2, we find that the periods of non-constant periodic solutions of system (4.7), (4.8) satisfy inequality (3.3) with \( L = \sum_{i=1}^{m} L_i \).

Note that for a specific system (1.4), the Euclidean and supremum Lipschitz constants have, in general, different values. As a result, the corresponding bounds for the period, provided by inequality (1.5), are different (clearly, the better bound is offered by the smaller constant).

The value \( \alpha(k) \) in (3.1) can be easily determined. Taking into account conditions (3.2) for \( t = 0 \), we write the solution of problem (3.2) in the form
\[ y(t, k) = \sum_{i=1}^{2k-1} y^{(i)}(0) t^i \frac{i!}{i!} + \frac{t^{2k}}{(2k)!} \] (4.9)

The values \( y^{(i)}(0), i = 2k - 1, 2k - 3, \ldots, 1 \) are successively determined by conditions (3.2) for \( t = 0.5 \); \( y^{(20)}(0.5) = 0 \), \( i = k - 1, k - 2, \ldots, 0 \). Then the required value \( y(0.25, k) \) is found from (4.9). In particular, the corresponding calculations give \( \alpha(1) = 5.667, \alpha(2) = 5.921, \alpha(3) = 6.033, \alpha(4) = 5.146 \).

References