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The Fuzzy Tychonoff Theorem

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Much of topology can be done in a setting where open sets have "fuzzy boundaries." To render this precise, the paper first describes cl_∞-monoids, which are used to measure the degree of membership of points in sets. Then L- or "fuzzy" sets are defined, and suitable collections of these are called L-topological spaces. A number of examples and results for such spaces are given. Perhaps most interesting is a version of the Tychonoff theorem which gives necessary and sufficient conditions on L for all collections with given cardinality of compact L-spaces to have compact product.

There seems to be use in certain infinite valued logics [4], in constructive analysis [1], and in mathematical psychology [8] for notions of proximity less restrictive than those found in ordinary topology. This paper develops some basic theory for spaces in which open sets are fuzzy. First, we need sufficiently general sets of truth values with which to measure degree of membership. The main thing is to get enough algebraic structure.

DEFINITION. A *cl_∞-monoid* is a complete lattice L with an additional associative binary operation $*$ such that the lattice zero 0 is a zero for $*$, the lattice infinity 1 is an identity for $*$, and the complete distributive laws

$$a * \left(\bigvee_{i \in I} b_i\right) = \bigvee_{i \in I} (a * b_i), \quad \text{and}$$

(D)

$$\left(\bigvee_{i \in I} b_i\right) * a = \bigvee_{i \in I} (b_i * a)$$

hold for all $a, b_i \in L$, and all index sets I.

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Recall that $0 \in L$ is a *zero* for $*$ iff $0 * a = a * 0 = 0$ for all $a \in L$. Note that $*$ is not assumed commutative. The term "closg" was used in [3, 4, 5], but has been replaced at the suggestion of Saunders Mac Lane so as to conform with standard terminology for ordered monoids. Actually, the phrase "integral *cl*-monoid" would be even more standard, but it seems too long. Since any completely distributive lattice is a cl_∞-monoid with $* = \wedge$, the closed unit interval $\mathbb{I} = [0, 1]$, the finite chains $[k] = \{0, 1,..., k - 1\}$ for $k \in \mathbb{P}$ (the set of positive integers), and all completely distributive Boolean algebras, are cl_∞-monoids. In particular the classical truth set $2 = \{0, 1\}$ is a cl_∞-monoid with the usual ordering. $\mathbb{I}$ is also a cl_∞-monoid with $*$ ordinary multiplication, and this is perhaps the most typical example. The set of all two sided ideals of a ring with unit is a cl_∞-monoid with $*$ "ideal multiplication," i.e., $A * B$ is the ideal generated by $\{a \cdot b \mid a \in A, b \in B\}$. The set $\mathbb{N}$ of nonnegative integers ordered by "is divisible by" with $*$ ordinary multiplication is a cl_∞-monoid ($a \vee b$ turns out to be greatest common divisor). It is not hard to check that a product of cl_∞-monoids is a cl_∞-monoid, so that $\mathbb{I}^n$ and $[k]^n$ are cl_∞-monoids for any $n \in \mathbb{P}$. 2 is a sub-cl_∞-monoid of any cl_∞-monoid with $0 \neq 1$. The set of all open subsets of any topological space forms a completely distribute lattice (recall that any upper complete lattice is lower complete; in this example $\bigwedge_{i \in I} A_i$ turns out to be $(\bigcap_{i \in I} A_i)^o$, that is, the infinimum is the interior of the intersection). In particular, the open subsets of $\mathbb{I}$ with the usual topology form a cl_∞-monoid (with $* = \wedge$) in which the distributive laws (D) hold, but their duals (replace $\bigvee$ by $\bigwedge$) fail. The following is needed for our main theorem.

LEMMA. *For a, b elements of a cl_∞-monoid L, $a * b \leqslant a \wedge b$.*

Proof. $a \leqslant 1$, so $1 * b = (a \vee 1) * b = a * b \vee 1 * b$, that is $b = (a * b) \vee b$, and, therefore, $a * b \leqslant b$. Similarly $a * b \leqslant a$. □

Actually, we can consider complete monoidal categories with a suitable distributive law as truth sets, and essentially all the material of this paper goes through (see the Appendix).

DEFINITION. An *L-set* A on a set X is a function $A: X \to L$. X is called the *carrier* of A, L is called the *truth* set of A, and for $x \in X$, $A(x)$ is called the *degree of membership* of x in A.

Although the definition makes sense for any sort of L, we shall always assume L is a cl_∞-monoid so as to get a nice algebra of L-sets. Let Y^X denote the set of all functions from X to Y, and let a_X denote the constant function on X with value a, for $a \in Y$. Then the class of all L-sets on X is a cl_∞-monoid under the pointwise lattice and semigroup operations, with zero 0_X and identity 1_X. For, the class of L-sets on X is L^X, which is (isomorphic to) the X-fold product of L with itself, $\prod_{x\in X} L_x$, where $L_x = L$; and any product of cl_∞-monoids is another. To be explicit about the operations, let A_i be L-sets on X for $i \in I$; then $\bigvee_{i\in I} A_i$ is the L-set on X defined by

$$\left(\bigvee_{i\in I} A_i\right)(x) = \bigvee_{i\in I} A_i(x) \qquad \text{for } x \in X.$$

If $\mathscr{C} \subseteq L^X$, it will be convenient for us to write $\bigvee\mathscr{C}$ for $\bigvee_{C\in\mathscr{C}} C$. Similarly $A * B$ is defined pointwise by $(A * B)(x) = A(x) * B(x)$, and $A \leqslant B$ iff $A(x) \leqslant B(x)$ for all $x \in X$. In this latter case, we say B *contains* A. It is sometimes convenient to write X for 1_X and ϕ for 0_X, for obvious reasons. The reader should think of $\bigvee_i A_i$ and $\bigwedge_i A_i$ for $A_i \in L^X$ as the *union* and *intersection* of the sets A_i. (See the Appendix for justification.)

More generally (as remarked in [3]) if $\mathscr{L}$ is any equational class of general algebras and $L \in \mathscr{L}$, then $L^X \in \mathscr{L}$, since L^X is $\prod_{x\in X} L_x$ as before, and any equational class is closed under products (see Cohn [1] or Linton [7] who also treats infinitary operations). For example, if L is a Boolean algebra, so is L^X, and all the laws of Boolean algebra are valid for L-sets; or when L is a cl_∞-monoid, the foregoing lemma applies to L-sets.[1]

We now have all the notation we need to give the main concept of this paper (L is assumed to be a cl_∞-monoid).

DEFINITION. An *L-topological* (or just *L-*) *space* is a pair $\langle X, \mathcal{O}\rangle$ such that X is a set, $\mathcal{O} \subseteq L^X$, and

(1) $\mathscr{C} \subseteq \mathcal{O} \Rightarrow \bigvee\mathscr{C} \in \mathcal{O}$,

(2) $A, B \in \mathcal{O} \Rightarrow A * B \in \mathcal{O}$,

(3) $0_X, 1_X \in \mathcal{O}$.

It should be noted that 2-topological spaces are just the ordinary topological spaces. Moreover, the axioms imply that $\mathcal{O}$ is a cl_∞-monoid. Elements of $\mathcal{O}$ are called *open sets*, and $\mathcal{O}$ is called an *L-topology* for X. Any sub-cl_∞-monoid of L^X is an L-space; in fact, the L-spaces on X could be more compactly defined[2] as the sub-cl_∞-monoids of L^X (it is required that the inclusion preserve $\vee$ and $*$, but not necessarily $\wedge$; more generally, we require the same of a *cl_∞-monoid homomorphism*). Let us give one example now (others will occur

[1] This observation that the L-sets on X satisfy the same equational laws that L does gives very short proofs of many results in the literature, especially in BROWN, J. G., A note on Fuzzy sets, *Information and Control* **18** (1971), 32–39, and DELUCA, A. AND S. TERMINI, Algebraic Properties of Fuzzy Sets, preprint from Laboratorio di Cibernetica del C.N.R., Napoli, Italy, 1970.

[2] This observation is also valid when $L = 2$ and seems to illuminate even the classical case: a topology on X is a sub-cl_∞-monoid of 2^X.

later). Let $X = L = \mathbb{I}$, and let $\mathcal{O}$ be all monotone (nondecreasing) functions to $(0, \frac{1}{2}] \cup \{1\}$, plus the function 0_X. This class is closed under $\bigvee$ and $*$ (in $\mathbb{I}^X$), but not under $\wedge$ (in $\mathbb{I}^X$), although $\mathcal{O}$ has an infimum of its own.

Most of the concepts of ordinary topology generalize. For example, the *interior* of $B \in L^X$ is $\bigvee \{A \in \mathcal{O} \mid A \leqslant B\}$; then infimum in $\mathcal{O}$ is given by interior of infimum in L^X. $\mathscr{C} \subseteq \mathcal{O}$ is a *cover* iff $\bigvee \mathscr{C} = 1_X$, and $\langle X, \mathcal{O} \rangle$ is *compact* iff every open cover has a finite subcover. $\mathscr{B} \subseteq \mathcal{O}$ is a *basis* iff each $A \in \mathcal{O}$ is of the form $\bigvee \mathscr{C}$ for some $\mathscr{C} \subseteq \mathscr{B}$, and $\mathscr{S} \subseteq \mathcal{O}$ is a *subbasis* iff $\mathscr{S}^* = \{S_1 * \cdots * S_n \mid S_i \in \mathscr{S}, n \in \mathbb{P}\}$ is a basis. Any $\mathscr{S} \subseteq L^X$ serves as a subbasis for a unique L-topology on X, said to be *generated* by $\mathscr{S}$, namely $\mathcal{O} = \{\bigvee \mathscr{C} \mid \mathscr{C} \subseteq \mathscr{S}^*\} \cup \{0_X, 1_X\}$; the proof of condition (2) for $\mathcal{O}$ requires a generalization $(\bigvee_{i \in I} a_i) * (\bigvee_{j \in J} b_j) = \bigvee_{ij \in I \times J} a_i * b_j$ of the distributive law for L, but this generalization is actually equivalent to the original (see [3]). A function $f: X_1 \to X_2$ is *continuous* for topologies $\mathcal{O}_i$ on X_i iff $A_2 \in \mathcal{O}_2 \Rightarrow f^{-1}(A_2) \in \mathcal{O}_1$, where $f^{-1}(A_2)(x_1) = A_2(f(x_1))$ for $x_1 \in X_1$ defines the *inverse image* of A_2 under f. It is easy to verify that the continuous image of a compact set is compact (the *image* of $A \in L^{X_1}$, also written $A \leqslant X_1$, under $f: X_1 \to X_2$, is defined by $f(A)(x_2) = \bigvee \{A(x_1) \mid f(x_1) = x_2\}$ for $x_2 \in X_2$). Here one uses the more general notion $\mathscr{C} \subseteq \mathcal{O}$ *covers* $A \in L^X$ iff $\bigvee \mathscr{C} \geqslant A$, so that A is compact iff every cover of A has a finite subcover. The fact that $f^{-1}(f(A)) \geqslant A$ is also needed.

If A and B are L-sets with carriers X and Y, we define the *product* $A \times B$ on $X \times Y$ by $(A \times B)(x, y) = A(x) * B(y)$. Arbitrary finite products are defined similarly. If $\langle X_i, \mathcal{O}_i \rangle$ are L-topological spaces for $i \in I$, we define their product $\prod_{i \in I} \langle X_i, \mathcal{O}_i \rangle$ to be $\langle X, \mathcal{O} \rangle$, where $X = \prod_{i \in I} X_i$ is the ordinary set product and $\mathcal{O}$ is the topology on X generated by the subbase

$$\mathscr{S} = \{p_i^{-1}(A_i) \mid A_i \in \mathcal{O}_i, i \in I\},$$

where $p_i: X \to X_i$ is the projection onto the ith coordinate. $\mathcal{O}$ is the weakest topology such that each p_i is continuous.

Definition. Let α be a cardinal. We say that the identity 1 in a cl_∞-monoid L is *α-isolated* iff whenever $|I| \leqslant \alpha$, $a_i \in L$ for $i \in I$ and $a_i < 1$, then $\bigvee_{i \in I} a_i < 1$.

Here $|I|$ denotes the cardinality of I. For example, 1 is α-isolated in $[k]$ for every α (let us say that 1 is *absolutely isolated*), and 1 is α-isolated in $\mathbb{I}$ for every finite α (let us say 1 is *finitely isolated*), but 1 is not ω-isolated in $\mathbb{I}$ (where ω is the cardinality of $\mathbb{N}$, i.e., countable). Also, 1 is α-isolated in $\mathbb{I}^J$ iff $\alpha < |J|$ and $\alpha < \omega$, and in $[k]^J$ iff $\alpha < |J|$.

Theorem 1. *Let 1 be α-isolated in L and let $|I| \leqslant \alpha$. Then the product $\prod_{i \in I} \langle X_i, \mathcal{O}_i \rangle$ of compact L-topological spaces $\langle X_i, \mathcal{O}_i \rangle$ is compact.*

The following converse to Theorem 1 shows it is, in a sense, the best possible result.

THEOREM 2. *If* 1 *is not* α*-isolated in L, then there is a collection* $\langle X_i, \mathcal{O}_i \rangle$ *for* $i \in I$ *and* $|I| = \alpha$ *of compact L-spaces such that the product* $\langle X, \mathcal{O} \rangle$ *is noncompact.*

Proof. Non-α-isolation gives I with $|I| = \alpha$ and $a_i < 1$ for $i \in I$ such that $\bigvee_i a_i = 1$. Let $X_i = \mathbb{N}$ for $i \in I$, and let $\mathcal{O}_i = \{0, (a_i)_{[n]}, 1\}^\star$, where a_S, for $a \in L$ and $S \subseteq X_i$, denotes the function equal to a on S and 0 on $X_i - S$, where $[n] = \{0, 1,..., n-1\}$ and where $\mathcal{S}^\star$ indicates the L-topology generated by $\mathcal{S}$ as a subbasis. Then $\langle \mathbb{N}, \mathcal{O}_i \rangle$ is compact, as every $A \in \mathcal{O}_i$ except 1 is contained in $(a_i)_{\mathbb{N}}$, and $(a_i)_{\mathbb{N}} < 1$, so that $\mathcal{C} \subseteq \mathcal{O}_i$ is a cover iff $1 \in \mathcal{C}$. Therefore every cover has $\{1\}$ as a subcover.

But $\langle X, \mathcal{O} \rangle = \prod_{i \in I} \langle X_i, \mathcal{O}_i \rangle$ is noncompact. Let A_{in} denote the open L-set $p_i^{-1}((a_i)_{[n]}) \in \mathcal{O}$, for $i \in I$ and $n \in \mathbb{N}$; it is given for $x \in X$ by the formula $A_{in}(x) = a_i$ if $x_i < n$, and $A_{in}(x) = 0$ otherwise. $\{A_{in} \mid i \in I, n \in \mathbb{N}\}$ is an open cover with no finite subcover. First, $\bigvee_n A_{in} = (a_i)_X$ since $\bigvee_n (a_i)_{[n]} = (a_i)_{\mathbb{N}}$ in $\langle X_i, \mathcal{O}_i \rangle$, and p_i^{-1} preserves suprema,[3] so that $\bigvee_{in} A_{in} = \bigvee_i (\bigvee_n A_{in}) = \bigvee_i (a_i)_X = 1$. Second, no finite subcollection $\{A_{in} \mid \langle i, n \rangle \in J\}$, $|J| < \omega$, can cover, since $\bigvee_{\langle i,n \rangle \in J} A_{in}(x) = 0$ for any x with $x_i \geqslant \bigvee \{n \mid \langle i, n \rangle \in J\}$ for any i, and infinitely many such x always exist. $\square$

Thus, we can state the *fuzzy Tychonoff theorem* as "Every product of α compact L-spaces is compact iff 1 is α-isolated in L," but we prefer to break the result into two pieces because the techniques of proof are so different. Before turning to the proof of Theorem 1, let us discuss some examples. Since $\mathbb{I}$ is finitely but not ω-isolated, finite products of compact $\mathbb{I}$-spaces are compact, but countable or larger products need not be. For $L = \mathbb{I}^2$, Theorem 2 provides an example of *two* compact L-spaces whose product is noncompact. In fact, for this case we do not need X_1 and X_2 to be countable but can take them to be the one point set $\{0\}$. Since the classical truth set 2 has 1 absolutely isolated, Theorem 1 gives the usual Tychonoff theorem, each product of compact 2-spaces is compact. This strong result also holds for $L = [k]$ for any $k \in \mathbb{P}$. However, there are two compact 2^2-spaces whose product is noncompact; these spaces must have infinitely many points.

These examples cover what are probably the most important truth sets.

[3] That inverse images preserve suprema can be shown by a small calculation; it also follows from the general category theoretic properties of the constructions discussed in the Appendix.

We do not have closed sets[4] available for proving Theorem 1, and we should try to avoid arguments involving points, as the fuzzification of a point is an L-set (see [3]). We, therefore, proceed by first generalizing Alexander's theorem on subbases and compactness as in Kelley [6]. This result requires Zorn's lemma, which we state first. A subset $\{B_i \mid i \in I\}$ of a partially ordered set $\mathscr{F}$ is a *chain* iff for each $i, j \in I$, either $B_i \leqslant B_j$ or $B_j \leqslant B_i$. $B \in \mathscr{F}$ is an *upper bound* for a subset $\{B_i \mid i \in I\}$ of $\mathscr{F}$ iff $B \geqslant B_i$ for all $i \in I$. An element M of $\mathscr{F}$ is *maximal* iff for no $B \in \mathscr{F}$ is it true that $M < B$.

ZORN'S LEMMA. *If each chain in a nonempty partially ordered set has an upper bound, then the set has a maximal element.*

THEOREM 3. *If $\mathscr{S}$ is a subbase for $\langle X, \mathcal{O}\rangle$ and every cover of X by sets in $\mathscr{S}$ has a finite subcover, then $\langle X, \mathcal{O}\rangle$ is compact.*

Proof. Let us say $\mathscr{C} \subseteq \mathcal{O}$ has the *finite union property*[5] (abbreviated FUP) iff $\bigvee \mathscr{C}_0 < 1_X$ for all $\mathscr{C}_0 \subseteq \mathscr{C}$ such that $|\mathscr{C}_0| < \omega$. Then $\langle X, \mathcal{O}\rangle$ is compact iff no $\mathscr{C} \subseteq \mathcal{O}$ with FUP is a cover. Let $\mathscr{C}$ have FUP, and let

$$\mathscr{F} = \{\mathscr{B} \mid \mathscr{C} \subseteq \mathscr{B} \subseteq \mathcal{O} \text{ and } \mathscr{B} \text{ has FUP}\};$$

we now prove from Zorn's lemma that $\mathscr{F}$, ordered by inclusion, has a maximal member, to be denoted $\tilde{\mathscr{C}}$. $\mathscr{F}$ is nonempty since it contains $\mathscr{C}$. Let $\mathscr{B}_i \in \mathscr{F}$ for $i \in I$ be a chain. Then clearly $\mathscr{C} \subseteq \bigcup_i \mathscr{B}_i$, and it remains to show that $\bigcup_i \mathscr{B}_i$ has FUP. Let $\mathscr{B}_0 \subseteq \bigcup_i \mathscr{B}_i$ be finite; then each element of $\mathscr{B}_0$ appears first in some $\mathscr{B}_i$, and therefore, all of $\mathscr{B}_0$ appears in the largest, say $\mathscr{B}_m$, of this finite set of $\mathscr{B}_i$. Since $\mathscr{B}_m$ has FUP, $\bigvee \mathscr{B}_0 < 1_X$. Therefore, $\bigcup_i \mathscr{B}_i$ is an upper bound for the chain $\mathscr{B}_i$, and $\tilde{\mathscr{C}}$ exists.

If $C \notin \tilde{\mathscr{C}}$ but $C \in \mathcal{O}$, then no open set containing C belongs to $\tilde{\mathscr{C}}$. For $C \notin \tilde{\mathscr{C}}$ if $\exists A_1, \ldots, A_m \in \mathcal{O}$ such that $C \vee \bigvee_{i=1}^m A_i = X$, because of maximality. Then $D \geqslant C$ would imply $D \vee \bigvee_{i=1}^m A_i = X$, so we must have $D \notin \tilde{\mathscr{C}}$ also if $D \in \mathcal{O}$.

If $C, D \notin \tilde{\mathscr{C}}$ but $C, D \in \mathcal{O}$, then $C * D \notin \tilde{\mathscr{C}}$. For $C \vee A = X$ and $D \vee B = X$, where $A = \bigvee_{i=1}^m A_i$ and $B = \bigvee_{i=1}^n B_i$ with $A_i, B_i \in \mathcal{O}$. Then

$$(C \vee A) * (D \vee B) = X * X = X$$

[4] Closed sets can be had if L has an involutory endomorphism satisfying the generalized de Morgan laws; $(\bigvee_i a_i)' = \bigwedge_i a_i'$ and $(\bigvee_i a_i)' = \bigvee_i a_i'$. For we can define A to be *closed* iff A' is open. However, it is in general false that $A \vee A' = 1_X$ and $A \wedge A' = 0_X$, so that this notion of closed will not in general be as useful as in the Boolean case. In most of the truth sets we considered as examples, a suitable operation is $a' = 1 - a$. In a Boolean algebra the ordinary compliment will do, and closed sets will be more natural. However, not all cl_∞-monoids admit such endomorphisms. See [4].

[5] I.e., iff $C_1, \ldots, C_n \in \mathscr{C} \Rightarrow C_1 \vee \cdots \vee C_n < 1_X$, for $n \in \mathbb{P}$.

gives

$$(C \vee A) * (D \vee B) = [(C \vee A) * D] \vee [(C \vee A) * B]$$
$$= (C * D) \vee (A * D) \vee (C * B) \vee (A * B) = X,$$

by distributivity. By the lemma after the definition of cl_∞-monoid, each of the last three terms is $\leqslant A$ or $\leqslant B$, so that $X \leqslant (C * D) \vee A \vee B$. Therefore, $(C * D) \vee A \vee B = X$, and this implies that $C * D \notin \tilde{\mathscr{C}}$. This result extends to $C_1, ..., C_n \notin \tilde{\mathscr{C}}$ and $C_1, ..., C_n \in \mathscr{O}\!\ell$ imply $C_1 * \cdots * C_n \notin \tilde{\mathscr{C}}$.

It now follows that if $C_1, ..., C_n \notin \tilde{\mathscr{C}}$ but are open, and $A \geqslant C_1 * \cdots * C_n$ and $A \in \mathscr{O}\!\ell$, then $A \notin \tilde{\mathscr{C}}$. The contrapositive of this says that if $A \in \tilde{\mathscr{C}}$ and $C_1 * \cdots * C_n \leqslant A$ for $C_i \in \mathscr{O}\!\ell$, then some $C_i \in \tilde{\mathscr{C}}$, and this is the form we use.

Now assume that $\mathscr{S}$ is a subbasis for $\langle X, \mathscr{O}\!\ell\rangle$ such that each subfamily of $\mathscr{S}$ with FUP is not a cover, and let $\mathscr{C} \subseteq \mathscr{O}\!\ell$ have FUP. We have to show that $\mathscr{C}$ is not a cover. By the preceding, there is a maximal $\tilde{\mathscr{C}} \supseteq \mathscr{C}$ and having FUP. It will suffice to show that $\tilde{\mathscr{C}}$ is not a cover.

$\mathscr{S} \cap \tilde{\mathscr{C}} \subseteq \mathscr{S}$ and $\mathscr{S} \cap \tilde{\mathscr{C}}$ has FUP, so $\mathscr{S} \cap \tilde{\mathscr{C}}$ is not a cover. It will, therefore, suffice to prove that $\bigvee\tilde{\mathscr{C}} \leqslant \bigvee (\mathscr{S} \cap \tilde{\mathscr{C}})$. Since $\mathscr{S}$ is a subbase, each $A \in \tilde{\mathscr{C}}$ is of the form $\bigvee_{i\in I} S_{i1} * \cdots * S_{in_i}$, for $n_i \in \mathbb{P}$ and $S_{ij} \in \mathscr{S}$. Then $S_{i1} * \cdots * S_{in_i} \leqslant A$ for all $i \in I$, and by the previous observation, for each $i \in I$ we must have some $j(i)$ with $S_{ij(i)} \in \mathscr{S} \cap \tilde{\mathscr{C}}$. It then follows that

$$A = \bigvee_i (S_{i1} * \cdots * S_{in_i}) \leqslant \bigvee_i S_{ij(i)},$$

for $S_{ij(i)} \in \mathscr{S} \cap \tilde{\mathscr{C}}$, that is, $\bigvee\tilde{\mathscr{C}} \leqslant \bigvee\mathscr{S} \cap \tilde{\mathscr{C}}$. □

Proof of Theorem 1. We have $\langle X_i, \mathscr{O}\!\ell_i\rangle$ compact for $i \in I$, $|I| \leqslant \alpha$, and $\langle X, \mathscr{O}\!\ell\rangle = \prod_{i\in I} \langle X_i, \mathscr{O}\!\ell_i\rangle$, where $\mathscr{O}\!\ell$ has the subbasis

$$\mathscr{S} = \{p_i^{-1}(A_i) \mid A_i \in \mathscr{O}\!\ell_i, i \in I\}.$$

By Theorem 3, it suffices to show that no $\mathscr{C} \subseteq \mathscr{S}$ with FUP is a cover. Let $\mathscr{C} \subseteq \mathscr{S}$ with FUP be given, and for $i \in I$ let $\mathscr{C}_i = \{A \in \mathscr{O}\!\ell_i \mid p_i^{-1}(A) \in \mathscr{C}\}$. Then each $\mathscr{C}_i$ has FUP; for if $\bigvee_{j=1}^n A_j = 1_{X_i}$, where $p_i^{-1}(A_j) \in \mathscr{C}$, then $\bigvee_{j=1}^n p_i^{-1}(A_j) = 1_X$, since $p_i^{-1}(1_{X_i}) = 1_X$ and p_i^{-1} preserves $\bigvee$. Therefore, $\mathscr{C}_i$ is a noncover for each $i \in I$, and there exists $x_i \in X_i$ such that $(\bigvee\mathscr{C}_i)(x_i) = a_i < 1$. Now let $x = \langle x_i\rangle \in X$ and $\mathscr{C}_i{}' = \{p_i^{-1}(A) \mid A \in \mathscr{O}\!\ell_i\} \cap \mathscr{C}$, Then $\mathscr{C} \subseteq \mathscr{S}$ implies $\mathscr{C} = \bigcup_i \mathscr{C}_i{}'$; and $(\bigvee\mathscr{C}_i)(x_i) = a_i$ implies $(\bigvee\mathscr{C}_i{}')(x) = a_i$, since

$$\left(\bigvee \mathscr{C}_i{}'\right)(x) = \bigvee \{(p_i^{-1}(A))(x) \mid A \in \mathscr{O}\!\ell_i \text{ and } p_i^{-1}(A) \in \mathscr{C}\}$$
$$= \bigvee \{A(p_i(x)) \mid p_i^{-1}(A) \in \mathscr{C} \text{ and } A \in \mathscr{O}\!\ell_i\} = \left(\bigvee \mathscr{C}_i\right)(x_i).$$

Therefore,

$$\left(\bigvee \mathscr{C}\right)(x) = \bigvee_{i \in I} \left(\left(\bigvee \mathscr{C}_i{}'\right)(x)\right) = \bigvee_i a_i < 1,$$

since 1 is α-isolated in L, each $a_i < 1$, and $|I| \leqslant \alpha$. □

Appendix

Those already acquainted with categorical algebra[6] may find these further remarks on matters in footnotes and matters in general, of some interest. First, there is a category **Sets**(L) of L-sets, for any partially ordered set L: its objects are L-sets; a morphism $f: A \to B$, where $A: X \to L$ and $B: Y \to L$ is a (set) function $\bar{f}: X \to Y$ such that $A(x) \leqslant B(\bar{f}(x))$ for all $x \in X$; and composition is just composition of the underlying set functions. It has been shown in [5] that a category **C** is equivalent to **Sets**(L) for some completely distributive lattice L iff **C** satisfies certain simple axioms. It has also been shown that these axioms are plausible assertions about inexact concepts (though this argument is part of philosophy rather than mathematics); see [5]. It is fairly easy to show that when L is a completely distributive lattice, **Sets**(L) has images, inverse images, unions, intersections, and products in the usual categorical sense, and that these universal constructions are given by the formulas given in this paper; **Sets**(L) also has exponentials (and is, therefore, Cartesian closed) for these L. These results extend in various ways to weaker L. For example, if L is a cl_∞-monoid, then **Sets**(L) is monoidal with the product using $*$ as multiplication, and this operation has a right adjoint; thus, **Sets**(L) is closed, and in particular there is a distributive law for multiplication over the usual (categorical) coproduct. That is, **Sets**(L) an instance of the categorical generalization of a cl_∞-monoid, a completely distributive monoidal category.[7] Such categories can be used as truth sets for more general species of fuzzy sets.

There is also a category **Cl**$_\infty$**-mon** of cl_∞-monoids with their morphisms. This category is complete and even has an exponential. For each cl_∞-monoid L, there is a functor E_L: **Sets** $\to$ (**Cl**$_\infty$**-mon**)^{op}, assigning to a set X the cl_∞-monoid L^X, and to a function $f: X \to Y$ the cl_∞-monoid morphism $f^{-1}: L^Y \to L^X$.

It is amusing, though not particularly important, to notice that the category **Top**(L) of L-topological spaces with continuous maps can be described as the full subcategory of the comma category (**Cl**$_\infty$**-mon**)$^{op}\downarrow E_L$ whose objects are

[6] Otherwise, see Mac Lane [9] first.

[7] In fact, a cl_∞-monoid is merely a small completely distributive monoidal category.

just the sub-cl_∞-monoid inclusions. This category has for products the construction we have given, provided L is a completely distributive lattice. When L is a cl_∞-monoid, our product renders **Top**(L) a monoidal category. This paper can be described as a determination of the extent of closure under multiplication of the subcategory of compact L-spaces, as a function of L.

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