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Semiclassical non-concentration near hyperbolic orbits

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Abstract

For a large class of semiclassical pseudodifferential operators, including Schrödinger operators, $P(h) = -h^2\Delta_g + V(x)$, on compact Riemannian manifolds, we give logarithmic lower bounds on the mass of eigenfunctions outside neighbourhoods of generic closed hyperbolic orbits. More precisely we show that if A is a pseudodifferential operator which is microlocally equal to the identity near the hyperbolic orbit and microlocally zero away from the orbit, then

$$\|u\| \leqslant C\left(\sqrt{\log(1/h)}/h\right)\left\|P(h)u\right\| + C\sqrt{\log(1/h)}\left\|(I-A)u\right\|.$$

This generalizes earlier estimates of Colin de Verdière and Parisse [Y. Colin de Verdière, B. Parisse, Équilibre instable en régime semi-classique: I – Concentration microlocale, Comm. Partial Differential Equations 19 (1994) 1535–1563; Équilibre instable en régime semi-classique: II – Conditions de Bohr–Sommerfeld, Ann. Inst. H. Poincaré Phys. Theor. 61 (1994) 347–367] obtained for a special case, and of Burq and Zworski [N. Burq, M. Zworski, Geometric control in the presence of a black box, J. Amer. Math. Soc. 17 (2004) 443–471] for real hyperbolic orbits.

Keywords: Loxodromic orbit; Complex hyperbolic orbit; Hamiltonian flow; Semiclassical estimates; Non-concentration

1. Introduction

To motivate the general result, we first present two applications. If (X, g) is a Riemannian manifold with Laplacian Δ_g, we consider the eigenvalue problem

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$$-\Delta_g u = \lambda^2 u, \qquad \|u\|_{L^2(X)} = 1.$$

If U is a small neighbourhood of a closed *hyperbolic* geodesic γ, we show that

$$\int_{X \setminus U} |u|^2 \, dx \geqslant \frac{c}{\log|\lambda|},$$

that is, if u concentrates near γ, the rate is logarithmic. This generalizes results of Colin de Verdière and Parisse [7], and Burq and Zworski [6].

As another application of our main results we consider the damped wave equation

$$\begin{cases} \left(\partial_t^2 - \Delta + 2a(x)\partial_t\right)u(x,t) = 0, & (x,t) \in X \times (0,\infty), \\ u(x,0) = 0, \quad \partial_t u(x,0) = f(x). \end{cases}$$

We prove in Section 7 that if $a(x) > 0$ outside a neighbourhood of a closed hyperbolic geodesic γ, we have the following energy estimate:

$$\|\partial_t u\|^2_{L^2(X)} + \|\nabla u\|^2_{L^2(X)} \leqslant C e^{-t/C} \|f\|^2_{H^\epsilon(X)}$$

for all $\epsilon > 0$. (In Section 7 a weaker geometric control condition in the spirit of Rauch and Taylor [21] is considered.) This application was suggested to us by M. Hitrik, and it generalizes an example of Lebeau [19].

We now turn to the general case. Let X be a compact n-dimensional manifold without boundary. We consider a selfadjoint pseudodifferential operator, $P(h)$, with real principal symbol p. We assume throughout if $p = 0$ then $dp \neq 0$, and that p is elliptic outside of a compact subset of T^*X. Assume that

$$\gamma \subset p^{-1}(0)$$

is a closed loxodromic orbit of the Hamiltonian flow of p. Let $N \subset \{p = 0\}$ be a Poincaré section for γ and let S be the Poincaré map. The assumption that γ be loxodromic means that no eigenvalue of $dS(0,0)$ lies on the unit circle. We assume also that $dS(0,0)$ has no real negative eigenvalues.

Main Theorem. *Let $A \in \Psi_h^{0,0}$ be a pseudodifferential operator whose principal symbol is* 1 *near γ and* 0 *away from γ. Then, there exist constants $h_0 > 0$ and $0 < C < \infty$ so that we have uniformly in $0 < h < h_0$,*

$$\|u\| \leqslant C \frac{\sqrt{\log(1/h)}}{h} \left\| P(h)u \right\| + C\sqrt{\log(1/h)} \left\| (I - A)u \right\|, \tag{1.1}$$

where the norms are L^2 norms on X. In particular if a family, $u = u(h)$ satisfies

$$P(h)u = \mathcal{O}_{L^2}(h^\infty), \quad \|u\|_{L^2(X)} = 1,$$

then

$$\left\| (I - A)u \right\|_{L^2(X)} \geqslant \frac{1}{C} \log\left((1/h)\right)^{-\frac{1}{2}}, \quad 0 < h < h_0. \tag{1.2}$$

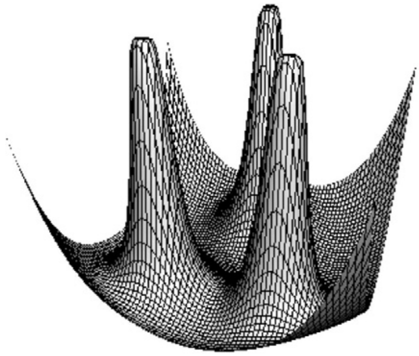

Fig. 1. A confining potential $V(x)$ with three bumps at the lowest energy level $E < 0$.

We note that the assumptions on A imply that $\mathrm{WF}_h(A)$ is contained in a neighbourhood of γ, while $\mathrm{WF}_h(I - A)$ is away from γ, see Section 2 for definitions.

Colin de Verdière and Parisse [7] have shown that the estimates (1.1)–(1.2) are sharp in the case where X is a segment of a hyperbolic cylinder and $P(h) = -h^2 \Delta_g$ is its Dirichlet Laplacian. Even though the closed orbit at the "neck" of the cylinder is hyperbolic, the flow is completely integrable in that case. This shows that eliminating the $\log(h^{-1})$ factor requires global conditions on the classical flow.

The assumption that the Poincaré map has no negative eigenvalues is standard in the literature on quantum Birkhoff normal forms (see, for example [16,17,30]), and in the present work serves to eliminate cases in which current techniques seem to break down. It is important to note that this case does arise, as in the example in [18, Section 3.4].

There are many examples in which the hypotheses of the theorem are satisfied, the simplest of which is the case in which $p = |\xi|^2 - E(h)$ for $E(h) > 0$. Then the Hamiltonian flow of p is the geodesic flow, so if the geodesic flow has a closed hyperbolic orbit, there is non-concentration of eigenfunctions, $u(h)$, for the equation

$$-h^2 \Delta u(h) = E(h)u(h).$$

Another example of such a p is the case $p = |\xi|^2 + V(x)$, where $V(x)$ is a confining potential with three "bumps" or "obstacles" in the lowest energy level (see Fig. 1). In the appendix to [23] it is shown that for an interval of energies $V(x) \sim 0$, there is a closed hyperbolic orbit γ of the Hamiltonian flow which "reflects" off the bumps (see Fig. 2). Loxodromic orbits may be constructed by considering 3-dimensional hyperbolic billiard problems (see, for example [2]), although in the present work we are assuming the orbit does not intersect the boundary of the manifold. In addition, Proposition 4.1 gives a somewhat artificial means of constructing a manifold diffeomorphic to a neighbourhood in $T^*\mathbb{S}^1_{(t,\tau)} \times T^*\mathbb{R}^{n-1}_{(x,\xi)}$ which contains a loxodromic orbit γ by starting with the Poincaré map γ is to have.

In order to prove the Main Theorem, we will first prove that the principal symbol of $P(h)$ can be put into a normal form near γ. This will allow analysis of small complex perturbations of $P(h)$. These are defined as follows. Let $a \in \mathcal{C}^\infty(T^*X, [0,1])$ be equal to 0 in a neighbourhood of γ and 1 outside of a larger neighbourhood of γ. For $z \in [-1,1] + i[-\delta, \delta]$, define

$$Q(z) := P(h) - z - ihCa^w \tag{1.3}$$

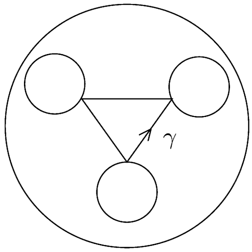

Fig. 2. The level set $V(x) = 0$ and the closed hyperbolic orbit γ.

for a constant C to be fixed later. The following theorem states that by perturbing $P(h)$ into $Q(z)$ we are able to push the spectrum of $P(h)$ into the lower half-plane.

Theorem 1. *There exist constants $c_0 > 0$, $h_0 > 0$, and N_0 such that for u with $\mathrm{WF}_h(u)$ in a sufficiently small neighbourhood of γ, $z \in [-1, 1] + i(-c_0 h, +\infty)$, and $0 < h < h_0$ we have*

$$\| Q(z)u \|_{L^2(X)} \geqslant C^{-1} h^{N_0} \|u\|_{L^2(X)} \tag{1.4}$$

for some constant C.

Using Theorem 1 and a semiclassical adaptation of the "three-lines" theorem from complex analysis, we will be able to deduce the following estimate.

Theorem 2. *Suppose $Q(z)$ is given by* (1.3)*, and $z \in I \Subset (-\infty, \infty)$. Then there is $h_0 > 0$ and $0 < C < \infty$ such that for $0 < h < h_0$,*

$$\| Q(z)^{-1} \|_{L^2(X) \to L^2(X)} \leqslant C \frac{\log(1/h)}{h}. \tag{1.5}$$

If $\varphi \in \mathcal{C}_c^\infty(X)$ is supported away from γ, then

$$\| Q(z)^{-1} \varphi \|_{L^2(X) \to L^2(X)} \leqslant C \frac{\sqrt{\log(1/h)}}{h}. \tag{1.6}$$

In order to apply the results of Theorems 1 and 2 to the Main Theorem, we observe that for A as in the statement of the Main Theorem we have $Q(0)A = P(h)A$ microlocally and apply a commutator argument.

This note is organized as follows. Section 2 recalls basic facts about h-pseudodifferential operators on manifolds. This is followed in Section 3 with a review of some standard results from the theory of h-Fourier integral operators. In Section 4 we present some symplectic geometry and prove the principal symbol can be put into a normal form in the case all the eigenvalues of $dS(0)$ are distinct. Section 5 contains the proof of Theorem 1 in the case of distinct eigenvalues, then re-examines the normal form of the principal symbol to show how it may be extended to the case when the eigenvalues are not distinct, and contains the details of the more general case of Theorem 1. Finally, in Section 6 we prove Theorem 2 and the Main

Theorem. In Section 7 we apply the techniques of Sections 4–6 to the damped wave equation.

The impetus for this paper came when M. Zworski suggested generalizing results from the appendix of [6], as well as correcting a mistake which was discovered by J.-F. Bony, S. Fujiie, T. Ramond, and M. Zerzeri (see [4] for their closely related work). This paper generalizes the statements of the theorems from the case of real hyperbolic trajectories to complex hyperbolic or loxodromic trajectories as well as correcting the mistake.

2. Preliminaries

This section contains some basic definitions and results from semiclassical and microlocal analysis which we will be using throughout the paper. This is essentially standard, but we include it for completeness. We will follow the presentation in [6, Section 2]. Let X be a smooth, compact manifold. We will be operating on half-densities,

$$u(x)\,|dx|^{\frac{1}{2}} \in \mathcal{C}^\infty\big(X, \Omega_X^{\frac{1}{2}}\big),$$

with the informal change of variables formula

$$u(x)\,|dx|^{\frac{1}{2}} = v(y)\,|dy|^{\frac{1}{2}}, \quad \text{for } y = \kappa(x) \quad \Longleftrightarrow \quad v\big(\kappa(x)\big)\big|\kappa'(x)\big|^{\frac{1}{2}} = u(x).$$

By symbols on X we mean

$$\mathcal{S}^{k,m}\big(T^*X, \Omega_{T^*X}^{\frac{1}{2}}\big) := \big\{a \in \mathcal{C}^\infty\big(T^*X \times (0,1], \Omega_{T^*X}^{\frac{1}{2}}\big)\colon \big|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)\big| \leqslant C_{\alpha\beta} h^{-m} \langle\xi\rangle^{k-|\beta|}\big\}.$$

There is a corresponding class of pseudodifferential operators $\Psi_h^{k,m}(X, \Omega_X^{\frac{1}{2}})$ acting on half-densities defined by the local formula (Weyl calculus) in $\mathbb{R}^n$:

$$\mathrm{Op}_h^w(a)u(x) = \frac{1}{(2\pi h)^n} \iint a\left(\frac{x+y}{2}, \xi; h\right) e^{i\langle x-y,\xi\rangle/h} u(y)\,dy\,d\xi.$$

We will occasionally use the shorthand notations $a^w := \mathrm{Op}_h^w(a)$ and $A := \mathrm{Op}_h^w(a)$ when there is no ambiguity in doing so.

We have the principal symbol map

$$\sigma_h : \Psi_h^{k,m}\big(X, \Omega_X^{\frac{1}{2}}\big) \to \mathcal{S}^{k,m}\big/\mathcal{S}^{k,m-1}\big(T^*X, \Omega_{T^*X}^{\frac{1}{2}}\big),$$

which gives the left inverse of Op_h^w in the sense that

$$\sigma_h \circ \mathrm{Op}_h^w : \mathcal{S}^{k,m} \to \mathcal{S}^{k,m}/\mathcal{S}^{k,m-1}$$

is the natural projection. Acting on half-densities in the Weyl calculus, the principal symbol is actually well defined in $\mathcal{S}^{k,m}/\mathcal{S}^{k,m-2}$, that is, up to $\mathcal{O}(h^2)$ in h (see, for example, [10, Appendix D]).

We will use the notion of wave front sets for pseudodifferential operators on manifolds. If $a \in \mathcal{S}^{k,m}(T^*X, \Omega^{\frac{1}{2}}_{T^*X})$ we define the singular support or essential support for a:

$$\text{ess-supp}_h\, a \subset T^*X \sqcup \mathbb{S}^*X,$$

where $\mathbb{S}^*X = (T^*X \setminus \{0\})/\mathbb{R}_+$ is the cosphere bundle (quotient taken with respect to the usual multiplication in the fibers), and the union is disjoint, $\text{ess-supp}_h\, a$ is defined using complements:

$$\begin{aligned}
&\text{ess-supp}_h\, a \\
&\quad := \complement\big\{(x,\xi) \in T^*X\colon \exists \epsilon > 0,\ \partial_x^\alpha \partial_\xi^\beta a(x',\xi') = \mathcal{O}(h^\infty),\ d(x,x') + |\xi - \xi'| < \epsilon\big\} \\
&\qquad \cup \complement\big\{(x,\xi) \in T^*X \setminus 0\colon \exists \epsilon > 0,\ \partial_x^\alpha \partial_\xi^\beta a(x',\xi') = \mathcal{O}\big(h^\infty \langle \xi \rangle^{-\infty}\big), \\
&\qquad d(x,x') + 1/|\xi'| + \big|\xi/|\xi| - \xi'/|\xi'|\big| < \epsilon\big\}\big/\mathbb{R}_+.
\end{aligned}$$

We then define the wave front set of a pseudodifferential operator $A \in \Psi_h^{k,m}(X, \Omega_X^{\frac{1}{2}})$:

$$\text{WF}_h(A) := \text{ess-supp}_h(a) \quad \text{for } A = \text{Op}_h^w(a).$$

Finally for distributional half-densities $u \in \mathcal{C}^\infty((0,1]_h, \mathcal{D}'(X, \Omega_X^{\frac{1}{2}}))$ such that there is N_0 so that $h^{N_0}u$ is bounded in $\mathcal{D}'(X, \Omega_X^{\frac{1}{2}})$, we can define the semiclassical wave front set of u, again by complement:

$$\begin{aligned}
\text{WF}_h(u) := \complement\big\{(x,\xi)\colon \exists A \in \Psi_h^{0,0},\ \text{with } \sigma_h(A)(x,\xi) \neq 0, \\
\text{and } Au \in h^\infty \mathcal{C}^\infty\big((0,1]_h, \mathcal{C}^\infty\big(X, \Omega_X^{\frac{1}{2}}\big)\big)\big\}.
\end{aligned}$$

For $A = \text{Op}_h^w(a)$ and $B = \text{Op}_h^w(b)$, $a \in \mathcal{S}^{k,m}$, $b \in \mathcal{S}^{k',m'}$ we have the composition formula (see, for example [8])

$$A \circ B = \text{Op}_h^w(a \# b), \tag{2.1}$$

where

$$\mathcal{S}^{k+k',m+m'} \ni a \# b(x,\xi) := e^{\frac{ih}{2}\omega(D_x, D_\xi; D_y, D_\eta)}\big(a(x,\xi)b(y,\eta)\big)\Big|_{\substack{x=y \\ \xi=\eta}}, \tag{2.2}$$

with ω the standard symplectic form.

We will need the definition of microlocal equivalence of operators. Suppose $T : \mathcal{C}^\infty(X, \Omega_X^{\frac{1}{2}}) \to \mathcal{C}^\infty(X, \Omega_X^{\frac{1}{2}})$ and that for any seminorm $\|\cdot\|_1$ on $\mathcal{C}^\infty(X, \Omega_X^{\frac{1}{2}})$ there is a second seminorm $\|\cdot\|_2$ on $\mathcal{C}^\infty(X, \Omega_X^{\frac{1}{2}})$ such that

$$\|Tu\|_1 = \mathcal{O}\big(h^{-M_0}\big)\|u\|_2$$

for some M_0 fixed. Then we say T is *semiclassically tempered*. We assume for the rest of this paper that all operators satisfy this condition. Let $U, V \subset T^*X$ be open precompact sets. We think of operators defined microlocally near $V \times U$ as equivalence classes of tempered operators. The equivalence relation is

$$T \sim T' \quad \Longleftrightarrow \quad A(T - T')B = \mathcal{O}(h^\infty) : \mathcal{D}'\big(X, \Omega_X^{\frac{1}{2}}\big) \to \mathcal{C}^\infty\big(X, \Omega_X^{\frac{1}{2}}\big)$$

for any $A, B \in \Psi_h^{0,0}(X, \Omega_X^{\frac{1}{2}})$ such that

$$\mathrm{WF}_h(A) \subset \widetilde{V}, \qquad \mathrm{WF}_h(B) \subset \widetilde{U}, \quad \text{with } \widetilde{V}, \widetilde{U} \text{ open} \quad \text{and}$$
$$\overline{V} \Subset \widetilde{V} \Subset T^*X, \qquad \overline{U} \Subset \widetilde{U} \Subset T^*X.$$

In the course of this paper, when we say $P = Q$ *microlocally* near $U \times V$, we mean for any A, B as above,

$$APB - AQB = \mathcal{O}_{L^2 \to L^2}(h^\infty),$$

or in any other norm by the assumed precompactness of U and V. Similarly, we say $B = T^{-1}$ on $V \times V$ if $BT = I$ microlocally near $U \times U$ and $TB = I$ microlocally near $V \times U$.

For this paper, we will need the following semiclassical version of Beals's theorem (see [8] for a proof). Recall for operators A and B, the notation $\mathrm{ad}_B A$ is defined as

$$\mathrm{ad}_B A = [B, A].$$

Theorem *(Beals's theorem). Let $A : \mathcal{S} \to \mathcal{S}'$ be a continuous linear operator. Then $A = \mathrm{Op}_h^w(a)$ for a symbol $a \in \mathcal{S}^{0,0}$ if and only if for all $N \in \mathbb{N}$ and all linear symbols $l_1, \ldots, l_N$,*

$$\mathrm{ad}_{\mathrm{Op}_h^w(l_1)} \circ \mathrm{ad}_{\mathrm{Op}_h^w(l_2)} \circ \cdots \circ \mathrm{ad}_{\mathrm{Op}_h^w(l_N)} A = \mathcal{O}\big(h^N\big)_{L^2 \to L^2}.$$

The following lemma (given more generally in [3]) will be used in the proof of Theorem 1. We include a sketch of the proof from [27] here for completeness. It is easiest to phrase in terms of order functions. A smooth function $m \in \mathcal{C}^\infty(T^*X; \mathbb{R})$ is called an order function if it satisfies

$$m(x, \xi) \leqslant C m(y, \eta) \big\langle \mathrm{dist}(x - y) + |\xi - \eta| \big\rangle^N$$

for some $N \in \mathbb{N}$. We say $a \in \mathcal{S}^l(m)$ if

$$|\partial^\alpha a| \leqslant C_\alpha h^{-l} m.$$

If $l = 0$, we write $\mathcal{S}(m) := \mathcal{S}^0(m)$.

Lemma 2.1. *Let m be an order function, and suppose $G \in \mathcal{C}^\infty(T^*X; \mathbb{R})$ satisfies*

$$G(x, \xi) - \log\big(m(x, \xi)\big) = \mathcal{O}(1) \tag{2.3}$$

and

$$\partial_x^\alpha \partial_\xi^\beta G(x,\xi) = \mathcal{O}(1) \quad \text{for } (\alpha,\beta) \neq (0,0). \tag{2.4}$$

Then for $G^w = \mathrm{Op}_h^w(G)$ *and* $|t|$ *sufficiently small,*

$$\exp(tG^w) = \mathrm{Op}_h^w(b_t)$$

for $b_t \in \mathcal{S}(m^t)$. *Here* e^{tG^w} *is defined as the unique solution to the evolution equation*

$$\begin{cases} \partial_t\big(U(t)\big) - G^w U(t) = 0, \\ U(0) = \mathrm{id}. \end{cases}$$

Sketch of the proof. The conditions on G (2.3) and (2.4) are equivalent to saying $e^{tG} \in \mathcal{S}(m^t)$. We will compare $\exp tG^w$ and $\mathrm{Op}_h^w(\exp tG)$, which we do in the following claims.

Claim 2.2. *Set* $U(t) := \mathrm{Op}_h^w(e^{tG}) : \mathcal{S} \to \mathcal{S}$. *For* $|t| < \epsilon_0$, $U(t)$ *is invertible and* $U(t)^{-1} = \mathrm{Op}_h^w(b_t)$ *for* $b_t \in \mathcal{S}(m^{-t})$, *where* ϵ_0 *depends only on* G.

Proof. Using the composition law, we see $U(-t)U(t) = \mathrm{id} + \mathrm{Op}_h^w(E_t)$, with $E_t = \mathcal{O}(t)$. Hence $\mathrm{id} + \mathrm{Op}_h^w(E_t)$ is invertible and using Beals's theorem, we get $(\mathrm{id} + \mathrm{Op}_h^w(E_t))^{-1} = \mathrm{Op}_h^w(c_t)$ for $c_t \in \mathcal{S}(1)$. Thus $\mathrm{Op}_h^w(c_t)U(-t)U(t) = \mathrm{id}$, so

$$U(t)^{-1} = \mathrm{Op}_h^w\big(c_t \,\#\, \exp(-tG)\big),$$

and subsequently $b_t \in \mathcal{S}(m^{-t})$. □

Now observe that

$$\frac{d}{dt}U(-t) = -\mathrm{Op}_h^w\big(G\exp(-tG)\big) \quad \text{and} \quad U(-t)G^w = \mathrm{Op}_h^w\big(e^{-tG} \,\#\, G\big),$$

so that

$$\begin{aligned} \frac{d}{dt}\big(U(-t)e^{tG^w}\big) &= -\mathrm{Op}_h^w\big(G\exp(-tG)\big)e^{tG^w} + \mathrm{Op}_h^w\big(e^{-tG} \,\#\, G\big)e^{tG^w} \\ &= \mathrm{Op}_h^w(A_t)e^{tG^w} \end{aligned} \tag{2.5}$$

for $A_t \in \mathcal{S}(m^{-1})$. To see (2.5), recall that by the composition law,

$$e^{-tG} \,\#\, G = e^{-tG}G + (\text{terms with } G \text{ derivatives}).$$

Then the first terms in (2.5) will cancel and the remaining terms will all involve at least one derivative of G, which is then bounded by (2.4).

Set $C(t) := -\mathrm{Op}_h^w(A_t)U(-t)^{-1}$. Claim 2.2 implies $C(t) = \mathrm{Op}_h^w(c_t)$ for a family $c_t \in \mathcal{S}(1)$. The composition law implies c_t depends smoothly on t. Then

$$\left(\frac{\partial}{\partial t} + C(t)\right)\big(U(-t)e^{tG^w}\big) = \mathrm{Op}_h^w(A_t)e^{tG^w} - \mathrm{Op}_h^w(A_t)e^{tG^w} = 0,$$

so we have reduced the problem to proving the following claim.

Claim 2.3. *Suppose $C(t) = \mathrm{Op}_h^w(c_t)$ with $c_t \in \mathcal{S}(1)$ depending smoothly on $t \in (-\epsilon_0, \epsilon_0)$. If $Q(t)$ solves*

$$\begin{cases} \left(\frac{\partial}{\partial t} + C(t)\right) Q(t) = 0, \\ Q(0) = \mathrm{Op}_h^w(q), \quad \text{with } q \in \mathcal{S}(1), \end{cases}$$

then $Q(t) = \mathrm{Op}_h^w(q_t)$ with $q_t \in \mathcal{S}(1)$ depending smoothly on $t \in (-\epsilon_0, \epsilon_0)$.

Proof. The Picard existence theorem for ODEs implies $Q(t)$ exists and is bounded on L^2. We want to use Beals's theorem to show $Q(t)$ is actually a quantized family of symbols. Let $l_1, \ldots, l_N$ be linear symbols. We will use induction to show that for any N and any choice of the l_j, $\mathrm{ad}_{\mathrm{Op}_h^w(l_1)} \circ \cdots \circ \mathrm{ad}_{\mathrm{Op}_h^w(l_N)} Q(t) = \mathcal{O}(h^N)_{L^2 \to L^2}$. Since we are dealing with linear symbols, we take $h = 1$ for convenience. First note

$$\frac{d}{dt} \mathrm{ad}_{\mathrm{Op}_h^w(l_1)} \circ \cdots \circ \mathrm{ad}_{\mathrm{Op}_h^w(l_N)} Q(t) + \mathrm{ad}_{\mathrm{Op}_h^w(l_1)} \circ \cdots \circ \mathrm{ad}_{\mathrm{Op}_h^w(l_N)} \big(C(t) Q(t)\big) = 0.$$

For the induction step, assume $\mathrm{ad}_{\mathrm{Op}_h^w(l_1)} \circ \cdots \circ \mathrm{ad}_{\mathrm{Op}_h^w(l_k)} Q(t) = \mathcal{O}(1)$ is known for $k < N$ and observe

$$\mathrm{ad}_{\mathrm{Op}_h^w(l_1)} \circ \cdots \circ \mathrm{ad}_{\mathrm{Op}_h^w(l_N)} \big(C(t) Q(t)\big) = C(t)\, \mathrm{ad}_{\mathrm{Op}_h^w(l_1)} \circ \cdots \circ \mathrm{ad}_{\mathrm{Op}_h^w(l_N)} Q(t) + R(t),$$

where $R(t)$ is a sum of terms of the form $A_k(t)\, \mathrm{ad}_{\mathrm{Op}_h^w(l_1)} \circ \cdots \circ \mathrm{ad}_{\mathrm{Op}_h^w(l_k)} Q(t)$ for each $k < N$ and $A_k(t) = \mathrm{Op}_h^w(a_k(t))$ with $a_k(t) \in \mathcal{S}(1)$. Set $\widetilde{Q}(t) = \mathrm{ad}_{\mathrm{Op}_h^w(l_1)} \circ \cdots \circ \mathrm{ad}_{\mathrm{Op}_h^w(l_N)} Q(t)$ and note that $\widetilde{Q}$ solves

$$\begin{cases} \left(\frac{\partial}{\partial t} + C(t)\right) \widetilde{Q}(t) = -R(t), \\ \widetilde{Q}(0) = \mathcal{O}(1)_{L^2 \to L^2}. \end{cases}$$

Since $R(t) = \mathcal{O}(1)_{L^2 \to L^2}$ by the induction hypothesis, Picard's theorem implies $\widetilde{Q}(t) : L^2 \to L^2$ as desired. $\square$

We will need to review some basic facts about the calculus of symbols with two parameters. We will only use symbol spaces with two parameters in the context of microlocal estimates, in which case we may assume we are working in an open subset of $\mathbb{R}^{2n}$. We define the following spaces of symbols with two parameters:

$$\mathcal{S}^{k,m,\tilde{m}}\big(\mathbb{R}^{2n}\big) := \big\{ a \in \mathcal{C}^\infty\big(\mathbb{R}^{2n} \times (0,1]^2\big) \colon \big|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h,\tilde{h})\big| \leqslant C_{\alpha\beta} h^{-m} \tilde{h}^{-\tilde{m}} \langle \xi \rangle^{k - |\beta|} \big\}.$$

For the applications in this paper, we assume $\tilde{h} > h$ and define the scaled spaces:

$$\mathcal{S}_\delta^{k,m,\tilde{m}}(\mathbb{R}^{2n}) := \Big\{ a \in \mathcal{C}^\infty(\mathbb{R}^{2n} \times (0,1]^2): \\ \big|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h,\tilde{h})\big| \leqslant C_{\alpha\beta} h^{-m} \tilde{h}^{-\tilde{m}} \left(\frac{\tilde{h}}{h}\right)^{\delta(|\alpha|+|\beta|)} \langle \xi \rangle^{k-|\beta|} \Big\}.$$

As before, we have the corresponding spaces of semiclassical pseudodifferential operators $\Psi^{k,m,\tilde{m}}$ and $\Psi_\delta^{k,m,\tilde{m}}$, where we will usually add a subscript of h or $\tilde{h}$ to indicate which parameter is used in the quantization. The relationship between Ψ_h and $\Psi_{\tilde{h}}$ is given in the following lemma.

Lemma 2.4. *Let $a \in \mathcal{S}_0^{k,m,\tilde{m}}$, and set*

$$b(X,\Xi) = a\big((h/\tilde{h})^{\frac{1}{2}} X, (h/\tilde{h})^{\frac{1}{2}} \Xi\big) \in \mathcal{S}_{-\frac{1}{2}}^{k,m,\tilde{m}}.$$

There is a linear operator $T_{h,\tilde{h}}$, unitary on L^2, and an operator such that

$$\mathrm{Op}_{\tilde{h}}^w(b) T_{h,\tilde{h}} u = T_{h,\tilde{h}} \mathrm{Op}_h^w(a) u.$$

Proof. For $u \in L^2(\mathbb{R}^n)$, define $T_{h,\tilde{h}}$ by

$$T_{h,\tilde{h}} u(X) := (h/\tilde{h})^{\frac{n}{4}} u\big((h/\tilde{h})^{\frac{1}{2}} X\big). \tag{2.6}$$

We see immediately that $T_{h,\tilde{h}}$ conjugates operators $a^w(x, hD_x)$ and $b^w(X, \tilde{h} D_X)$. □

We have the following microlocal commutator lemma.

Lemma 2.5. *Suppose $a \in \mathcal{S}_0^{-\infty,0,0}$, $b \in \mathcal{S}_{-\frac{1}{2}}^{-\infty,m,\tilde{m}}$, and $\tilde{h} > h$.*

(a) *If $A = \mathrm{Op}_{\tilde{h}}^w(a)$ and $B = \mathrm{Op}_{\tilde{h}}^w(b)$,*

$$[A,B] = \frac{\tilde{h}}{i} \mathrm{Op}_{\tilde{h}}^w\big(\{a,b\}\big) + \mathcal{O}\big(h^{3/2} \tilde{h}^{3/2}\big).$$

(b) *More generally, for each $l > 1$,*

$$\mathrm{ad}_A^l B = \mathcal{O}_{L^2 \to L^2}\big(h \tilde{h}^{l-1}\big).$$

Proof. Without loss of generality, $m = \tilde{m} = 0$, so for (a) we have from the Weyl calculus:

$$[A,B] = \frac{\tilde{h}}{i} \mathrm{Op}_{\tilde{h}}^w\big(\{a,b\}\big) + \tilde{h}^3 \mathcal{O}\Big(\sum_{|\alpha|=|\beta|=3} \partial^\alpha a \partial^\beta b \Big),$$

since the second order term vanishes in the Weyl expansion of the commutator. Note $\partial^\alpha a$ is bounded for all α, and observe for $|\beta| = 3$,

$$\tilde{h}^3 \partial^\beta b = \tilde{h}^3 \mathcal{O}\big(h^{3/2} \tilde{h}^{-3/2}\big).$$

For part (b) we again assume $m = \tilde{m} = 0$, and we observe that for $l > 1$ we no longer have the same gain in powers of h as in part (a). This follows from the fact that the $\tilde{h}$-principal symbol for the commutator $[A, [A, B]]$, $-i\tilde{h}\{a, -i\tilde{h}\{a, b\}\}$, satisfies

$$-i\tilde{h}\big\{a, -i\tilde{h}\{a, b\}\big\} = -\tilde{h}^2\big(\partial_\Xi a \partial_X (\partial_\Xi a \partial_X b - \partial_X a \partial_\Xi b) - \partial_X a \partial_\Xi (\partial_\Xi a \partial_X b - \partial_X a \partial_\Xi b)\big) \tag{2.7}$$

$$\in \mathcal{S}_0^{-\infty,-1,-1}, \tag{2.8}$$

since $\{a, b\}$ involves products of derivatives of both a and b.

For general $l > 1$, assume

$$\sigma_{\tilde{h}}\big(\mathrm{ad}_A^l B\big) \in \mathcal{S}_0^{0,-1,1-l}$$

and a calculation similar to (2.7)–(2.8) finishes the induction. □

3. *h*-Fourier integral operators

In this section we review some facts about h-Fourier integral operators (h-FIOs). See [9] for a comprehensive introduction to general FIOs without h, or [10, Section 10.1] with the addition of the h parameter. For this note, we are only interested in a special class of h-FIOs, namely those associated to a symplectomorphism. In order to motivate this, suppose $f : X \to Y$ is a diffeomorphism. Then we write

$$f^* u(x) = u\big(f(x)\big) = \frac{1}{(2\pi h)^n} \int e^{i\langle f(x)-y,\xi\rangle/h} u(y)\, dy\, d\xi,$$

and $f^* : \mathcal{C}^\infty(Y) \to \mathcal{C}^\infty(X)$ is an h-FIO associated to the nondegenerate phase function $\varphi = \langle f(x) - y, \xi\rangle$. We recall the notation from [9]: if $A : \mathcal{C}_c^\infty(Y) \to \mathcal{D}'(X)$ is a continuous mapping with distributional kernel $K_A \in \mathcal{D}'(X \times Y)$,

$$\mathrm{WF}_h'(A) = \big\{\big((x,\xi),(y,\eta)\big) \in (T^*X \times T^*Y) \setminus 0\colon\ (x, y; \xi, -\eta) \in \mathrm{WF}_h(K_A)\big\}.$$

In this notation, we note

$$\mathrm{WF}_h' f^* \subset \big\{\big((x,\xi),(y,\eta)\big)\colon\ y = f(x),\ \xi = {}^t D_x f \cdot \eta\big\},$$

which is the graph of the induced symplectomorphism

$$\kappa(x,\xi) = \big(f(x), ({}^t D_x f)^{-1}(\xi)\big).$$

To continue, we follow [26], and let $A(t)$ be a smooth family of pseudodifferential operators: $A(t) = \mathrm{Op}_h^w(a(t))$ with

$$a(t) \in \mathcal{C}^\infty\big([-1,1]_t; \mathcal{S}^{-\infty,0}(T^*X)\big),$$

such that for each t, $\mathrm{WF}_h(A(t)) \Subset T^*X$. Let $U(t): L^2(X) \to L^2(X)$ be defined by

$$\begin{cases} hD_t U(t) + U(t)A(t) = 0, \\ U(0) = U_0 \in \Psi_h^{0,0}(X), \end{cases} \tag{3.1}$$

where $D_t = -i\partial/\partial t$ as usual. If we let $a_0(t)$ be the real-valued h-principal symbol of $A(t)$ and let $\kappa(t)$ be the family of symplectomorphisms defined by

$$\begin{cases} \dfrac{d}{dt}\kappa(t)(x,\xi) = \big(\kappa(t)\big)_*\big(H_{a_0(t)}(x,\xi)\big), \\ \kappa(0)(x,\xi) = (x,\xi), \end{cases}$$

for $(x,\xi) \in T^*X$, then $U(t)$ is a family of h-FIOs associated to $\kappa(t)$. We have the following well-known theorem of Egorov (see, for example [10, Section 10.1]).

Theorem *(Egorov's theorem). Suppose $B \in \Psi_h^{k,m}(X)$, and $U(t)$ defined as above. Suppose further that U_0 in* (3.1) *is elliptic* ($\sigma_h(U_0) \geqslant c > 0$). *Then there exists a smooth family of pseudodifferential operators $V(t)$ such that*

$$\begin{cases} \sigma_h\big(V(t)BU(t)\big) = \big(\kappa(t)\big)^*\sigma_h(B), \\ V(t)U(t) - I,\ U(t)V(t) - I \in \Psi_h^{-\infty,-\infty}(X). \end{cases} \tag{3.2}$$

Proof. As U_0 is elliptic, there exists an approximate inverse V_0, such that $U_0V_0 - I$, $V_0U_0 - I \in \Psi_h^{-\infty,-\infty}$. Let $V(t)$ solve

$$\begin{cases} hD_t V(t) - A(t)V(t) = 0, \\ V(0) = V_0. \end{cases}$$

Write $B(t) = V(t)BU(t)$, so that

$$hD_t B(t) = A(t)V(t)BU(t) - V(t)BU(t)A(t) = \big[A(t), B(t)\big]$$

modulo $\Psi_h^{-\infty,-\infty}$. But the principal symbol of $[A(t), B(t)]$ is

$$\sigma_h\big(\big[A(t), B(t)\big]\big) = \frac{h}{i}\big\{\sigma_h\big(A(t)\big), \sigma_h\big(B(t)\big)\big\} = \frac{h}{i}H_{a_0(t)}\sigma_h\big(B(t)\big),$$

so (3.2) follows from the definition of $\kappa(t)$. □

Let $U := U(1)$, and suppose the graph of κ is denoted by C. Then we introduce the standard notation

$$U \in I_h^0(X \times X; C'), \quad \text{with } C' = \big\{(x,\xi; y,-\eta):\ (x,\xi) = \kappa(y,\eta)\big\},$$

meaning U is the h-FIO associated to the graph of κ. The next few results when taken together will say that locally all h-FIOs associated to symplectic graphs are of the same form as $U(1)$. First a well-known lemma.

Lemma 3.1. *Suppose* $\kappa : \mathrm{neigh}(0,0) \to \mathrm{neigh}(0,0)$ *is a symplectomorphism fixing* $(0,0)$. *Then there exists a smooth family of symplectomorphisms* κ_t *fixing* $(0,0)$ *such that* $\kappa_0 = \mathrm{id}$ *and* $\kappa_1 = \kappa$. *Further, there is a smooth family of functions* g_t *such that*

$$\frac{d}{dt}\kappa_t = (\kappa_t)_* H_{g_t}.$$

The proof of Lemma 3.1 is standard, but we include a sketch here, as it will be used in the proof of Proposition 4.1 (see [10, Section 10.1] for details).

Sketch of the proof. First suppose $K : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a linear symplectic transformation. Write the polar decomposition of K, $K = QP$ with Q orthogonal and P positive definite. It is standard that K symplectic implies Q and P are both symplectic as well. Identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ on which Q is unitary. Write $Q = \exp iB$ for B Hermitian and $P = \exp A$ for A real symmetric and $JA + AJ = 0$, where

$$J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

is the standard matrix of symplectic structure on $\mathbb{R}^{2n}$. Then $K_t = \exp(itB)\exp(tA)$ satisfies $K_0 = \mathrm{id}$ and $K_1 = K$.

In the case κ is nonlinear, set $K = \partial\kappa(0,0)$ and choose K_t such that $K_0 = \mathrm{id}$ and $K_{1/2} = K$. Then set

$$\tilde{\kappa}_t(x,\xi) = \frac{1}{t}\kappa\big(t(x,\xi)\big),$$

and note that $\tilde{\kappa}_t$ satisfies $\tilde{\kappa}_0 = K$, $\tilde{\kappa}_1 = \kappa$. Rescale $\tilde{\kappa}_t$ in t, so that $\tilde{\kappa}_t \equiv K$ near $1/2$ and $\tilde{\kappa}_1 = \kappa$. Rescale K_t so that $K_0 = \mathrm{id}$ and $K_t \equiv K$ near $1/2$. Then κ_t is defined for $0 \leqslant t \leqslant 1$ by taking K_t for $0 \leqslant t \leqslant 1/2$ and $\tilde{\kappa}_t$ for $1/2 \leqslant t \leqslant 1$.

To show $\frac{d}{dt}\kappa_t = (\kappa_t)_* H_{g_t}$, set $V_t = \frac{d}{dt}\kappa_t$. Cartan's formula then gives for ω the symplectic form

$$\mathcal{L}_{V_t}\omega = d\omega \lrcorner V_t + d(\omega \lrcorner V_t),$$

but $\mathcal{L}_{V_t}\omega = \frac{d}{dt}\kappa_t^*\omega = 0$ since κ_t is symplectic for each t. Hence $\omega \lrcorner V_t = dg_t$ for some smooth function g_t by the Poincaré lemma, in other words, $V_t = (\kappa_t)_* H_{g_t}$. □

We have the following version of Egorov's theorem.

Proposition 3.2. *Suppose* U *is an open neighbourhood of* $(0,0)$ *and* $\kappa : U \to U$ *is a symplectomorphism fixing* $(0,0)$. *Then there is a bounded operator* $F : L^2 \to L^2$ *such that for all* $A = \mathrm{Op}_h^w(a)$,

$$AF = FB \quad \textit{microlocally on } U \times U,$$

where $B = \mathrm{Op}_h^w(b)$ *for a Weyl symbol* b *satisfying*

$$b = \kappa^* a + \mathcal{O}(h^2).$$

F *is microlocally invertible in* $U \times U$ *and* $F^{-1}AF = B$ *microlocally in* $U \times U$.

Proposition 3.2 is a standard result, however we include a proof as we will be using it for the proof of Theorem 3.

Proof. For $0 \leqslant t \leqslant 1$ let κ_t be a smooth family of symplectomorphisms satisfying $\kappa_0 = \mathrm{id}$, $\kappa_1 = \kappa$, and let g_t satisfy $\frac{d}{dt}\kappa_t = (\kappa_t)_* H_{g_t}$. Let $G_t = \mathrm{Op}_h^w(g_t)$, and solve the following equations:

$$\begin{cases} hD_t F(t) + F(t)G(t) = 0 & (0 \leqslant t \leqslant 1), \\ F(0) = I, \end{cases}$$

$$\begin{cases} hD_t \widetilde{F}(t) - G(t)\widetilde{F}(t) = 0 & (0 \leqslant t \leqslant 1), \\ F(0) = I. \end{cases}$$

Then $F(t), \widetilde{F}(t) = \mathcal{O}(1): L^2 \to L^2$ and

$$hD_t\big(F(t)\widetilde{F}(t)\big) = -F(t)G(t)\widetilde{F}(t) + F(t)G(t)\widetilde{F}(t) = 0,$$

so $F(t)\widetilde{F}(t) = I$ for $0 \leqslant t \leqslant 1$. Similarly, $E(t) = \widetilde{F}F - I$ satisfies

$$hD_t E(t) = G(t)\widetilde{F}(t)F(t) - \widetilde{F}(t)F(t)G(t) = \big[G(t), E(t)\big] \tag{3.3}$$

with $E(0) = 0$. But Eq. (3.3) has unique solution $E(t) \equiv 0$ for the initial condition $E(0) = 0$. Hence $\widetilde{F}(t)F(t) = I$ microlocally.

Now set $B(t) = \widetilde{F}(t)AF(t)$. We would like to show $B(t) = \mathrm{Op}_h^w(b_t)$, for $b_t = \kappa_t^* a + \mathcal{O}(h^2)$. Set $\widetilde{B}(t) = \mathrm{Op}_h^w(\kappa_t^* a)$. Then

$$\begin{aligned} hD_t\widetilde{B}(t) &= \frac{h}{i}\mathrm{Op}_h^w\left(\frac{d}{dt}\kappa_t^* a\right) = \frac{h}{i}\mathrm{Op}_h^w\big(\{g_t, \kappa_t^* a\}\big) \\ &= \big[G(t), \widetilde{B}(t)\big] + E_1(t), \end{aligned}$$

where $E_1(t) = \mathrm{Op}_h^w(e_1(t))$ for $e_1(t)$ a smooth family of symbols. Note if we take $g_t \# (\kappa_t^* a) - (\kappa_t^* a) \# g_t$, the composition formula (2.2) implies the h^2 term vanishes for the Weyl calculus since ω^2 is symmetric while

$$g_t(x,\xi)\kappa_t^* a(y,\eta) - \kappa_t^* a(x,\xi) g_t(y,\eta)$$

is antisymmetric. Thus $E_1(t) \in \Psi_h^{0,-3}$, since we are working microlocally. We calculate:

$$
\begin{aligned}
&hD_t\big(F(t)\widetilde{B}(t)\widetilde{F}(t)\big) && (3.4)\\
&\quad = -F(t)G(t)\widetilde{B}(t)\widetilde{F}(t) + F(t)\big(\big[G(t), \widetilde{B}(t)\big] + E_1(t)\big)\widetilde{F}(t) && (3.5)\\
&\qquad + F(t)\widetilde{B}(t)G(t)\widetilde{F}(t)\\
&\quad = F(t)E_1(t)\widetilde{F}(t) && (3.6)\\
&\quad = \mathcal{O}\big(h^3\big).
\end{aligned}
$$

Integrating in t and dividing by h we get

$$
F(t)\widetilde{B}(t)\widetilde{F}(t) = A + \frac{i}{h}\int_0^t F(s)E_1(s)\widetilde{F}(s)\,ds = A + \mathcal{O}\big(h^2\big), \tag{3.7}
$$

so that $\widetilde{B}(t) - B(t) = \mathcal{O}(h^2)$.

We will construct families of pseudodifferential operators $B_k(t)$ so that for each m

$$
B(t) = \widetilde{B}(t) + B_1(t) + \cdots + B_m(t) + \mathcal{O}\big(h^{m+2}\big). \tag{3.8}
$$

Let

$$
\tilde{e}_1(t) = (\kappa_t)^* \int_0^t \big(\kappa_s^{-1}\big)^* e_1(s)\,ds,
$$

and set $\widetilde{E}_1(t) = \mathrm{Op}_h^w(\tilde{e}_1(t))$. Observe

$$
hD_t\widetilde{E}_1 = \big[G(t), \widetilde{E}_1\big] + \frac{h}{i}\big(E_1(t) + E_2(t)\big),
$$

where $E_2(t) \in \Psi_h^{0,-4}$ by the Weyl calculus, since $[G, \widetilde{E}_1] = \mathcal{O}(h^4)$. Then as in (3.4)–(3.6)

$$
\begin{aligned}
hD_t\big(F(t)\widetilde{E}_1(t)\widetilde{F}(t)\big) &= -F(t)\big[G(t), \widetilde{E}_1(t)\big]\widetilde{F}(t) + F(t)hD_t\big(\widetilde{E}_1(t)\big)\widetilde{F}(t)\\
&= \frac{h}{i}\big(F(t)E_1(t)\widetilde{F}(t) + F(t)E_2(t)\widetilde{F}(t)\big).
\end{aligned}
$$

Integrating in t gives

$$
F(t)\widetilde{E}_1(t)\widetilde{F}(t) = \int_0^t F(s)E_1(s)\widetilde{F}(s)\,ds + \frac{i}{h}\int_0^t F(s)E_2(s)\widetilde{F}(s)\,ds,
$$

and substituting in (3.7) gives

$$\widetilde{B}(t) - B(t) = \frac{i}{h}\widetilde{E}_1(t) - \widetilde{F}(t)\left(\frac{i}{h}\int_0^t F(s)E_2(s)\widetilde{F}(s)\,ds\right)F(t)$$
$$= \frac{i}{h}\widetilde{E}_1(t) + \mathcal{O}(h^3).$$

Setting $B_1(t) = i\widetilde{E}_1(t)/h$ and continuing inductively gives $B_k(t)$ satisfying (3.8).

Let l be a linear symbol, and $L = \mathrm{Op}_h^w(l)$. Then

$$\mathrm{ad}_L(\widetilde{B} - B) = [\widetilde{B} - B, L] = \mathcal{O}(h^2).$$

Fix N. From (3.8) we can choose $B_1, \dots, B_N$ so that replacing $\widetilde{B}$ with $\widetilde{B} + B_1 + \cdots + B_N$, we have for $l_1, \dots, l_N$ linear symbols, $L_k = \mathrm{Op}_h^w(l_k)$,

$$\mathrm{ad}_{L_1} \circ \cdots \circ \mathrm{ad}_{L_N}(\widetilde{B} - B) = \mathcal{O}(h^{N+2}),$$

so Beals theorem implies $B(t) = \mathrm{Op}_h^w(b(t))$ for $b(t) = \kappa_t^* a + \mathcal{O}(h^2)$. □

The next proposition is essentially a converse to Proposition 3.2.

Proposition 3.3. *Suppose $U = \mathcal{O}(1) : L^2 \to L^2$ and for all pseudodifferential operators $A, B \in \Psi_h^{0,0}(X)$ such that $\sigma_h(B) = \kappa^*\sigma_h(A)$, $AU = UB$ microlocally near (ρ_0, ρ_0), where $\kappa : \mathrm{neigh}(\rho_0, \rho_0) \to \mathrm{neigh}(\rho_0, \rho_0)$ is a symplectomorphism fixing (ρ_0, ρ_0). Then $U \in I_h^0(X \times X; C')$ microlocally near (ρ_0, ρ_0).*

Proof. Choose κ_t a smooth family of symplectomorphisms such that $\kappa_0 = \mathrm{id}$, $\kappa_1 = \kappa$, and $\kappa_t(\rho_0) = \rho_0$. Choose $a(t)$ a smooth family of functions satisfying $\frac{d}{dt}\kappa_t = (\kappa_t)_* H_{a(t)}$, and let $A(t) = \mathrm{Op}_h^w(a(t))$. Let $U(t)$ be a solution to

$$\begin{cases} hD_t U(t) - U(t)A(t) = 0, \\ U(1) = U \end{cases}$$

for $0 \leqslant t \leqslant 1$. Next let A and B satisfy the assumptions of the proposition. Since $AU = UB$, we can find $V(t)$ satisfying

$$\begin{cases} AU(t)V(t) = U(t)BV(t), \\ V(0) = \mathrm{id}. \end{cases} \tag{3.9}$$

By Egorov's theorem, the right-hand side of (3.9) is equal to

$$U(t)V(t)\big(V(t)^{-1}BV(t)\big) = U(t)V(t)A + \mathcal{O}(h).$$

Setting $t = 0$, we see $[U(0), A] = \mathcal{O}(h)$. Applying the same argument to $[U(t), A]$ and another choice of $\widetilde{A}, \widetilde{B}$ satisfying the hypotheses of the proposition yields by induction,

$$\mathrm{ad}_{A_1} \circ \cdots \circ \mathrm{ad}_{A_N} U(0) = \mathcal{O}(h^N) \tag{3.10}$$

for any choice of $A_1, \dots, A_N \in \Psi_h^{0,0}(X)$. Since we are only interested in what $U(t)$ looks like microlocally, (3.10) is sufficient to apply Beals's theorem and conclude that $U(0) \in \Psi_h^{0,0}(X)$. Thus $U(t)$ and hence $U(1) = U$ is in $I_h^0(X \times X; C')$ for the twisted graph

$$C' = \{(x, \xi, y, -\eta)\colon\ (y, \eta) = \kappa(x, \xi)\}. \qquad \square$$

Using the following more general version of the Poincaré lemma from [29], we will be able to generalize Proposition 3.2 to a neighbourhood of a periodic orbit.

Lemma 3.4. *Let $N \subset T^*X$ be a closed submanifold, and assume $(x, \xi) \in N$ implies $(x, 0) \in N$. Then if ω is a closed k-form such that $\omega|_N = 0$, then there is a $(k-1)$-form $I(\omega)$ in a neighbourhood of N such that $\omega = dI(\omega)$.*

Proof. Let $m_s : T^*X \to T^*X$, $m_s : (x, \xi) \mapsto (x, s\xi)$, be multiplication by s in the fibres, and define

$$X_s = \left(\frac{d}{dr} m_r\right)\bigg|_{r=s}.$$

That is, in coordinates,

$$X_s = \frac{1}{s} \sum_j \xi_j \frac{\partial}{\partial_{\xi_j}}$$

is just $1/s$ times the radial vector field. Then

$$\frac{d}{dr}\left(m_r^* \omega\right)\bigg|_{r=s} = m_s^*(X_s \lrcorner d\omega) + d\left(m_s^*(X \lrcorner \omega)\right),$$

and integrating in r gives

$$\omega - m_0^* \omega = I(d\omega) + dI(\omega)$$

for

$$I(\omega) = \int_0^1 m_r^*(X_r \lrcorner \omega)\, dr.$$

Now $\omega|_N = 0$ and $d\omega = 0$ finishes the proof. $\square$

Theorem 3. *Suppose $N \subset T^*X$ is a closed submanifold such that $(x, \xi) \in N$ implies $(x, 0) \in N$, and assume $\kappa : \mathrm{neigh}(N) \to \kappa(\mathrm{neigh}(N))$ is a symplectomorphism which is smoothly homotopic in the symplectic group to identity on N. Then there is a bounded linear operator $F : L^2(\mathrm{neigh}(N)) \to L^2(\kappa(\mathrm{neigh}(N)))$ such that for all $A = \mathrm{Op}_h^w(a)$,*

$$AF = FB \quad \textit{microlocally on } \mathrm{neigh}(N) \times \kappa\left(\mathrm{neigh}(N)\right),$$

where $B = \mathrm{Op}_h^w(b)$ for a Weyl symbol $b = \kappa^* a + \mathcal{O}(h^2)$. Further, F is microlocally invertible and $F^{-1} A F = B$ in $N \times \kappa(N)$.

Proof. The proof will follow from the proof of Proposition 3.2. Let κ_t be the homotopy in the proposition, $\kappa_0 = \mathrm{id}$ and $\kappa_1 = \kappa$. We need only verify that κ_t is generated by a Hamiltonian. Set $V_t = \frac{d}{dt}\kappa_t$, and calculate

$$0 = \frac{d}{dt}\kappa_t^* \omega = \mathcal{L}_{V_t}\omega = V_t \lrcorner d\omega + d(V_t \lrcorner \omega).$$

Hence $\lambda_t = V_t \lrcorner \omega$ is closed and further $\lambda_t|_N = 0$ so we may apply Lemma 3.4 to obtain a 0-form $I(\lambda_t)$ so that

$$dI(\lambda_t) = \lambda_t$$

or

$$V_t = H_{I(\lambda_t)}. \qquad \square$$

We will make use of the following proposition (see [10, Section 10.5] for a proof).

Proposition 3.5. *Let $P \in \Psi_h^{k,0}(X)$ be a semiclassical operator of real principal type ($p = \sigma_h(P)$ is real and independent of h), and assume $dp \neq 0$ whenever $p = 0$. Then for any $\rho_0 \in \{p^{-1}(0)\}$, there exists a symplectomorphism $\kappa : T^*X \to T^*\mathbb{R}^n$ defined from a neighbourhood of ρ_0 to a neighbourhood of $(0,0)$ and an h-FIO T associated to its graph such that:*

(i) $\kappa^* \xi_1 = p$,
(ii) $TP = hD_{x_1}T$ *microlocally near* $(\rho_0; (0,0))$,
(iii) T^{-1} *exists microlocally near* $((0,0); \rho_0)$.

4. Symplectic geometry and quadratic forms

We now return to the setup of the introduction. Let $P(h)$ satisfy all the assumptions from Section 1. The main tool at our disposal is to use symplectomorphisms to transform the Weyl principal symbol into a different Weyl principal symbol which is in a more tractible form. Then by Propositions 3.2 and 3.3, any estimates we prove about the quantization of the transformed principal symbol will hold for the original operator modulo $\mathcal{O}(h^2)$.

It is classical (see, for example [1]) that using our assumptions on p, the Implicit Function theorem guarantees that there is an $\epsilon_0 > 0$ such that for $\epsilon \in [-\epsilon_0, \epsilon_0]$, the energy surface $\{p^{-1}(\epsilon)\}$ is regular and contains a closed loxodromic orbit γ^ϵ. Further,

$$\bar{\gamma} := \bigcup_{-\epsilon_0 \leqslant \epsilon \leqslant \epsilon_0} \gamma^\epsilon$$

is a smooth, 2-dimensional symplectic manifold diffeomorphic to $\mathbb{S}^1 \times [-\epsilon_0, \epsilon_0] \subset T^*\mathbb{S}^1$. Choose symplectic coordinates (t, τ, x, ξ) in a neighbourhood of $\bar{\gamma}$ so that γ is the image of the unit circle, $\mathbb{S}^1 \ni t \mapsto \gamma(t)$, t parametrizes γ^ϵ and $\gamma = \{t, 0; 0, 0\}$. In [1] it is shown that $S = \{t = 0\}$ is a contact manifold with the contact form $\tilde{\omega}_{(x,\xi)} = i^*\omega$, where $i : S \hookrightarrow X$ is the inclusion. Then the Poincaré map preserves p and $\tilde{\omega}$, modulo a term encompassing the period shift for $\epsilon \in [-\epsilon_0, \epsilon_0]$ different from zero and $(x, \xi) \neq (0, 0)$. This motivates our next change of variables. Similar to [25], we observe that τ depends only on the energy surface in which γ^ϵ lies: $\tau = g(\epsilon)$. H_p is tangent to the energy surface $\{p^{-1}(\epsilon)\}$ for each $\epsilon \in [-\epsilon_0, \epsilon_0]$, so that

$$\partial_t p(t,\tau,x,0) = \partial_t p(t,\tau,0,\xi) = 0 \qquad \text{and} \qquad \partial_x p(t,\tau,0,0) = 0, \quad \partial_\xi p(t,\tau,0,0) = 0,$$

so that

$$p(t,\tau,0,0) = f(\tau) \quad \text{and} \quad p(t,0,x,\xi) = f(0) + \mathcal{O}_t\big(x^2+\xi^2\big).$$

Thus, there exists a smooth nonvanishing function $a(t,\tau,x,\xi)$ defined in a neighbourhood of $\bar{\gamma}$ such that

$$a(t,\tau,x,\xi)p(t,\tau,x,\xi) = f(\tau) + \mathcal{O}_t\big(x^2+\xi^2\big).$$

Since the Hamiltonian vector field of p, H_p is tangent to $\{p = 0\}$, we can choose a Poincaré section contained in $\{p = 0\}$, that is, a $(2n-2)$-dimensional submanifold N, transverse to H_p on $\{p = 0\}$ centered at $\gamma(0)$. Let $S : N \to N$ be the Poincaré (first return) map near $\gamma(0)$. Note that $\omega = dt \wedge d\tau + \tilde{\omega}_{(x,\xi)}$ is the symplectic form on T^*X in our choice of coordinates, so S preserves the $(2n-2)$-dimensional symplectic form $\tilde{\omega}$ on N. Thus S is a symplectic mapping $N \to N$, with $S(0) = 0$. That γ is loxodromic means none of the eigenvalues of $dS(0)$ lie on the unit circle. In this section for simplicity we consider only the case where all the eigenvalues are distinct (the general case is handled in Section 5.2). We think of $dS(0)$ as the linearization of S near $0 \in N$, with N identified with T_0N near 0.

We want to put p into a normal form in a neighbourhood of γ. Inspiration for this construction comes from [12,25]. Let $q(\rho)$ be defined near $0 \in N$ and quadratic such that $dS(0) = \exp H_q$. Let κ_t be a smooth family of symplectomorphisms such that $\kappa_0 = \mathrm{id}$ while $\kappa_1 = S$. Then from the proof of Lemma 3.1 we can find $q_t(\rho)$ defined near $0 \in N$ so that

$$q_t(\rho) = q(\rho) + f_t(\rho)$$

with $f_t(\rho) = \mathcal{O}_t(|\rho|^3)$ and

$$\frac{d}{dt}\kappa_t = (\kappa_t)_* H_{q_t}.$$

Remark. Here we see the first obstacle to extending these techniques to include negative real eigenvalues: we want to write $dS(0) = \exp H_q$ for a real quadratic form q. But this is impossible for some linear symplectic transformations with negative eigenvalues as the example

$$dS(0) = \begin{pmatrix} -e^2 & 0 \\ 0 & -e^{-2} \end{pmatrix}$$

shows. Here $dS(0)$ is symplectic, but cannot be written as $\exp H_q$ with q real. Roughly, negative eigenvalues may be realized only by deforming a family of symplectomorphisms κ_t *through an elliptic component*.

Set $\tilde{p}(s, \sigma, \rho) = \sigma + q_s(\rho)$. We will show p and $\tilde{p}$ are equivalent under a symplectic change of coordinates on the set $p^{-1}(0)$. Then since both p and $\tilde{p}$ have nonvanishing differentials, we can write

$$\kappa^* p = b(t, \tau, x, \xi)\tilde{p} \tag{4.1}$$

for a smooth, positive function b and a symplectomorphism κ. Indeed, we claim

$$\exp(tH_p)(s, \sigma, \rho) = \bigl(s + t, \sigma_t(\rho, s, \sigma), \kappa_{t+s} \circ \kappa_s^{-1}(\rho)\bigr)$$

for some $\sigma_t(s, \sigma, \rho)$, giving (4.1). To see this, set

$$\Phi_t(s, \rho) := \bigl(s + t, \kappa_{t+s} \circ \kappa_s^{-1}(\rho)\bigr).$$

We need to check that $\Phi_t|_{N\times\mathbb{S}^1}$ is a 1-parameter group. We compute

$$\Phi_{t_1+t_2}|_{N\times\mathbb{S}^1}(s, \rho) = \bigl(s + t_1 + t_2, \kappa_{t_1+t_2+s} \circ \kappa_s^{-1}(\rho)\bigr).$$

But we check

$$\begin{aligned}\Phi_{t_1}|_{N\times\mathbb{S}^1} \circ \Phi_{N\times\mathbb{S}^1}(s, \rho) &= \Phi_{t_1}|_{N\times\mathbb{S}^1}\bigl(s + t_2, \kappa_{t_2+s} \circ \kappa_s^{-1}(\rho)\bigr)\\ &= \bigl(s + t_1 + t_2, \kappa_{t_1+t_2+s} \circ \kappa_{t_2+s}^{-1}\bigl(\kappa_{t_2+s} \circ \bigl(\kappa_{t_2+s} \circ \kappa_s^{-1}(\rho)\bigr)\bigr)\bigr),\end{aligned}$$

so the group law holds. We need only verify that p and $\tilde{p}$ have the same Poincaré map, so we check:

$$\left(\frac{d}{dt}\Phi_t|_{N\times\mathbb{S}^1}(s, \rho)\right)\bigg|_{t=0} = \bigl(1, H_{q_s}(\rho)\bigr),$$

which is clear. Note this construction depends only on the Poincaré map S and is unique up to symplectomorphism.

Next we want to examine what form the quadratic part $q(\rho)$ can take. The fact that $S(0) = 0$ implies we can write

$$q(\rho) = \frac{1}{2}\bigl\langle q''(0)\rho, \rho\bigr\rangle. \tag{4.2}$$

Now we define the Hamilton matrix B by

$$q(\rho) = \frac{1}{2}\tilde{\omega}(\rho, B\rho) \tag{4.3}$$

so that the symplectic transpose of B, ${}^{\tilde{\omega}}B$, is equal to $-B$. Note that B is the matrix representation of H_q, and so has eigenvalues which are the logarithms (with a suitably chosen branch cut) of the eigenvalues of $dS(0)$. Thus the condition that γ be loxodromic implies none of the eigenvalues of B have nonzero real part. Recall that since $dS(0)$ is a symplectic transformation, if μ is an eigenvalue of $dS(0)$, then so are $\bar{\mu}$, μ^{-1}, and $\bar{\mu}^{-1}$. This implies for the corresponding Hamilton matrix B in (4.3), if λ is an eigenvalue of B, then so are $-\lambda$, $\bar{\lambda}$, and $-\bar{\lambda}$. Thus the analysis of

B in the loxodromic, or complex hyperbolic case amounts to analyzing the eigenvalues in sets of 2 or 4. For this we follow the appendix in [16], and recall for this section we are assuming the eigenvalues are distinct. There are 2 cases. First, assume $\lambda_j > 0$ is real. Then $-\lambda_j$ is also an eigenvalue. Let e_j and f_j be the respective eigenvectors such that $\tilde{\omega}(e_j, f_j) = 1$. Then e_j and f_j span a real symplectic vector space of dimension 2. For a point ρ in this vector space, write $\rho = x_j e_j + \xi_j f_j$. Then (x_j, ξ_j) are symplectic coordinates, in which $q_j(\rho)$, the projection of q onto the jth coordinates becomes $q_j(\rho) = \lambda_j x_j \xi_j$. We call the

$$\lambda_j x_j \xi_j$$

the *action variables*.

Now we would like to see what these actions look like when the eigenvalues have nonzero imaginary part. Suppose λ_j is an eigenvalue with $\operatorname{Re}\lambda_j > 0$, $\operatorname{Im}\lambda_j > 0$. Then $-\lambda_j$, $\bar{\lambda}_j$, and $-\bar{\lambda}_j$ are eigenvalues. Let e_j, f_j, $\bar{e}_j$, and $\overline{f}_j$ be the respective eigenvectors. Note $\tilde{\omega}(e_j, \bar{e}_j) = \tilde{\omega}(e_j, \overline{f}_j) = \tilde{\omega}(f_j, \overline{f}_j) = 0$. Scale f_j so that $\tilde{\omega}(e_j, f_j) = 1$. Then $\{e_j, f_j\}$ and $\{\bar{e}_j, \overline{f}_j\}$ span complex conjugate symplectic vector spaces of complex dimension 2. Thus $\{e_j, \bar{e}_j, f_j, \overline{f}_j\}$ span a symplectic vector space of complex dimension 4 which is the complexification of a real symplectic vector space. Write a point ρ in this space in this basis, $\rho = z_j e_j + \zeta_j f_j + w_j \bar{e}_j + \eta_j \overline{f}_j$. Then $(z_j, \zeta_j, w_j, \eta_j)$ become symplectic coordinates, in which the projection q_j becomes $q_j(\rho) = \lambda_j z_j \zeta_j + \bar{\lambda}_j w_j \eta_j$. Now write

$$e_j = \frac{1}{\sqrt{2}}\left(e_j^1 + i e_j^2\right), \qquad f_j = \frac{1}{\sqrt{2}}\left(f_j^1 - i f_j^2\right)$$

for real e_j^k, f_j^k. This is a symplectic change of basis, and writing ρ in this basis:

$$\rho = z_j e_j + \zeta_j f_j + w_j \bar{e}_j + \eta_j \overline{f}_j = \sum_{k=1}^{2} \left(x_j^k e_j^k + \xi_j^k f_j^k\right),$$

we have

$$q_j(\rho) = \operatorname{Re}\lambda_j \left(x_j^1 \xi_j^1 + x_j^2 \xi_j^2\right) - \operatorname{Im}\lambda_j \left(x_j^1 \xi_j^2 - x_j^2 \xi_j^1\right).$$

This is summarized in the following proposition (using the notation of [16]). Let n_{hc} be the number of complex hyperbolic eigenvalues μ_j of $dS(0)$ with $|\mu_j| > 1$, and n_{hr} the number of real hyperbolic eigenvalues μ_j of $dS(0)$ such that $\mu_j > 1$. Thus we have $2n - 2 = 4n_{hc} + 2n_{hr}$.

Proposition 4.1. *Let $p \in \mathcal{C}^\infty(T^*X)$, $\gamma \subset \{p = 0\}$ as in Section* 1, *with the linearized Poincaré map having distinct eigenvalues μ_j not on the unit circle. Assume for $1 \leqslant j \leqslant n_{hc}$ we have $|\mu_j| > 1$ and $\operatorname{Im}\mu_j > 0$, and for $2n_{hc} + 1 \leqslant j \leqslant 2n_{hc} + n_{hr}$ we have $\mu_j > 1$. Then there exists a neighbourhood, U, of γ in T^*X, a smooth positive function $b \geqslant C^{-1} > 0$ defined in U, and a symplectomorphism $\kappa : U \to \kappa(U) \subset T^*\mathbb{S}^1_{(t,\tau)} \times T^*\mathbb{R}^{n-1}_{(x,\xi)}$ such that*

$$\kappa(\gamma) = \left\{(t, 0; 0, 0)\colon\ t \in \mathbb{S}^1\right\},$$

and $b(t, \tau, x, \xi) p = \kappa^*(g + r)$, with

$$\begin{aligned} g(t,\tau,x,\xi) \\ &= \tau + \sum_{j=1}^{n_{hc}} \big(\operatorname{Re}\lambda_j(x_{2j-1}\xi_{2j-1} + x_{2j}\xi_{2j}) - \operatorname{Im}\lambda_j(x_{2j-1}\xi_{2j} - x_{2j}\xi_{2j-1})\big) && (4.4)\\ &\quad + \sum_{j=2n_{hc}+1}^{2n_{hc}+n_{hr}} \lambda_j x_j \xi_j, \quad \textit{with } 2n_{hc}+n_{hr} = n-1 \quad \textit{and} && (4.5)\\ r &= \mathcal{O}\big(|x|^3 + |\xi|^3\big). \end{aligned}$$

Here $\lambda_j = \log(\mu_j)$ *for* $|\mu_j| > 1$ *and* $\operatorname{Im}\lambda_j \geqslant 0$.

Remark. The quadratic form (4.4)–(4.5) in Proposition 4.1 is the leading part of the real Birkhoff normal form for a symplectomorphism near a loxodromic fixed point. With a non-resonance condition and the addition of some higher order "action" variables (see, for example, [14,16]), the error r could be taken to be

$$r = \mathcal{O}\big(|x|^4 + |\xi|^4\big),$$

or even $\mathcal{O}(|x|^\infty + |\xi|^\infty)$.

Remark. We think of $p(t,\tau,x,\xi) \in \mathcal{C}^\infty(\mathbb{R}^4)$, $p = \tau + \lambda x\xi$, for $\lambda > 0$ as our "model case." The feature we are going to exploit about this model case is that if $G(t,\tau,x,\xi) = \frac{1}{2}(x^2 - \xi^2)$, then

$$H_p G = \lambda\big(x^2 + \xi^2\big), \tag{4.6}$$

which is a positive definite quadratic form. However, the growth of $x^2 - \xi^2$ will force us to use instead $G(x,\xi) = \log(1+x^2) - \log(1+\xi^2)$. Suppose $p = \tau + \lambda x\xi + x^3 - \xi^3 = \tau + \lambda x\xi + \mathcal{O}(x^3 + \xi^3)$ in a neighbourhood of γ of size $\epsilon > 0$ as in Proposition 4.1. Then

$$H_p G = \lambda\frac{x^2}{1+x^2} + \lambda\frac{\xi^2}{1+\xi^2} + 3\frac{\xi^2 x}{1+x^2} + 3\frac{x^2\xi}{1+\xi^2}.$$

Motivated by (4.6), we would like to write this as

$$H_p G = \lambda\frac{x^2}{1+x^2}\big(1+\mathcal{O}(\epsilon)\big) + \lambda\frac{\xi^2}{1+\xi^2}\big(1+\mathcal{O}(\epsilon)\big),$$

which we clearly cannot do in this example.

As the last remark indicates, in order to deal with the error terms, we will need a more refined form than that given in Proposition 4.1. Inspiration for this development, and in particular Proposition 4.3 comes from [11,22].

Let $\{\mu_j\}$ be the eigenvalues of the linearized Poincaré map at $\gamma(0)$. They come in pairs μ_j, μ_j^{-1} for the real μ_j and in sets of four μ_j, $\overline{\mu_j}$, μ_j^{-1}, and $\overline{\mu_j}^{-1}$ for the complex μ_j. The Stable/Unstable Manifold theorem guarantees we will get two n-dimensional, transversal, flow-invariant sub-manifolds Λ_+ and Λ_- such that $\exp tH_p$ is expanding on Λ_+ and contracting on Λ_-. Since the $\Lambda_\pm$ are invariant under the flow $\Phi_t = \exp tH_p$ which is symplectic, the symplectic form ω vanishes on the $\Lambda_\pm$, that is, the $\Lambda_\pm$ are Lagrangian sub-manifolds.

Lemma 4.2. *Assume p is in the form of Proposition* 4.1. *Then there exists a local symplectic coordinate system (t, τ, x, ξ) near γ such that $\Lambda_+ = \{\tau = 0, \xi = 0\}$ and $\Lambda_- = \{\tau = 0, x = 0\}$.*

Proof. We claim the $\Lambda_\pm$ are orientable and embedded in $T^*\mathbb{S}^1 \times T^*\mathbb{R}^{n-1}$. Since $dS(0)$ describes how the flow of H_p has acted at time $t = 1$, we know the evolution of a tangent frame of $\Lambda_\pm$ will be described by $dS(0)$. Using the action variables in Proposition 4.1, we have

$$dS(0) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

with

$$A = \operatorname{diag}\big(\mu_1, \overline{\mu_1}, \ldots, \mu_{n_{hc}}, \overline{\mu_{n_{hc}}}; \mu_{2n_{hc}+1}, \ldots, \mu_{2n_{hc}+n_{hr}}\big),$$

describing the time 1 evolution of Λ_+ and $|\mu_j| > 1$ for each $1 \leqslant j \leqslant n_{hr} + n_{hc}$ by our choice of coordinates. Similarly,

$$B = \operatorname{diag}\big(\mu_1^{-1}, \overline{\mu_1}^{-1}, \ldots, \mu_{n_{hc}}^{-1}, \overline{\mu_{n_{hc}}}^{-1}; \mu_{2n_{hc}+1}^{-1}, \ldots, \mu_{2n_{hc}+n_{hr}}^{-1}\big)$$

describes the time 1 evolution of Λ_- with $|\mu_j^{-1}| < 1$ for each j. But we have assumed there are no negative real eigenvalues, so $\det A > 0$ implies Λ_+ is orientable. Similarly, $\det B > 0$ and Λ_- is orientable. Now our assumptions on p mean the flow has no critical points in a neighbourhood of γ so the $\Lambda_\pm$ can have no self-intersections and hence are embedded.

Let $\tilde{\Lambda} \subset T^*\mathbb{S}^1 \times T^*\mathbb{R}^{n-1}$, $\tilde{\Lambda} = \{\tau = 0, \xi = 0\}$. Since Λ_+ is a closed, n-dimensional submanifold of T^*X, the tubular neighbourhood theorem guarantees there is a diffeomorphism f (not necessarily symplectic) taking a neighbourhood U of γ into itself so that f fixes t and

$$f(\Lambda_+ \cap U) = \tilde{\Lambda} \cap U.$$

Further, since $T_{\gamma(t)}\Lambda_+ = T_{\gamma(t)}\tilde{\Lambda}$ for $0 \leqslant t \leqslant 1$, we can choose f satisfying

$$\big[\big(f^{-1}\big)^*\tilde{\omega}\big]_{\gamma(t)} = \tilde{\omega}_{\gamma(t)}, \quad 0 \leqslant t \leqslant 1. \tag{4.7}$$

The statement in the lemma about Λ_+ now follows directly from the more general Theorem 4.1 in [29], but we include a proof of this concrete case. We have $\tilde{\Lambda} \subset T^*\mathbb{S}^1 \times T^*\mathbb{R}^{n-1}$, a Lagrangian submanifold with two distinct symplectic structures, $\omega_0 = (f^{-1})^*\tilde{\omega}$ and the standard symplectic structure ω_1 inherited from $T^*\mathbb{S}^1 \times T^*\mathbb{R}^{n-1}$. We want to find a diffeomorphism $g : U \to U$ such that $g(\tilde{\Lambda}) = \tilde{\Lambda}$ and $g^*\omega_1 = \omega_0$.

Set $\omega_s = s\omega_0 + (1-s)\omega_1$. We have $d\omega_s = 0$ and $\omega_s|_{\tilde{\Lambda}} = 0$. Note (4.7) implies ω_s is nondegenerate in a neighbourhood of γ for $0 \leqslant s \leqslant 1$. Let $\hat{\omega}_s : TX \to T^*X$ denote the isomorphism generated by ω_s, $\hat{\omega}_s : Z \mapsto Z \lrcorner \omega_s$. We use the general Poincaré Lemma 3.4 to obtain a 1-form $\varphi = i(\omega_0 - \omega_1)$ so that $d\varphi = \omega_0 - \omega_1$ and set $Y_s = \hat{\omega}_s^{-1}(\varphi)$. Then $\varphi|_{\tilde{\Lambda}} = 0$ implies

$$Y_s \lrcorner \omega_s = \hat{\omega}(Y_s) = \varphi,$$

so that Y_s is tangent to $\tilde{\Lambda}$. Thus if $g_s = \exp(sY_s)$ for $0 \leqslant s \leqslant 1$ is the integral of Y_s, $g_s(\tilde{\Lambda}) = \tilde{\Lambda}$. We calculate:

$$\begin{aligned} \frac{d}{dr}\big(g_r^* \omega_r\big)\Big|_{r=s} &= g_s^*\left(\frac{d}{dr}\omega_r\right)\Big|_{r=s} + g_s^*\big(d(Y_s \lrcorner \omega_s)\big) \\ &= g_s^*\big(\omega_0 - \omega_1 + d(-\varphi)\big) \\ &= 0. \end{aligned}$$

Setting $g = g_1$ gives $g^*\omega_1 = \omega_0$ as desired. Now taking $g^{-1} \circ f$ gives a diffeomorphism of a neighbourhood of γ taking Λ_+ to $\tilde{\Lambda}$ such that $g^* \circ (f^{-1})^* \tilde{\omega} = \tilde{\omega}$.

After this change of coordinates, we still need to put Λ_- in the desired form. Since Λ_- is transversal to Λ_+ and all of our transformations so far leave $\{\tau = 0\}$ invariant, we can write Λ_- as a graph over $\{x = 0\}$:

$$\Lambda_- = \big\{(t, 0, x, \xi)\colon\ x = g(\xi, t)\big\}. \tag{4.8}$$

Further, since for each fixed t, (4.8) is Lagrangian and the first de Rham cohomology group $H^1_{dR}(\{\tau = 0, x = 0\}) \simeq H^1_{dR}(\mathbb{R}^{n-1})$ vanishes, it is classical that we can write $g(\xi, t) = \partial_\xi h(\xi, t)$ for a smooth $h(\xi, t)$ (see, for example [20]). Then we write

$$\Lambda_- = \big\{(t, 0, x, \xi)\colon\ x = \partial_\xi h(\xi, t)\big\},$$

and observe h must satisfy $\partial_\xi h(0, t) = 0$. This determines h up to a constant, which we take to be 0 so that $h(0, t) = 0$. Now let $b(\xi, t)$ be a smooth function satisfying $b(\xi, t) = \partial_t h(\xi, t)$, and note $b(0, t) = 0$. Then we perform the following change of variables:

$$\begin{cases} t' = t, \\ \tau' = \tau + b(\xi, t), \\ x' = x - \partial_\xi h(\xi, t), \\ \xi' = \xi. \end{cases}$$

We calculate:

$$\begin{aligned} d\tau' \wedge dt' + d\xi' \wedge dx' &= \left(d\tau + \sum_j \partial_{\xi_j} b(\xi, t)\, d\xi_j + \partial_t b(\xi, t)\, dt\right) \wedge dt \\ &\quad + \sum_j d\xi_j \wedge \left(dx_j - \sum_i \partial_{\xi_i}\partial_{\xi_j} h(\xi, t)\, d\xi_i - \partial_t \partial_{\xi_j} h(\xi, t)\, dt\right) \\ &= d\tau \wedge dt + d\xi \wedge dx, \end{aligned}$$

by the symmetry of the Hessian $\partial_{\xi_i}\partial_{\xi_j} h(\xi, t)$. Thus this change of variables is symplectic and the lemma is proved. □

Using the change of variables in Lemma 4.2, we have the following proposition.

Proposition 4.3. *Let $p \in \mathcal{C}^\infty(T^*X)$, $\gamma \subset \{p = 0\}$ as above, with the Poincaré map having distinct eigenvalues μ_j not on the unit circle. Then there exists a neighbourhood U of γ in T^*X, a smooth positive function $b \geqslant C^{-1} > 0$ defined in U, a symplectomorphism $\kappa : U \to \kappa(U) \subset T^*\mathbb{S}^{-1}_{(t,\tau)} \times T^*\mathbb{R}^{n-1}_{(x,\xi)}$, and a smooth, $n \times n$-matrix valued function B_t such that*

$$\kappa(\gamma) = \{(t, 0; 0, 0): t \in \mathbb{S}^1\}, \quad \text{and} \quad b(t, \tau, x, \xi)p = \kappa^* g, \quad \text{with}$$
$$g(t, \tau; x, \xi) = \tau + \langle B_t(x, \xi)x, \xi\rangle, \tag{4.9}$$

with B_t satisfying

$$\langle B_t(0, 0)x, \xi\rangle = \sum_{j=1}^{n_{hc}} \big(\operatorname{Re}\lambda_j(x_{2j-1}\xi_{2j-1} + x_{2j}\xi_{2j}) - \operatorname{Im}\lambda_j(x_{2j-1}\xi_{2j} - x_{2j}\xi_{2j-1})\big) \tag{4.10}$$
$$+ \sum_{j=2n_{hc}+1}^{2n_{hc}+n_{hr}} \lambda_j x_j \xi_j. \tag{4.11}$$

Here $\lambda_j = \log(\mu_j)$ for $|\mu_j| > 1$ and $\operatorname{Im}\lambda_j \geqslant 0$.

Proof. Recall that the Poincaré map S is linear in lowest order, and let $dS(0)$ be the linearized map. Let q_0 satisfy $dS(0) = \exp H_{q_0}$. After a linear symplectic change of variables, q_0 can be written in block-diagonal form

$$\begin{aligned} q_0(x, \xi) &= \langle bx, \xi\rangle \\ &= \sum_{j=1}^{n_{hc}} \big(\operatorname{Re}\lambda_j(x_{2j-1}\xi_{2j-1} + x_{2j}\xi_{2j}) - \operatorname{Im}\lambda_j(x_{2j-1}\xi_{2j} - x_{2j}\xi_{2j-1})\big) \\ &\quad + \sum_{j=2n_{hc}+1}^{2n_{hc}+n_{hr}} \lambda_j x_j \xi_j, \quad \text{with } 2n_{hc} + n_{hr} = 2n - 2. \end{aligned}$$

According to Lemma 4.2, we may symplectically change variables so $\Lambda_+ = \{\tau = 0, \xi = 0\}$ and $\Lambda_- = \{\tau = 0, x = 0\}$. The linearization of the Hamiltonian vector field of p is H_{q_0}, which implies we have a quadratic form as in the proposition. □

5. Proof of Theorem 1

Proof of Theorem 1 with distinct eigenvalues. First we assume $P(h)$ has principal symbol given by

$$p(t, \tau; x, \xi) = \tau + \langle B_t(x, \xi)x, \xi\rangle, \tag{5.1}$$

with B_t satisfying (4.10), (4.11) as in Proposition 4.3. Let U be a neighbourhood of γ, $U \subset T^*\mathbb{S}^1 \times T^*\mathbb{R}^{n-1}$, and assume

$$U \subset U_{\epsilon/2} := \left\{ (t, \tau, x, \xi)\colon \left(d\bigl(x, x(\gamma(t))\bigr)^2 + \bigl|\xi - \xi(\gamma(t))\bigr|^2 + \tau^2 \right)^{\frac{1}{2}} < \frac{\epsilon}{2} \right\}$$

for $\epsilon > 0$. Let ψ_0 be a microlocal cutoff function to a neighbourhood of U, that is, take $\psi_0 \in \mathcal{C}_c^\infty(\mathbb{R}^{2n})$, $\psi_0 \equiv 1$ on $U_{\epsilon/2}$ with support in U_ϵ. Then we assume throughout that we are working in U_ϵ. With $\tilde{h}$ small (fixed later in the proof), we do the following rescaling:

$$X := (\tilde{h}/h)^{\frac{1}{2}} x, \qquad \varXi = (\tilde{h}/h)^{\frac{1}{2}} \xi, \tag{5.2}$$

and assume for the remainder of the proof that $|(X, \varXi)| \leqslant (\tilde{h}/h)^{1/2}\epsilon$. We use the unitary operator $T_{h,\tilde{h}}$ defined in (2.6) to introduce the second parameter into $P(h)$. Following [6] we define the operator $\widetilde{P}(h)$ by

$$\widetilde{P}(h) = T_{h,\tilde{h}} P(h)(h) T_{h,\tilde{h}}^{-1},$$

so that the principal symbol of $\widetilde{P}(h)$ is

$$\tilde{p}(t, \tau; X, \varXi) = \tau + \bigl\langle B_t\bigl((h/\tilde{h})^{\frac{1}{2}}(X, \varXi)\bigr)(h/\tilde{h})^{\frac{1}{2}} X, (h/\tilde{h})^{\frac{1}{2}} \varXi \bigr\rangle, \tag{5.3}$$

and $\tilde{p} \in \mathcal{S}_{-1/2}^{-\infty,0,0}$ microlocally. We have

$$\bigl|\partial_{X,\varXi}^\alpha \tilde{p}\bigr| \leqslant C_\alpha (h/\tilde{h})^{|\alpha|/2} \tag{5.4}$$

for $(X, \varXi) \in U_{(\tilde{h}/h)^{1/2}\epsilon}$ by Lemma 2.4.

We will use the following escape function, which we define in the $(X, \varXi)$ coordinates:

$$G(X, \varXi) := \frac{1}{2}\bigl(\log\bigl(1 + |X|^2\bigr) - \log\bigl(1 + |\varXi|^2\bigr)\bigr).$$

G satisfies

$$\bigl|\partial_X^\alpha \partial_X^\beta G(X, \varXi)\bigr| \leqslant C_{\alpha\beta} \langle X\rangle^{-|\alpha|} \langle \varXi\rangle^{-|\beta|} \quad \text{for } (\alpha, \beta) \neq (0, 0),$$

and since $\langle X\rangle^2 \langle \varXi\rangle^{-2}$ is an order function, G satisfies the assumptions of Lemma 2.1 so we may construct the family e^{sG^ω} for sufficiently small s.

Now for $|(X, \varXi)| \leqslant (\tilde{h}/h)^{1/2}\epsilon$ we have

$$\begin{aligned} &H_{\tilde{p}} G(X, \varXi) \\ &\quad = (h/\tilde{h}) \left[\left\langle B_t X, \frac{\partial}{\partial X} \right\rangle - \left\langle B_t \frac{\partial}{\partial \varXi}, \varXi \right\rangle \right] G(X, \varXi) &(5.5)\\ &\qquad + (h/\tilde{h})^{\frac{3}{2}} \left[\sum_{j=1}^{n-1} \left\langle \frac{\partial}{\partial \varXi_j} B_t(\cdot,\cdot) X, \varXi \right\rangle \frac{\partial}{\partial X_j} G(X, \varXi) \right] &(5.6)\\ &\qquad - (h/\tilde{h})^{\frac{3}{2}} \left[\sum_{j=1}^{n-1} \left\langle \frac{\partial}{\partial X_j} B_t(\cdot,\cdot) X, \varXi \right\rangle \frac{\partial}{\partial \varXi_j} G(X, \varXi) \right]. &(5.7) \end{aligned}$$

For s sufficiently small, we define a family of operators

$$\begin{aligned}\widetilde{P}_s(h) &= e^{-sG^\omega}\widetilde{P}(h)\mathrm{Op}_{\tilde{h}}^w\big(\psi_0\big((h/\tilde{h})^{\frac{1}{2}}\bullet\big)\big)e^{sG^\omega}\\ &= \exp(-s\,\mathrm{ad}_{G^\omega})\widetilde{P}(h)\mathrm{Op}_{\tilde{h}}^w\big(\psi_0\big((h/\tilde{h})^{\frac{1}{2}}\bullet\big)\big),\end{aligned} \tag{5.8}$$

where $\mathrm{Op}_{\tilde{h}}^w$ and G^w are quantizations in the $\tilde{h}$-Weyl calculus. Now owing to Lemma 2.5 and (5.4) we have microlocally to leading order in h:

$$\mathrm{ad}_{G^\omega}^k\big(\widetilde{P}\mathrm{Op}_{\tilde{h}}^w\big(\psi_0\big((h/\tilde{h})^{\frac{1}{2}}\bullet\big)\big)\big) = \mathcal{O}_{L^2\to L^2}\big(h\tilde{h}^{k-1}\big),$$

and in particular,

$$\big[\widetilde{P}(h), G^w\big] = -i\tilde{h}\,\mathrm{Op}_{\tilde{h}}^w(H_{\tilde{p}}G) + \mathcal{O}\big(h^{3/2}\tilde{h}^{3/2}\big). \tag{5.9}$$

Now near $(0,0)$, B_t is positive definite, $\langle B_tX, X\rangle \geqslant C^{-1}|X|^2$, so

$$\langle B_tX, X\rangle^{-1} \leqslant C|X|^{-2}.$$

Applying this to the errors (5.6), (5.7) we get

$$(h/\tilde{h})^{\frac{3}{2}}\left[\sum_{j=1}^{n-1}\left\langle\frac{\partial}{\partial\Xi_j}B_t(\cdot,\cdot)X, \Xi\right\rangle\frac{\partial}{\partial X_j}G(X,\Xi)\right] = (h/\tilde{h})^{\frac{3}{2}}\frac{|X|^2}{1+|X|^2}\mathcal{O}\big(|\Xi|\big)$$

and similarly for (5.7). Adding these to (5.5), we get

$$H_{\tilde{p}}G = (h/\tilde{h})\left[\frac{\langle B_tX, X\rangle}{1+|X|^2}\right]\big(1+(h/\tilde{h})^{\frac{1}{2}}\mathcal{O}\big(|\Xi|\big)\big) \tag{5.10}$$

$$+ (h/\tilde{h})\left[\frac{\langle B_t\Xi, \Xi\rangle}{1+|\Xi|^2}\right]\big(1+(h/\tilde{h})^{\frac{1}{2}}\mathcal{O}\big(|X|\big)\big). \tag{5.11}$$

Now we expand B_t in a Taylor series about $(0,0)$ to get

$$\begin{aligned}H_{\tilde{p}}G &= (h/\tilde{h})\left[\frac{\langle B_t(0,0)X, X\rangle}{1+|X|^2} + (h/\tilde{h})^{\frac{1}{2}}\frac{|X|^2}{1+|X|^2}\mathcal{O}\big(\big|(X,\Xi)\big|\big)\right]\big(1+(h/\tilde{h})^{\frac{1}{2}}\mathcal{O}\big(|\Xi|\big)\big)\\ &\quad+ (h/\tilde{h})\left[\frac{\langle B_t(0,0)\Xi, \Xi\rangle}{1+|\Xi|^2} + (h/\tilde{h})^{\frac{1}{2}}\frac{|\Xi|^2}{1+|\Xi|^2}\mathcal{O}\big(\big|(X,\Xi)\big|\big)\right]\big(1+(h/\tilde{h})^{\frac{1}{2}}\mathcal{O}\big(|X|\big)\big),\end{aligned}$$

which can again be written as (5.10), (5.11). Recalling that $B_t(0,0)$ is block diagonal of the form (4.10), (4.11), we get for $|(X,\Xi)| \leqslant (\tilde{h}/h)^{1/2}\epsilon$,

$$\begin{aligned} &H_{\tilde{p}}G(X,\Xi)\\ &= \left[\sum_{j=1}^{n_{hc}} \operatorname{Re}\lambda_j \left(\frac{X_{2j}^2 + X_{2j-1}^2}{1+|X|^2} + \frac{\Xi_{2j}^2 + \Xi_{2j-1}^2}{1+|\Xi|^2} \right)\right] \big(1 + \tilde{h}^{-\frac{1}{2}}\mathcal{O}(\epsilon)\big) \qquad (5.12)\\ &\quad + \left[\sum_{j=2n_{hc}+1}^{2n_{hc}+n_{hr}} \lambda_j \left(\frac{X_j^2}{1+|X|^2} + \frac{\Xi_j^2}{1+|\Xi|^2} \right)\right] \big(1 + \tilde{h}^{-\frac{1}{2}}\mathcal{O}(\epsilon)\big). \qquad (5.13) \end{aligned}$$

Thus

$$\widetilde{P}_s(h) = \widetilde{P}(h) - ish\big(A(1+E_0)\big)^w + sE_1^w + s^2E_2^w \tag{5.14}$$

with $E_0 = \mathcal{O}(\tilde{h}^{-1/2}\epsilon)$, $E_1 = \mathcal{O}(h^{3/2}/\tilde{h}^{3/2})$, $E_2 = \mathcal{O}(h\tilde{h})$, and $A^w = \mathrm{Op}_{\tilde{h}}^w(A)$ for

$$\begin{aligned} &A(X,\Xi)\\ &:= \sum_{j=1}^{n_{hc}} \operatorname{Re}\lambda_j \left(\frac{X_{2j}^2 + X_{2j-1}^2}{1+|X|^2} + \frac{\Xi_{2j}^2 + \Xi_{2j-1}^2}{1+|\Xi|^2} \right) \qquad (5.15)\\ &\quad + \sum_{j=2n_{hc}+1}^{2n_{hc}+n_{hr}} \lambda_j \left(\frac{X_j^2}{1+|X|^2} + \frac{\Xi_j^2}{1+|\Xi|^2} \right). \qquad (5.16) \end{aligned}$$

We claim that for $\tilde{h}$ sufficiently small,

$$\langle A^w U, U\rangle \geqslant \frac{\tilde{h}}{C}\|U\|^2 \tag{5.17}$$

for some constant $C > 0$, which is essentially the lower bound for the harmonic oscillator $\tilde{h}^2 D_X^2 + X^2$. Clearly it suffices to prove this inequality for individual j for the real hyperbolic terms (5.16), and in pairs for the complex hyperbolic terms (5.15), which is the content of Lemma 5.1.

Now fix $\tilde{h} > 0$ and $|s| > 0$ sufficiently small so that the estimate (5.17) holds and the errors E_1 and E_2 satisfy

$$\|shA^wU\|_{L^2} \gg \big\|sE_1^wU\big\|_{L^2} + \big\|s^2E_2^wU\big\|_{L^2},$$

and fix $\epsilon > 0$ sufficiently small that the error $|E_0| \ll 1$, independent of $h > 0$.

We now have for smooth U satisfying $\mathrm{Op}_{\tilde{h}}^w(\psi_0(h^{\frac{1}{2}}\bullet))U = U + \mathcal{O}(h^\infty)$,

$$-\operatorname{Im}\big\langle \widetilde{P}_s(h)U, U\big\rangle \geqslant \frac{h\tilde{h}}{C}\|U\|^2. \tag{5.18}$$

Now define the operator K_h^w by $e^{sK_h^w} := T_{h,\tilde{h}}^{-1} e^{sG_{\tilde{h}}^\omega} T_{h,\tilde{h}}$. Translating back into original coordinates, and with $z \in [-1,1] + i(-c_0h + \infty)$ for sufficiently small $c_0 > 0$, (5.18) gives

$$-\operatorname{Im}\big\langle e^{sK_h^w}\big(P(h) - z\big)e^{-sK_h^w}u, u\big\rangle \geqslant \frac{h}{C_1}\|u\|^2.$$

Finally, since $\|\exp(\pm s K_h^w)\| = \mathcal{O}(h^{-N})$ for some N, the theorem follows in the case where p is of the form (5.1).

For general p, by Proposition 4.3, there is a symplectomorphism κ so that up to an elliptic factor, $\kappa^* p$ is of the form (5.1). Using Theorem 3 to quantize κ as an h-FIO F, we get

$$\mathrm{Op}_h^w(\kappa^* p + E_1) = F^{-1} P(h) F,$$

where $E_1 = \mathcal{O}(h^2)$ is the error arising from Theorem 3. We may then use the previous argument for $\kappa^* p$ getting an additional error of $\mathcal{O}(h^2)$ from Theorem 3 in (5.18), which is of the same order as E_1. □

Remark. The error arising at the end of the proof of Theorem 1 from the use of Theorem 3 is of order $\mathcal{O}(h^2)$ and hence negligible compared to our lower bound of h for A. However, the estimate of A is used for the imaginary part of $\widetilde{P}_s$, and the error in Theorem 3 is real, so $\mathcal{O}(h)$ would have been sufficient.

Lemma 5.1. *Let*

$$a_0(y, \eta) := \frac{y_j^2}{\langle y \rangle^2} + \frac{\eta_j^2}{\langle \eta \rangle^2},$$

for $(y, \eta) \in \mathbb{R}^{2n-2}$*, and* $\langle y \rangle = (1 + |y|^2)^{1/2}$*, and let*

$$a_1(y, \eta) := \frac{y_{2j}^2 + y_{2j-1}^2}{\langle y \rangle^2} + \frac{\eta_{2j}^2 + \eta_{2j-1}^2}{\langle \eta \rangle^2}.$$

Then a_i*,* $i = 0, 1$*, satisfies*

$$\left\langle \mathrm{Op}_{\tilde{h}}^w(a_i) U, U \right\rangle \geqslant \frac{\tilde{h}}{C} \|U\|^2 \tag{5.19}$$

or $\tilde{h} > 0$ *sufficiently small and a constant* $0 < C < \infty$*.*

Proof. The idea of the proof is that a_i is essentially the harmonic oscillator which satisfies the inequality (5.19). We write each a_i as a $a_i = |b|^2$ for b a complex symbol. Observe $a_0(y, \eta) = |b(y, \eta)|^2$ with

$$b(y, \eta) := \frac{y_j}{\langle y \rangle} + i \frac{\eta_j}{\langle \eta \rangle}.$$

Thus, using the $\tilde{h}$-Weyl calculus,

$$a_0^w(y, \tilde{h} D_y) = b^w(y, \tilde{h} D_y)^* b^w(y, \tilde{h} D_y) + c^w(y, \tilde{h} D_y), \tag{5.20}$$

where

$$c(y,\eta)=\tilde{h}\left\{\frac{\eta_j}{\langle\eta\rangle},\frac{y_j}{\langle y\rangle}\right\}+\mathcal{O}(\tilde{h}^2)$$
$$=\tilde{h}\langle y\rangle^{-3}\langle\eta\rangle^{-3}\big(1+\mathcal{O}(|y|^2+|\eta|^2)\big)+\mathcal{O}(|y|^2|\eta|^2)+\mathcal{O}(\tilde{h}^2). \tag{5.21}$$

For (y,η) small, c is bounded from below by $\tilde{h}$ as in (5.17), and for large (y,η) we have

$$C^{-1}\leqslant a_0\leqslant C$$

for some constant $C>0$. Hence for large (y,η), (5.20) is bounded from below independent of $\tilde{h}$. Observe $a_1(y,\eta)=|b_{2j}(y,\eta)|^2+|b_{2j-1}(y,\eta)|^2$ for

$$b_k(y,\eta)=\frac{y_k}{\langle y\rangle}-i\frac{\eta_k}{\langle\eta\rangle},$$

and the same argument applies to a_1 as to a_0. □

Remark. It is interesting to note that the estimate (1.4) depends only on the real parts of the eigenvalues λ_j above. Unraveling the definitions, the eigenvalues λ_j are logarithms of the eigenvalues of the linearized Poincaré map $dS(0)$ from above. Then (1.4) depends only on the moduli of the eigenvalues of $dS(0)$.

5.1. A return to quadratic forms

Recall the only place we have used that the eigenvalues are distinct is in determining the possible form of the quadratic form $q(\rho)$ defined by $dS(0)=\exp H_q$. We then considered the Hamilton, or fundamental matrix B defined by

$$q(\rho)=:\frac{1}{2}\tilde{\omega}(\rho,B\rho). \tag{5.1.1}$$

We follow [15] and return to the setup for Proposition 4.1. All of the following changes of variables will be linear, so we may assume we are working in $\mathbb{R}^{2n-2}$ and choose local symplectic coordinates in which $\tilde{\omega}$ is the standard symplectic form

$$\tilde{\omega}=\sum_{j=1}^{n-1}d\xi_j\wedge dx_j.$$

Then we can write (5.1.1) in a more easily manipulated form:

$$q(\rho)=:\frac{1}{2}\langle\rho,JB\rho\rangle,$$

where J is the matrix of symplectic structure on $\mathbb{R}^{2n-2}$,

$$J=\begin{pmatrix}0&-I\\I&0\end{pmatrix}.$$

As mentioned previously, the eigenvalues of B are the logarithms of the eigenvalues of $dS(0)$ (with a suitably chosen branch cut), hence have nonzero real part, and come in pairs λ, $-\lambda$ for the positive real hyperbolic eigenvalues, and 4-tuples λ, $-\lambda$, $\bar{\lambda}$, $-\bar{\lambda}$ for the complex hyperbolic. If we allow ρ to be complex for the moment, and denote by V_λ the generalized eigenspace for λ real or complex, we see

$$\tilde{\omega}(V_\lambda, V_{\lambda'}) = 0$$

unless $\lambda + \lambda' = 0$. We then consider the spaces $V_\lambda \oplus V_{-\lambda}$, which is symplectic with the restricted symplectic form $\tilde{\omega}|_{V_\lambda \oplus V_{-\lambda}}$, since $\lambda \neq 0$. As in Section 4 we choose the pairs and 4-tuples of eigenvalues so that $\operatorname{Re}\lambda > 0$ and $\operatorname{Im}\lambda \geqslant 0$. We thus have a decomposition of $\mathbb{R}^{2n-2}$ into symplectic subspaces

$$\mathbb{R}^{2n-2} = \left(\bigoplus_{j=1}^{n_{hc}} V_{\lambda_j} \oplus V_{-\lambda_j} \oplus V_{\overline{\lambda_j}} \oplus V_{-\overline{\lambda_j}} + \bigoplus_{j=n_{hc}+1}^{n_{hc}+n_{hr}} V_{\lambda_j} \oplus V_{-\lambda_j} \right),$$

where n_{hr} is the number of real eigenvalues with $\lambda > 0$ and n_{hc} is the number of complex eigenvalues with $\operatorname{Re}\lambda > 0$, $\operatorname{Im}\lambda > 0$. Our notation here means if λ_j has multiplicity k_j, then

$$\sum_{j=1}^{n_{hc}} 4k_j + \sum_{j=n_{hc}+1}^{n_{hc}+n_{hr}} 2k_j = 2n - 2.$$

Fix λ real or complex, $\operatorname{Re}\lambda > 0$, $\operatorname{Im}\lambda \geqslant 0$, with multiplicity greater than 1 and consider the complex symplectic subspace $V_\lambda \oplus V_{-\lambda}$. Assume V_λ has dimension m. Note B restricts to a linear map in V_λ, $T := B|_{V_\lambda}$, such that $T - \lambda I$ is nilpotent. Our definitions equip $V_\lambda \oplus V_{-\lambda}$ with a symplectic structure in which $V_{-\lambda}$ is dual and isomorphic to V_λ. We abuse notation and write a point $(x, \xi) \in V_\lambda \oplus V_{-\lambda}$. Then if we put T into Jordan form in V_λ so that $Tx = \lambda x + (x_2, x_3, \ldots, x_m, 0)$, we obtain a symplectic change of coordinates by writing

$$B|_{V_\lambda \oplus V_{-\lambda}}(x, \xi) = \big(\lambda x + (x_2, \ldots, x_m, 0), -\lambda\xi - (0, \xi_1, \xi_2, \ldots, \xi_{m-1})\big),$$

by the symplectic skew symmetry of B. In these coordinates we then have q_λ, the projection of q onto $V_\lambda \oplus V_{-\lambda}$,

$$q_\lambda(x, \xi) = \lambda \sum_{l=1}^{k} x_l \xi_l + \sum_{l=1}^{k-1} x_{l+1} \xi_l, \tag{5.1.2}$$

where k is the multiplicity of λ. This is the normal form in complex variables, with the "actions" $\lambda x_j \xi_j$ as in Section 4, but with the additional terms coming from the Jordan form. In order to understand the real normal form, there are two cases to examine.

Case 1: $\lambda > 0$ is real. Then the space $V_\lambda \oplus V_{-\lambda}$ is real, the change of variables above is real, and we get q_λ exactly as in (5.1.2). Let the real matrix Q_λ be defined by the real normal form:

$$q_\lambda(x, \xi) =: \frac{1}{2}\big\langle (x, \xi), Q(x, \xi)\big\rangle. \tag{5.1.3}$$

Then Q takes the special form

$$Q = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix},$$

where A is the $k \times k$ matrix

$$A = \begin{pmatrix} \lambda & 0 & \cdots & \cdots \\ 1 & \lambda & 0 & \cdots \\ 0 & \ddots & \ddots & 0 \\ \vdots & \cdots & 1 & \lambda \end{pmatrix} \tag{5.1.4}$$

and A^T denotes the transpose of A.

Case 2: λ complex, $\operatorname{Re}\lambda > 0$, $\operatorname{Im}\lambda > 0$. We use a similar change of variables to that in Section 4. That is, let $\{e_l, f_l\}$ be the generalized eigenvectors for λ, $-\lambda$, respectively. Here, $1 \leqslant l \leqslant k$ where k is the multiplicity of λ. Then $\{e_l, f_l, \bar{e}_l, \overline{f}_l\}$ forms a basis for a complex vector space which is the complexification of a real symplectic vector space. We then consider the projection q_λ of q onto the space

$$W = V_\lambda \oplus V_{-\lambda} \oplus V_{\bar{\lambda}} \oplus V_{-\bar{\lambda}}.$$

Write a point ρ in W as

$$\rho = \sum_{l=1}^{k} z_l e_l + \zeta_l f_l + w_l \bar{e}_l + \eta_l \overline{f}_l,$$

so that

$$q_\lambda(\rho) = \lambda \sum_1^k z_l \zeta_l + \bar{\lambda} \sum_1^k w_l \eta_l + \sum_1^{k-1} z_{l+1} \zeta_l + \sum_1^{k-1} w_{l+1} \eta_l.$$

We define as in Section 4 a real symplectic basis $\{e_l^1, e_l^2, f_l^1, f_l^2\}$ for $1 \leqslant l \leqslant k$ by

$$e_l = \frac{1}{\sqrt{2}}\big(e_l^1 + i e_l^2\big), \qquad f_l = \frac{1}{\sqrt{2}}\big(f_l^1 + i f_l^2\big),$$

and write in these new coordinates

$$\rho = \sum_{l=1}^{k} \sum_{r=1}^{2} x_l^r e_l^r + \xi_l^r r_l^r.$$

Then we get the real normal form of q_λ in these coordinates:

$$q_\lambda(\rho) = \operatorname{Re}\lambda \sum_1^k (x_{2l-1}\xi_{2l-1} + x_{2l}\xi_{2l}) - \operatorname{Im}\lambda \sum_1^k (x_{2l}\xi_{2l-1} - x_{2l-1}\xi_{2l}) \\ + \sum_1^{k-1} (x_{2l+1}\xi_{2l-1} + x_{2l+2}\xi_{2l}).$$

We again define the real matrix Q in terms of the real quadratic normal form q_λ by (5.1.3), which now takes the form

$$Q = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix},$$

where A is the $2k \times 2k$ matrix

$$\begin{pmatrix} \Lambda & 0 & \cdots & \cdots \\ 1 & \Lambda & 0 & \cdots \\ 0 & \ddots & \ddots & 0 \\ \vdots & \cdots & I & \Lambda \end{pmatrix} \tag{5.1.5}$$

with I the 2×2 identity matrix and

$$\Lambda = \begin{pmatrix} \operatorname{Re}\lambda & -\operatorname{Im}\lambda \\ \operatorname{Im}\lambda & \operatorname{Re}\lambda \end{pmatrix}.$$

Putting this discussion together with the proof of Proposition 4.1, we have proved the following proposition.

Proposition 5.2. *Let $p \in \mathcal{C}^\infty(T^*X)$, $\gamma \subset \{p = 0\}$ as above, with the linearized Poincaré map $dS(0)$ having eigenvalues $\{\mu_j\}$ not on the unit circle, and suppose μ_j has multiplicity k_j. Then there exists a neighbourhood, U, of γ in T^*X, a smooth positive function $b \geqslant C^{-1} > 0$ defined in U, and a symplectomorphism $\kappa : U \to \kappa(U) \subset T^*\mathbb{S}^1_{(t,\tau)} \times T^*\mathbb{R}^{n-1}_{(x,\xi)}$ such that*

$$\kappa(\gamma) = \{(t, 0; 0, 0)\colon\ t \in \mathbb{S}^1\},$$

and $b(t, \tau; x, \xi)p = \kappa^*(g + r)$, with

$$g(t, \tau; x, \xi) = \tau + \sum_{j=1}^{n_{hc}} \sum_{l=1}^{k_j} \big(\operatorname{Re}\lambda_j(x_{2l-1}\xi_{2l-1} + x_{2l}\xi_{2l}) - \operatorname{Im}\lambda_j(x_{2l-1}\xi_{2l} - x_{2l}\xi_{2l-1})\big) \\ + \sum_{j=1}^{n_{hc}} \sum_{l=1}^{k_j-1} (x_{2l+1}\xi_{2l-1} + x_{2l+2}\xi_{2l}) \\ + \sum_{j=2n_{hc}+1}^{2n_{hc}+n_{hr}} \left(\sum_{l=1}^{k_j} \lambda_j x_l \xi_l + \sum_{l=1}^{k_j-1} x_{l+1}\xi_l \right),$$

where $\lambda_j = \log \mu_j$ for each j (with a suitable branch cut) and $r = \mathcal{O}(|x|^3 + |\xi|^3)$.

The proof of Lemma 4.2 depends only on the moduli of the eigenvalues of $dS(0)$ restricted to the stable and unstable manifolds, hence does not depend on the multiplicities, or the Jordan form. Consequently we have the analogue of Proposition 4.3.

Proposition 5.3. *Under the assumptions of Proposition* 5.2*, there exists a neighbourhood U of γ in T^*X, a smooth positive function $b \geqslant C^{-1} > 0$ defined in U, a symplectomorphism $\kappa : U \to \kappa(U) \subset T^*\mathbb{S}^1_{(t,\tau)} \times T^*\mathbb{R}^{n-1}_{(x,\xi)}$, and a smooth, $n \times n$-matrix valued function B_t such that*

$$\kappa(\gamma) = \{(t,0;0,0)\colon t \in \mathbb{S}^1\}, \quad \textit{and} \quad b(t,\tau;x,\xi)p = \kappa^* g,$$
$$\textit{with} \quad g(t,\tau;x,\xi) = \tau + \langle B_t(x,\xi)x, \xi\rangle,$$

with B_t satisfying

$$\begin{aligned}\langle B_t(0,0)x,\xi\rangle &= \sum_{j=1}^{n_{hc}} \sum_{l=1}^{k_j} \big(\operatorname{Re}\lambda_j(x_{2l-1}\xi_{2l-1} + x_{2l}\xi_{2l}) - \operatorname{Im}\lambda_j(x_{2l-1}\xi_{2l} + x_{2l}\xi_{2l-1})\big)\\ &\quad + \sum_{j=1}^{n_{hc}} \sum_{l=1}^{k_j-1} (x_{2l+1}\xi_{2l-1} + x_{2l+2}\xi_{2l})\\ &\quad + \sum_{j=2n_{hc}+1}^{2n_{hc}+n_{hr}} \left(\sum_{l=1}^{k_j} \lambda_j x_l \xi_l + \sum_{l=1}^{k_j-1} x_{l+1}\xi_l \right).\end{aligned}$$

5.2. *End of the proof of Theorem 1*

Now we turn our attention to the proof of Theorem 1 in the case of non-distinct eigenvalues of $dS(0)$. Recall the key feature to the proof of Theorem 1 in the case of distinct eigenvalues was that the normal form given in Proposition 4.3 has quadratic part $q(x,\xi)$ with the property that there exists another quadratic form

$$w(x,\xi) = \frac{1}{2}\langle W(x,\xi),(x,\xi)\rangle$$

such that $H_q w(x,\xi)$ is a positive definite quadratic form. Then we would like our escape function to be $G(x,\xi) = w(x,\xi)$, however for technical reasons we had to use a logarithmic escape function and form the families $e^{\pm sG^\omega}$. With the following theorem, the proof of Theorem 1 is complete.

Theorem 4. *Suppose $q \in \mathcal{C}^\infty(\mathbb{R}^{2m})$ is quadratic of the form*

$$q(x,\xi) = \sum_{j=1}^{n_{hc}} \sum_{l=1}^{k_j} \big(\operatorname{Re}\lambda_j(x_{2l-1}\xi_{2l-1} + x_{2l}\xi_{2l}) - \operatorname{Im}\lambda_j(x_{2l-1}\xi_{2l} - x_{2l}\xi_{2l-1})\big) \tag{5.2.1}$$

$$+\sum_{j=1}^{n_{hc}}\sum_{l=1}^{k_j-1}(x_{2l+1}\xi_{2l-1}+x_{2l+2}\xi_{2l}) \tag{5.2.2}$$

$$+\sum_{j=2n_{hc}+1}^{2n_{hc}+n_{hr}}\left(\sum_{l=1}^{k_j}\lambda_j x_l\xi_l+\sum_{l=1}^{k_j-1}x_{l+1}\xi_l\right), \tag{5.2.3}$$

and

$$G(x,\xi)=\frac{1}{2}\big(\log\big(1+|x|^2\big)-\log\big(1+|\xi|^2\big)\big).$$

Then there exist $m\times m$ nonsingular matrices A and A', positive real numbers $0<r_1\leqslant r_2\leqslant\cdots\leqslant r_m<\infty$, and symplectic coordinates (x,ξ) such that

$$H_q(G)=\frac{\sum_{j=1}^m r_j^{-2}x_j^2}{1+|Ax|^2}+\frac{\sum_{j=1}^m r_j^{-2}\xi_j^2}{1+|A'\xi|^2}. \tag{5.2.4}$$

Proof. First, suppose

$$g(x,\xi)=\frac{1}{2}\left\langle\tilde g\begin{pmatrix}x\\ \xi\end{pmatrix},\begin{pmatrix}x\\ \xi\end{pmatrix}\right\rangle$$

is a real quadratic form with $\tilde g$ symmetric of the form

$$\tilde g=\begin{pmatrix}P & 0\\ 0 & -P\end{pmatrix},$$

where P is symmetric and nonsingular. Then

$$\partial_x\frac{1}{2}\log\big(1+\langle Px,x\rangle\big)=\frac{Px}{1+\langle Px,x\rangle},$$

and similarly for ξ so studying

$$H_q\left(\frac{1}{2}\big(\log\big(1+\langle Px,x\rangle\big)-\log\big(1+\langle P\xi,\xi\rangle\big)\big)\right)$$

can be reduced to studying $H_q g(x,\xi)$, modulo the positive terms $1+\langle P\,\cdot,\cdot\rangle$ in the denominator. If $q(x,\xi)$ is of the form (5.2.1)–(5.2.3), then we can write q in terms of the fundamental matrix B:

$$q(x,\xi)=\left\langle\begin{pmatrix}x\\ \xi\end{pmatrix},JB\begin{pmatrix}x\\ \xi\end{pmatrix}\right\rangle,$$

where

$$J=\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}$$

as usual. Then the vector field H_q can be written as

$$H_q = \left\langle B\begin{pmatrix} x \\ \xi \end{pmatrix}, \begin{pmatrix} \partial_x \\ \partial_\xi \end{pmatrix} \right\rangle,$$

and

$$\begin{aligned} H_q g &= \left\langle B\begin{pmatrix} x \\ \xi \end{pmatrix}, \begin{pmatrix} \partial_x \\ \partial_\xi \end{pmatrix} \right\rangle \left(\frac{1}{2} \left\langle \tilde{g}\begin{pmatrix} x \\ \xi \end{pmatrix}, \begin{pmatrix} x \\ \xi \end{pmatrix} \right\rangle \right) \\ &= \left\langle B\begin{pmatrix} x \\ \xi \end{pmatrix}, \tilde{g}\begin{pmatrix} x \\ \xi \end{pmatrix} \right\rangle = \left\langle B\begin{pmatrix} x \\ \xi \end{pmatrix}, \begin{pmatrix} x \\ \xi \end{pmatrix} \right\rangle, \end{aligned}$$

since $\tilde{g}$ is symmetric.

Now from the discussion preceding the statement of Theorem 4, we know $B = -JQ$ for Q of the form

$$Q = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}, \tag{5.2.5}$$

where A is block diagonal with diagonal elements of the form (5.1.4) or (5.1.5). Thus with the same A as (5.2.5),

$$B = \begin{pmatrix} A^T & 0 \\ 0 & -A \end{pmatrix}.$$

Now we have reduced the problem to finding nonsingular P such that PA^T and PA are both positive definite. But we know that if λ is an eigenvalue of A, then $\operatorname{Re}\lambda > 0$, so A is positive definite and $P = I$ suffices. (5.2.4) then follows immediately from Lemma 5.4. □

We have used the following classical lemma (see, for example [14] for a proof).

Lemma 5.4. *Let*

$$q(x,\xi) = \frac{1}{2}\langle Q(x,\xi), (x,\xi)\rangle$$

be a positive definite quadratic form, where Q is symmetric. Then there are positive numbers $0 < r_1 \leqslant r_2 \leqslant \cdots \leqslant r_m < \infty$ and a linear symplectic transformation T such that

$$q\big(T(x,\xi)\big) = \sum_{j=1}^{m} \frac{1}{r_j^2}\big(x_j^2 + \xi_j^2\big).$$

Further, if T' is another linear symplectic transformation such that

$$q\big(T'(x,\xi)\big) = \sum_{j=1}^{m} \frac{1}{r_j'^2}\big(x_j^2 + \xi_j^2\big)$$

for $0 < r_1' \leqslant \cdots < r_m' \leqslant \infty$, then $r_j = r_j'$ for all j and $T = T'$.

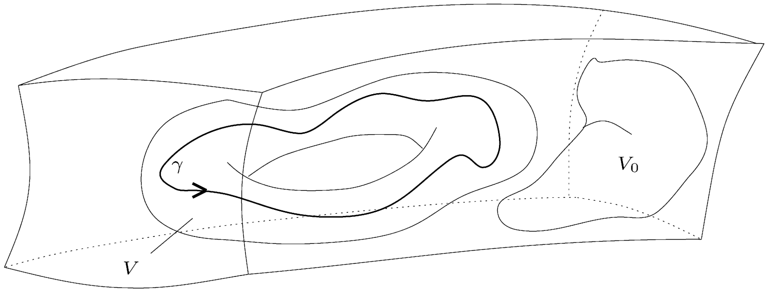

Fig. 3. The energy surface $\{p^{-1}(0)\}$.

6. Proof of Theorem 2 and the Main Theorem

6.1. Proof of Theorem 2

In this section we show how to use Theorem 1 with a few other results to deduce Theorem 2. This is similar to [6], with the generalization of the loxodromic assumption. First we need the following standard lemma.

Lemma 6.1. *Suppose $V_0 \Subset T^*X$, p is a symbol, $T > 0$, A an operator, and $V \Subset T^*X$ a neighbourhood of γ satisfying*

$$\forall \rho \in \{p^{-1}(0)\} \setminus V,\ \exists 0 < t < T \text{ and } \epsilon = \pm 1 \text{ such that } \exp(\epsilon s H_p)(\rho) \subset \{p^{-1}(0)\} \setminus V$$
$$\text{for } 0 < s < t, \text{ and } \exp(\epsilon t H_p)(\rho) \in V_0 \tag{6.1.1}$$

and A is microlocally elliptic in $V_0 \times V_0$. If $B \in \Psi^{0,0}(X, \Omega_X^{1/2})$ and $\mathrm{WF}_h(B) \subset T^*X \setminus V$, then

$$\|Bu\| \leqslant C\big(h^{-1}\|Pu\| + \|Au\|\big) + \mathcal{O}(h^\infty)\|u\|.$$

Figure 3 is a picture of the setup of Lemma 6.1.

Proof. Since $\{p^{-1}(0)\}$ is compact, we can replace V_0 with a precompact neighbourhood of $V_0 \cap \{p^{-1}(0)\}$. We will prove a local version which can be pasted together to get the global estimate. We may assume $\mathrm{WF}_h(A) \subset U$, where U is a small open neighbourhood of some point $\rho_0 \in V_0$, and

$$\mathrm{WF}_h(B) \Subset \bigcup_{0 \leqslant t \leqslant t_0} \exp(\epsilon t H_p)(U_1) \subset T^*X \setminus V, \tag{6.1.2}$$

where $U_1 \Subset U$ and A is microlocally elliptic on $U_1 \times U_1$. For $|t| \leqslant t_1$ sufficiently small, by Proposition 3.5 there is a microlocally invertible h-FIO T which conjugates P to hD_{x_1}. Set $\tilde{u} = Tu$, and let $\widetilde{B} \in \Psi^{0,0}$ be microlocally 1 on $\mathrm{WF}_h(B) \times \mathrm{WF}_h(B)$ and 0 microlocally outside $(\bigcup_{0 \leqslant t \leqslant t_1} \exp(\epsilon t H_p)(U_1))^2 \subset (T^*X \setminus V)^2$. We calculate:

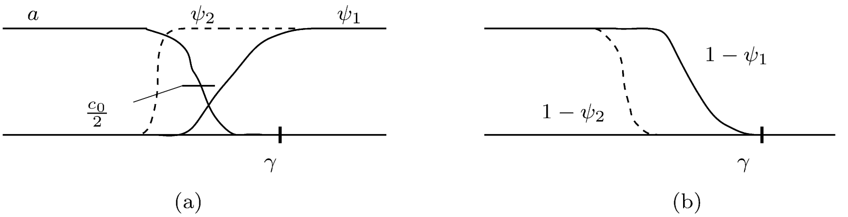

Fig. 4. (a) The cutoff functions a, ψ_1, and ψ_2. (b) $(1-\psi_2)^2 < (1-\psi_1)$.

$$\begin{aligned}
\frac{1}{2}\partial_{x_1}\|\tilde{u}\|^2 &= \langle \partial_{x_1}\tilde{u}, \tilde{u}\rangle \\
&\leqslant \|\partial_{x_1}\tilde{u}\|\|\tilde{u}\| \\
&\leqslant \frac{1}{4}h^{-1}\|TPT^{-1}\|^2 + \|\tilde{u}\|^2 + \mathcal{O}(h^\infty)\|u\|^2_{L^2(X)} \\
\Rightarrow \quad \|\widetilde{B}T^{-1}\tilde{u}\|^2_{L^2(X)} &\leqslant \big(C_{t_1}\big(h^{-1}\|TPT^{-1}\tilde{u}\|^2_{L^2(X)} + \|AT^{-1}\tilde{u}\|^2_{L^2(X)}\big) \\
&\quad + \mathcal{O}(h^\infty)\|u\|^2_{L^2(X)}\big),
\end{aligned}$$

where the last inequality follows from Gronwall's inequality. But

$$\|BT^{-1}\tilde{u}\|^2_{L^2(X)} \leqslant \|\widetilde{B}T^{-1}\tilde{u}\|^2_{L^2(X)}$$

gives the result for small t. Then we partition $[0, t_0]$ into finitely many subintervals and apply the small t argument to each one. □

Using this lemma, we can deduce the following proposition.

Proposition 6.2. *Suppose $\psi_0 \in \mathcal{S}^{0,0}(T^*X) \cap \mathcal{C}_c^\infty(T^*X)$ is a microlocal cutoff function to a small neighbourhood of $\gamma \subset \{p^{-1}(0)\}$. For $Q(z) = P(h) - z - iCha^w$ as above with $z \in [-1, 1] + i(-c_0h, \infty)$, $c_0 > 0$ and $C > 0$ sufficiently large, we have*

$$Q(z)u = f \quad \Rightarrow \quad \big\|(1-\psi_0)^w u\big\| \leqslant Ch^{-1}\|f\| + \mathcal{O}(h^\infty)\|u\|. \tag{6.1.3}$$

For this proposition and the proof, we use the convenient shorthand notation: for a symbol b, $b^w := \mathrm{Op}_h^w(b)$.

Remark. Note that Proposition 6.2 is the best possible situation. It says roughly that away from γ, Q^{-1} is bounded by Ch^{-1}. Thus the global statement in Theorem 2 represents a loss of $\sqrt{\log(1/h)}$.

Proof. Choose $c_0 > 0$ from Theorem 1, microlocal cutoff functions ψ_1, ψ_2 such that $\mathrm{WF}_h(1-\psi_j) \cap \gamma = \emptyset$, and $C > 0$ sufficiently large so that

$$(Ca - c_0)^w(1-\psi_1)^w \geqslant \begin{cases} c_0(1-\psi_1)^w/2, \\ c_0((1-\psi_2)^w)^*(1-\psi_2)^w/2, \end{cases}$$

and supp $\psi_1 \subset \{\psi_2 = 1\}$ (see Fig. 4). Then we calculate

$$\begin{aligned}\frac{1}{2}c_0 h \int_X \left|(1-\psi_2)^w u\right|^2 dx &\leqslant h \int_X (Ca^w + h^{-1}\operatorname{Im} z)u\overline{(1-\psi_1)^w u}\,dx \\ &= -\operatorname{Im}\int_X Q(z)u\overline{(1-\psi_1)^w u}\,dx \\ &= -\operatorname{Im}\int_X f\overline{(1-\psi_1)^w u}\,dx \\ &\leqslant \|f\|\big(\left\|(1-\psi_1)^w u\right\| + \mathcal{O}(h^\infty)\|u\|\big) \\ &\leqslant (4\epsilon h)^{-1}\|f\|^2 + \epsilon h\left\|(1-\psi_1)^w u\right\|^2 + \mathcal{O}(h^\infty)\|u\|^2.\end{aligned}$$

Now we use Lemma 6.1 with $A = (1-\psi_2)^w$, $B = (1-\psi_1)^w$, and $P = Q(z)$, which we may do since the perturbation terms in $Q(z)$ are all of lower order. Thus

$$\begin{aligned}&\left\|(1-\psi_1)^w u\right\| \leqslant Ch^{-1}\left\|Q(z)u\right\| + \left\|(1-\psi_2)^w u\right\| + \mathcal{O}(h^\infty)\|u\| \\ &\quad\Rightarrow\quad \left\|(1-\psi_1)^w u\right\|^2 \\ &\qquad \leqslant Ch^{-1}\|f\|\big(Ch^{-1}\|f\| + \left\|(1-\psi_2)^w u\right\|\big) + \left\|(1-\psi_2)^w u\right\|^2 + \mathcal{O}(h^\infty)\|u\|^2 \\ &\qquad \leqslant Ch^{-2}\|f\|^2 + \left\|(1-\psi_2)^w u\right\|^2 + \mathcal{O}(h^\infty)\|u\|^2 \\ &\qquad \leqslant Ch^{-2}\|f\|^2 + \epsilon\left\|(1-\psi_1)^w u\right\|^2 + \mathcal{O}(h^\infty)\|u\|^2,\end{aligned}$$

which gives (6.1.3) with ψ_0 replaced by ψ_1. Another application of Lemma 6.1 with $A = (1-\psi_2)^w$, $B = (\psi_1 - \psi_0)^w$, and $P = Q(z)$ shows the error $\|(\psi_1-\psi_0)^w u\|$ is bounded by the same estimate as in (6.1.3). □

We will need the next lemma, which is essentially an operator version of the classical Three-Line theorem from complex analysis. We include the proof here for the reader's convenience, collected from [5,6,28].

Lemma 6.3. *Let $\mathcal{H}$ be a Hilbert space, and assume $A, B : \mathcal{H} \to \mathcal{H}$ are bounded, self-adjoint operators satisfying $A^2 = A$ and $BA = AB = A$. Suppose $F(z)$ is a family of bounded operators satisfying $F(z)^* = F(\bar{z})$, $\operatorname{Re} F \geqslant C^{-1}\operatorname{Im} z$ for $\operatorname{Im} z > 0$, and further assume*

$$BF^{-1}(z)B \text{ is holomorphic in } \Omega := [-\epsilon, \epsilon] + i[-\delta, \delta] \quad \text{for } \frac{\delta}{\epsilon} \ll M^{-\frac{1}{N_1}} < 1,$$

for some $N_1 > 0$, where $\|BF^{-1}(z)B\| \leqslant M$. Then for $|z| < \epsilon/2$, $\operatorname{Im} z = 0$,

$$\text{(a)}\qquad \left\|BF^{-1}(z)B\right\| \leqslant C\frac{\log M}{\delta},$$

$$\text{(b)}\qquad \left\|BF^{-1}(z)A\right\| \leqslant C\sqrt{\frac{\log M}{\delta}}.$$

Proof. For the proof of part (a), consider the holomorphic operator-valued function $f(z) = BF(z)^{-1}B$. Choose $\psi \in \mathcal{C}_c^\infty([-3\epsilon/4, 3\epsilon/4])$, $\psi \equiv 1$ on $[-\epsilon/2, \epsilon/2]$, and for $z \in \Omega$, set

$$\varphi(z) = \delta^{-\frac{1}{2}} \int e^{-(x-z)^2/\delta} \psi(x)\,dx.$$

$\varphi(z)$ has the following properties:

(a) $\varphi(z)$ is holomorphic in Ω,
(b) $|\varphi(z)| \leqslant C$ in Ω,
(c) $|\varphi(z)| \geqslant C^{-1} > 0$ on $[-\epsilon/2, \epsilon/2]$, and
(d) $|\varphi(z)| \leqslant Ce^{-C/\delta}$ on $\Omega \cap \{\operatorname{Re} z = \pm\epsilon\}$.

Now for $z \in \tilde{\Omega} := [-\epsilon, \epsilon] + i[-\delta, \delta/\log M]$ set

$$g(z) = e^{-iNz\log M/\delta} \varphi(z) f(z),$$

and note that $g(z)$ satisfies:

(a) $|g(z)| \leqslant CM^{1-N}$ on $\tilde{\Omega} \cap \{\operatorname{Im} z = -\delta\}$,
(b) $|g(z)| \leqslant C_N e^{-C/\delta}$ on $\tilde{\Omega} \cap \{\operatorname{Re} z = \pm\epsilon\}$, and
(c) $|g(z)| \leqslant C_N \log(M)/\delta$ on $\tilde{\Omega} \cap \{\operatorname{Im} z = \delta/\log M\}$.

Then the classical maximum principle implies for δ sufficiently small and N sufficiently large, $|g(z)| \leqslant C\log(M)/\delta$, which in turn implies

$$\left|f(z)\right| \leqslant C\frac{\log M}{\delta} \quad \text{on} \left[-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right] \subset \mathbb{R}.$$

For part (b), note that our assumptions on $F(z)$ imply

$$\operatorname{Im} z\|u\|^2 \leqslant C \operatorname{Re}\langle F(z)u, u\rangle.$$

We have

$$\|BF^{-1}A\|_{L^2\to L^2} = \sup_{\{\|b\|_{L^2}=1\}} \|BF^{-1}Ab\|_{L^2} = \sup \|BF^{-1}A^2b\|_{L^2},$$

since $A^2 = A$. Suppose $F(z)u(x) = Ab(x,z)$. Then $u = F(z)^{-1}Ab$ and $Bu = BF^{-1}AAb$, and for $\operatorname{Im} z > 0$,

$$\begin{aligned}\|Bu\|^2 &\leqslant C\|u\|^2 \\ &\leqslant \frac{C}{\operatorname{Im} z}\langle \operatorname{Re} F(z)u, u\rangle \\ &\leqslant \frac{C}{\operatorname{Im} z}\left|\langle F(z)u, u\rangle\right|\end{aligned}$$

$$
\begin{aligned}
&= \frac{C}{\operatorname{Im} z} \left| \langle Ab, u \rangle \right| \\
&= \frac{C}{\operatorname{Im} z} \left| \langle Ab, Au \rangle \right| \\
&\leqslant \frac{C}{\operatorname{Im} z} \|Ab\|^2.
\end{aligned}
$$

where we have used $A^*A = A^2 = A$. Thus we have

$$
\begin{aligned}
\left\| BF(z)^{-1}A \right\|_{L^2 \to L^2} &\leqslant \frac{C}{\sqrt{\operatorname{Im} z}} \quad \text{for } \operatorname{Im} z > 0 \quad \text{and} \\
\left\| BF(z)^{-1}A \right\|_{L^2 \to L^2} &= \sup_{\{\|u\|=1\}} \|BF^{-1}Au\|_{L^2} \\
&= \sup_{\{\|u\|=1\}} \|BF^{-1}BAu\|_{L^2 \to L^2} \\
&\leqslant M \sup_{\{\|u\|=1\}} \|Au\|_{L^2} \\
&\leqslant CM,
\end{aligned}
$$

and we can apply the proof of part (a) to $f(z) = BF(z)^{-1}A$ to get (b). □

Proof of Theorem 2. Let ψ_0 satisfy the assumptions of Proposition 6.2. Then

$$
\left\| (1 - \psi_0)^w u \right\| \leqslant Ch^{-1} \left\| Q(z)u \right\| + \mathcal{O}(h^\infty)\|u\|.
$$

Further, since

$$
\left\| \left[Q, \psi_0^w \right] u \right\| \leqslant \left\| \left[Q, \psi_0^w \right] \left(1 - \tilde{\psi}_0^w \right) u \right\| + \mathcal{O}(h^\infty)\|u\|,
$$

for some $\tilde{\psi}_0$ satisfying the assumptions of Proposition 6.2 and $\mathrm{WF}_h \tilde{\psi}_0 \subset \{\psi_0 = 1\}$ so using Theorem 1 and the fact that $[Q, \psi_0^w]$ is compactly supported and of order h, we have

$$
\begin{aligned}
\|\psi_0^w u\| &\leqslant Ch^{-N_0} \left(\left\| \psi_0^w Qu \right\| + \left\| \left[Q, \psi_0^w \right] u \right\| \right) \\
&\leqslant Ch^{-N_0} \left(\left\| \psi_0^w Qu \right\| + h^{-1} \|hQu\| \right) + \mathcal{O}(h^\infty) \\
&\leqslant Ch^{-N_0} \|Qu\| + \mathcal{O}(h^\infty)\|u\|.
\end{aligned}
$$

Now let $F(w)$ be the family of operators $F(w) = ih^{-1}Q(z_0 + hw)$, $A = \chi^w_{\operatorname{supp} \varphi}$, $B = \mathrm{id}$. Fix $\delta > 0$ independent of h, $\epsilon = (Ch)^{-1}$, $M = h^{-N_0}$, and apply Lemma 6.3 to get

$$
\begin{aligned}
\|BF^{-1}B\| &\leqslant C \log\left(h^{-N_0}\right), \\
\|BF^{-1}A\| &\leqslant C \sqrt{\log\left(h^{-N_0}\right)},
\end{aligned}
$$

and (1.5), (1.6) follows. □

6.2. Proof of the Main Theorem

The Main Theorem is an easy consequence of Theorem 2.

Proof of the Main Theorem. Recall A is 0 microlocally away from $\gamma \times \gamma$. Let $\widetilde{A} \in \Psi_h^{0,0}$ be a pseudodifferential operator so that $\widetilde{A} = I$ microlocally on a neighbourhood of $\mathrm{WF}_h(A) \times \mathrm{WF}_h(A)$. Let a^w be as in Theorems 1 and 2. Choosing A and $\widetilde{A}$ so that $\mathrm{WF}_h(a^w)$ is disjoint from $\mathrm{WF}_h(\widetilde{A})$, we have for $Q = Q(0)$

$$Q\widetilde{A}u = P\widetilde{A}u. \tag{6.2.1}$$

The right-hand side of (6.2.1) is $[P, \widetilde{A}]u + \widetilde{A}Pu$. Now $[P, \widetilde{A}]$ is supported away from γ since $\widetilde{A}$ is constant near γ, so

$$\begin{aligned} \|P\widetilde{A}u\|_{L^2(X)} &\leqslant \big\|[P, \widetilde{A}]u\big\|_{L^2(X)} + \|Pu\|_{L^2(X)} \\ &\leqslant Ch\big\|(I - A)u\big\|_{L^2(X)} + \|Pu\|_{L^2(X)}. \end{aligned} \tag{6.2.2}$$

From Theorem 2, we have

$$\|Q\widetilde{A}u\|_{L^2(X)} \geqslant C^{-1}\frac{h}{\sqrt{\log(1/h)}}\|\widetilde{A}u\|_{L^2(X)}. \tag{6.2.3}$$

Combining (6.2.2) and (6.2.3), we have

$$\begin{aligned} C^{-1}\|u\|_{L^2(X)} &\leqslant C^{-1}\big(\|\widetilde{A}u\|_{L^2(X)} + \big\|(I - A)u\big\|_{L^2(X)}\big) \\ &\leqslant C\big(\sqrt{\log(1/h)} + C^{-1}\big)\big\|(I - A)u\big\|_{L^2(X)} + C\frac{\sqrt{\log(1/h)}}{h}\|Pu\|_{L^2(X)}, \end{aligned}$$

which for $0 < h < h_0$ is (1.1). □

Remark. In the calculation (6.2.2), we have only used $\|[P, \widetilde{A}]u\| \leqslant Ch\|(I - A)u\|$. If we could determine a global geometric condition which would allow us to choose A in a manner which improves this, but does not have too much interaction with a^w in the definition of $Q(z)$, we could eliminate the $\log(h^{-1})$ in (1.1).

7. An application: The damped wave equation

In this section we adapt the techniques from Sections 5, 6 to study the damped wave equation. Let X be a compact manifold without boundary, $a(x) \in \mathcal{C}^\infty(X)$, $a(x) \geqslant 0$, and consider the following problem:

$$\begin{cases} \big(\partial_t^2 - \Delta + 2a(x)\partial_t\big)u(x,t) = 0, & (x,t) \in X \times (0,\infty), \\ u(x,0) = 0, \quad \partial_t u(x,0) = f(x). \end{cases} \tag{7.1}$$

Let $p \in \mathcal{C}^\infty(T^*X)$, $p = |\xi|^2$, be the microlocal principal symbol of $-\Delta$ and suppose the classical flow (geodesic flow) of H_p admits a single closed, loxodromic orbit γ in the level set $\{p^{-1}(1)\}$.

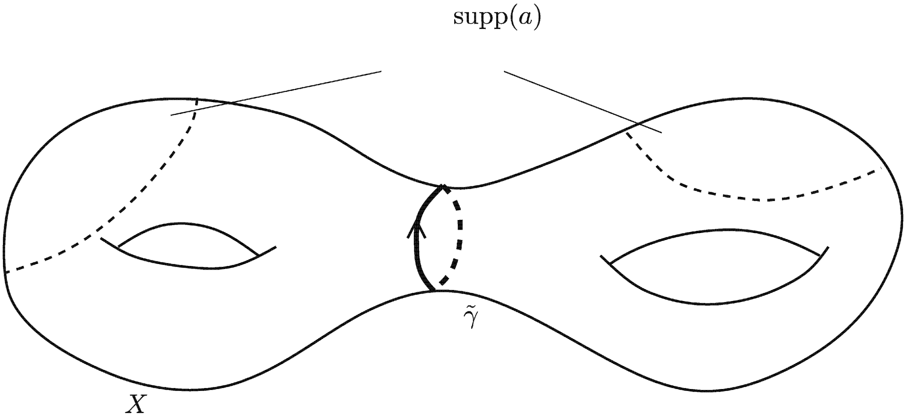

Fig. 5. The manifold X and the projection $\tilde{\gamma}$ of γ onto X.

Assume throughout that $a(x)$ is supported away from the projection $\tilde{\gamma}$ of γ onto X (see Fig. 5). We recall that the H^s inner product on X is given by the local formula

$$\langle u, v\rangle_{H^s} = \int\limits_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s \hat{u}\hat{\bar{v}}\, d\xi,$$

where $\hat{u}$ is the Fourier transform of u. If u solves (7.1), we define the *s-energy* $E^s(t)$ of u at time t to be

$$E^s(t) = \frac{1}{2}\left(\|\partial_t u\|^2_{H^s(X)} + \left\|\sqrt{-\Delta}u\right\|^2_{H^s(X)}\right).$$

Lemma 7.1. *If $a(x) \equiv 0$, $E^s(t)$ is constant. If $a(x)$ is not identically zero, then $E^s(t)$ is decreasing.*

Proof.

$$\begin{aligned}\frac{d}{dt}E^s(t) &= \left\langle \partial_t^2 u, \partial_t u\right\rangle_{H^s} + \left\langle \partial_t\sqrt{-\Delta}u, \sqrt{-\Delta}u\right\rangle_{H^s}\\ &= \left\langle \partial_t u, \left(\partial_t^2 - \Delta\right)u\right\rangle\\ &= -\left\langle \partial_t u, 2a(x)\partial_t u\right\rangle. \qquad \square\end{aligned}$$

We make an important dynamical assumption, which amounts to a geometric control condition similar to that given by Rauch and Taylor in [21]. We assume:

$$\begin{gathered}\text{There exists a time } T>0 \text{ and a neighbourhood } V \text{ of } \gamma \text{ such that for all } |\xi|=1,\\ (x,\xi)\in T^*X\subset V,\ \exp(tH_p)(x,\xi)\cap\{a>0\}\neq\emptyset \text{ for some } |t|\leqslant T.\end{gathered} \tag{7.2}$$

In [10, Section 5.3], it is shown that with a global Rauch–Taylor condition, we have exponential decay in zero-energy. Here we have a region without geometric control, so we expect some loss.

Theorem 5. *Assume* (7.2) *holds and* $a(x)$ *is not identically zero. Then for any* $\epsilon > 0$*, there is a constant* $C > 0$ *such that*

$$E^0(t) \leqslant Ce^{-t/C} \|f\|_{H^\epsilon}^2.$$

The damped wave equation in the context of a global Rauch–Taylor condition has been studied in [13,19,21,24]. The difference here is the presence of γ and a neighbourhood in which the Rauch–Taylor condition does not hold.

Formally, if $u \equiv 0$ for $t < 0$, we apply the Fourier transform to (7.1) in the t variable and integrating by parts motivates us to study the equation

$$P(\tau)\hat{u}(x,\tau) := \big(-\tau^2 - \Delta + 2ia(x)\tau\big)\hat{u}(x,\tau) = f. \tag{7.3}$$

We use the techniques of the previous sections to gain estimates on the resolvent $P(\tau)^{-1}$. We call the poles of $P(\tau)^{-1}$ *eigenfrequencies* for (7.1). Note if τ is an eigenfrequency, then $0 \leqslant \operatorname{Im}\tau \leqslant 2\|a\|_{L^\infty}$. Further, (7.3) is invariant under the transformation $(\hat{u},\tau) \mapsto (\bar{\hat{u}}, -\bar{\tau})$, so the set of eigenfrequencies is symmetric about the imaginary axis. We therefore study only those in the right half-plane. For $0 < h \leqslant h_0$ and $z \in \Omega := [\alpha,\beta] + i[-\gamma,\gamma]$ where $0 < \alpha < 1 < \beta < \infty$ and $\gamma > 0$, set $\tau = \sqrt{z}/h$. (7.3) becomes

$$\frac{1}{h^2} Q(z,h)\hat{u} = f, \tag{7.4}$$

where

$$Q(z,h) = P(h) - z + 2ih\sqrt{z}a(x) \tag{7.5}$$

and the principal symbol of $P(h)$ is $p(x,\xi) = |\xi|^2$. The next corollary follows directly from the proof of Theorem 1, replacing s in the conjugation (5.8) with $-s$.

Corollary 6. *Suppose* u *has wavefront set sufficiently close to* γ*. Then there exists* $c_0 > 0$*,* $C < \infty$*, and* $N \geqslant 0$ *such that for* $z \in [\alpha,\beta] + i[-c_0 h, c_0 h]$*,*

$$\operatorname{Im}\big\langle \big(P(h) - z\big)u, u\big\rangle \geqslant C^{-1}h^N|u|^2.$$

In particular, $\|Q(z,h)u\| \geqslant C^{-1}h^N\|u\|$.

We observe that for u as in the theorem and $-c_0 h < \operatorname{Im} z < 0$, $\|Q(z,h)u\| \geqslant C^{-1}\operatorname{Im} z\|u\|$.

The proof of Proposition 6.2 relies on the assumption that the symbol $a(x,\xi)$ in (1.3) is elliptic away from γ. The function $a(x)$ in (7.5) is not assumed to be elliptic anywhere, so we will use a technique from [19] to replace $a(x)$ with its average over trajectories of $\exp(tH_p)$.

For $T > 0$, we define the T-trajectory average of a smooth function b:

$$\langle b\rangle_T(x,\xi) = \frac{1}{T}\int_0^T b \circ \exp(tH_p)(x,\xi)\,dt.$$

Set $q(z) = 2\sqrt{z}a(x)$, and for $z \in \tilde{\Omega} := [\alpha, \beta] + i[0, c_1 h]$, where $c_1 > 0$ will be chosen later, and $(x, \xi) \in \{p^{-1}([\alpha - \delta, \beta + \delta])\}$ for $\delta > 0$, let $g_{\operatorname{Re} z} \in \mathcal{S}(1)$ depending on T solve

$$q(\operatorname{Re} z) - H_p g_{\operatorname{Re} z} = \big\langle q(\operatorname{Re} z)\big\rangle_T.$$

(See [24] for details on the construction of $g_{\operatorname{Re} z}$.) Now we form the elliptic operator $A := \operatorname{Op}_h^w(e^g) \in \Psi^{0,0}$, and observe

$$\begin{aligned} A^{-1}PA &= P + A^{-1}[P, A] \\ &= P - ih\operatorname{Op}_h^w(e^g)^{-1}\operatorname{Op}_h^w\big(\{p, e^g\}\big) \\ &= P - ihB, \end{aligned}$$

with $\sigma_h(B) = e^{-g}\{p, e^g\} + \mathcal{O}(h) = H_p g + \mathcal{O}(h)$. Thus

$$\begin{aligned} A^{-1}\big(P + ihq(z)\big)A &= P + ih\operatorname{Op}_h^w\big(q(\operatorname{Re} z) - H_p g\big) + \mathcal{O}\big(h^2\big) \\ &= P + ih\operatorname{Op}_h^w\big(\big\langle q(\operatorname{Re} z)\big\rangle_T\big), \end{aligned}$$

since $\operatorname{Im} z = \mathcal{O}(h)\operatorname{Re} z$. Following [13], we claim there exists a time $T > 0$ such that

$$\langle a\rangle_T(x, \xi) \geqslant C^{-1} > 0 \tag{7.6}$$

for $(x, \xi) \in \{p^{-1}([\alpha - \delta/2, \beta + \delta/2])\} \setminus V$, where V is as in the statement of Theorem 5. To see this, recall $p = |\xi|^2$ means $H_p = 2\langle\xi, \partial_x\rangle$ and $p^{-1}(E) = \{|\xi| = \sqrt{E}\}$, which means

$$\inf_{p^{-1}(E)} \langle a\rangle_T = \inf_{p^{-1}(1)} \langle a\rangle_{\sqrt{E}T}.$$

By assumption (7.2),

$$\inf_{p^{-1}(1)} \langle a\rangle_{\sqrt{E}T} \geqslant C^{-1} > 0$$

in $\{p^{-1}(1) \setminus V$ for T sufficiently large and $\sqrt{E}$ close to 1. Choosing α and β sufficiently close to 1 proves (7.6).

Corollary 7. *Suppose $\psi_0 \in \mathcal{S}^{0,0}(T^*X) \cap \mathcal{C}_c^\infty(T^*X)$ is a microlocal cutoff function to a small neighbourhood of $\gamma \subset \{p^{-1}(1)\}$. For $Q(z, h) = P(h) - z + 2ih\sqrt{z}a$ as above with $z \in [\alpha, \beta] + i(-c_1 h, c_1 h)$, $c_1 > 0$, we have*

$$Q(z, h)u = f \quad \Rightarrow \quad \big\|(1 - \psi_0)^w u\big\| \leqslant Ch^{-1}\|f\| + \mathcal{O}(h^\infty)\|u\|. \tag{7.7}$$

Proof. Selecting $T > 0$ sufficiently large and $c_1 > 0$ such that

$$0 < c_1 < \inf_{p^{-1}([\alpha - \delta/2, \beta + \delta/2])} \langle a\rangle_T(x, \xi),$$

we apply the proof of Proposition 6.2 to the conjugated operator $A^{-1}Q(z, h)A$. □

We now have good resolvent estimates for z in an h interval below the real axis, as well as weaker estimates above.

Corollary 8.

(i) *There exist constants* $C > 0$ *and* $N > 0$ *such that the resolvent* $Q(z,h)^{-1}$ *satisfies*

$$\|Q(z,h)^{-1}\|_{L^2\to L^2} \leqslant Ch^{-N}, \quad z \in [\alpha,\beta] + i(-c_0h, c_0h).$$

(ii) *In addition, there is a constant* C_1 *such that*

$$\|Q(z,h)^{-1}\|_{L^2\to L^2} \leqslant C_1\frac{\log(1/h)}{h}, \quad z \in [\alpha,\beta] + i\big[-C_1^{-1}h/\log(1/h), C_1^{-1}h\big].$$

This is an immediate consequence of the proof of Theorem 2, together with the slight modification of Lemma 6.3 given in Lemma 7.2.

Lemma 7.2. *Let* $f(z)$ *be a holomorphic function on* $\Omega = [-\epsilon,\epsilon] + i[-\delta,\delta]$, *with*

$$\frac{\delta}{\epsilon} \ll M^{-\frac{1}{N_1}}$$

for some $N_1 > 0$, *and suppose* f *satisfies* $\|f(z)\| \leqslant M$ *on* Ω *with* $|f(z)| \leqslant C|\mathrm{Im} z|$ *for* $\mathrm{Im}\, z < 0$. *Then there exists a constant* $0 < C_1 < \infty$ *such that if* $-C_1^{-1}\delta/\log M \leqslant \mathrm{Im}\, z \leqslant C_1^{-1}\delta$ *we have*

$$\big|f(z)\big| \leqslant C\frac{\log M}{\delta}.$$

Proof. Let $\psi(x)$ be as in the proof of Lemma 6.3, and for $C_1^{-1} \ll c_0$, let

$$\varphi(z) = \delta^{-\frac{1}{2}}\int e^{-(x-z+iC_1^{-1}\delta)^2/\delta}\psi(x)\,dx.$$

We observe if $C_1 > 0$ is sufficiently large, for $|\mathrm{Im}\, z| \leqslant C_1^{-1}\delta$,

$$\big(x - z + iC_1^{-1}\delta\big)^2 = (x - \mathrm{Re}\, z)^2 - \big(C_1^{-1}\delta - \mathrm{Im}\, z\big)^2 + 2i(x - \mathrm{Re}\, z)\big(C_1^{-1}\delta - \mathrm{Im}\, z\big)$$

and

$$\big|(x - \mathrm{Re}\, z)\big(C_1^{-1}\delta - \mathrm{Im}\, z\big)\big| \leqslant 4C_1^{-1}\epsilon\delta,$$

so if $C_1 > 0$ is sufficiently large,

$$\begin{aligned}\mathrm{Re}\, e^{-(x-z+iC_1^{-1}\delta)^2/\delta} &\geqslant e^{-(x-\mathrm{Re}\, z)^2/\delta + (C_1^{-1}\delta - \mathrm{Im}\, z)^2/\delta}\cos\big(4C_1^{-1}\epsilon\big)\\ &\geqslant C^{-1}e^{-(x-\mathrm{Re}\, z)^2/\delta + (C_1^{-1}\delta - \mathrm{Im}\, z)^2/\delta}.\end{aligned}$$

Thus $\varphi(z)$ satisfies:

(a) $\varphi(z)$ is holomorphic in Ω,
(b) $|\varphi(z)| \leqslant C$ in Ω,
(c) $|\varphi(z)| \geqslant C^{-1}$ for $z \in [-\epsilon/2, \epsilon/2] + i[C_1^{-1}\delta, C_1^{-1}\delta]$,
(d) $|\varphi(z)| \leqslant Ce^{-C/\delta}$ on $\{\pm\epsilon\} \times i[-C_1^{-1}\delta, C_1^{-1}\delta]$, if $C_1 > 0$ is chosen large enough.

Now similar to the proof of Lemma 6.3, for

$$z \in \tilde{\Omega} := [-\epsilon, \epsilon] + i\big[-C_1^{-1}\delta/\log M, C_1^{-1}\delta\big]$$

set

$$g(z) = e^{iNz\log M/\delta}\varphi(z) f(z).$$

Then as in the proof of Lemma 6.3, the classical maximum principle implies for δ sufficiently small and N sufficiently large, $|g(z)| \leqslant C\log(M)/\delta$, which in turn implies

$$\big|f(z)\big| \leqslant C\frac{\log M}{\delta} \quad \text{on} \left[-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right] + i\big[-C_1^{-1}\delta\big/\log M, C_1^{-1}\delta\big]. \qquad \square$$

With these resolvent estimates, we have the following estimates in terms of τ.

Proposition 7.3. *Fix* $\epsilon > 0$. *There exist constants* $0 < C,\ C_1 < \infty$ *such that if* $-(\log\langle\tau\rangle)^{-1} \leqslant \operatorname{Im}\tau \leqslant C_1^{-1}$, *then*

$$\big\|P(\tau)^{-1}\big\|_{L^2\to L^2} \leqslant \frac{C\log\langle\tau\rangle}{\langle\tau\rangle}, \tag{7.8}$$

$$\big\|P(\tau)^{-1}\big\|_{L^2\to H^2} \leqslant C\langle\tau\rangle\log\langle\tau\rangle, \quad \textit{and} \tag{7.9}$$

$$\big\|P(\tau)^{-1}\big\|_{H^s\to H^{s+1-\epsilon}} \leqslant C. \tag{7.10}$$

Proof. (7.8) follows directly from rescaling. To see (7.9), calculate

$$\begin{aligned}
\|u\|_{H^2} &\leqslant C\big(\|\Delta u\|_{L^2} + \|u\|_{L^2}\big) \\
&\leqslant C\left(\big\|P(\tau)u\big\|_{L^2} + \big\|\big(-\tau^2 + 2ia(x)\tau\big)u\big\|_{L^2} + \frac{\log\langle\tau\rangle}{\langle\tau\rangle}\big\|P(\tau)u\big\|_{L^2}\right) \\
&\leqslant C\big(1 + |\tau|\log\langle\tau\rangle + \langle\tau\rangle^{-1}\log\langle\tau\rangle\big)\big\|P(\tau)u\big\|_{L^2}.
\end{aligned}$$

For (7.10), let $\epsilon > 0$ be given. From Lemma 7.4, we have

$$\begin{aligned}
\big\|P(\tau)^{-1}u\big\|_{H^{1-\epsilon}}^2 &\leqslant C\big\|P(\tau)^{-1}u\big\|_{H^2}^{1-\epsilon}\big\|P(\tau)^{-1}u\big\|_{L^2}^{1+\epsilon} \\
&\leqslant C_\epsilon\|u\|_{L^2(X)}^2.
\end{aligned}$$

To get the estimates for $H^s \to H^{s+1-\epsilon}$, we conjugate $P(\tau)^{-1}$ by the operators

$$\Lambda^s = (1-\Delta)^{\frac{s}{2}}$$

and apply to $v = \Lambda^s u$:

$$\begin{aligned}\|P(\tau)^{-1}u\|_{H^{s+1-\epsilon}} &= \|\Lambda^{1-\epsilon}\Lambda^s P(\tau)^{-1}\Lambda^{-s}v\|_{L^2} \\ &= \|\Lambda^{1-\epsilon}\big(P(\tau)^{-1} + \Lambda^s\big[P(\tau)^{-1}, \Lambda^{-s}\big]\big)v\|_{L^2} \\ &\leqslant \|v\|_{L^2} \\ &\leqslant C\|u\|_{H^s}. \qquad \square\end{aligned}$$

We have used the following interpolation lemma.

Lemma 7.4. *Let $\epsilon > 0$ be given, and suppose $f \in H^2(X) \cap L^2(X)$. Then*

$$\|f\|^2_{H^{1-\epsilon}} \leqslant C\|f\|^{1-\epsilon}_{H^2}\|f\|^{1+\epsilon}_{L^2}.$$

Proof. We use the local formula for H^s norms and calculate:

$$\begin{aligned}\|f\|^2_{H^{1-\epsilon}} &= \int_{\mathbb{R}^n} \big(1+|\xi|^2\big)^{1-\epsilon}\hat{f}\bar{\hat{f}}\,d\xi \\ &= \int \big(\big(1+|\xi|^2\big)|\hat{f}|\big)^{1-\epsilon}|\hat{f}|^{1+\epsilon}\,d\xi \\ &\leqslant C\big\|\big(\big(1+|\xi|^2\big)|\hat{f}|\big)^{1-\epsilon}\big\|_{L^{\frac{2}{1-\epsilon}}}\big\||\hat{f}|^{1+\epsilon}\big\|_{L^{\frac{2}{1+\epsilon}}} \\ &\leqslant C\|f\|^{1-\epsilon}_{H^2}\|f\|^{1+\epsilon}_{L^2}. \qquad \square\end{aligned}$$

We are now in position to prove Theorem 5. This proof comes almost directly from [10, Section 5.3].

Proof of Theorem 5. Assume $u(x, f)$ solves (7.1). Choose $\chi \in \mathcal{C}^\infty(\mathbb{R})$, $0 \leqslant \chi \leqslant 1$, $\chi \equiv 1$ on $[1, \infty)$, and $\chi \equiv 0$ on $(-\infty, 0]$. Set $u_1(x, t) = \chi(t)u(x, t)$. We apply the damped wave operator to u_1:

$$\begin{aligned}&\big(\partial_t^2 - \Delta + 2a\partial_t\big)u_1 \\ &\quad = \chi''u + 2\chi'u_t + 2a\chi'u + \chi\big(\partial_t^2 - \Delta + 2a\delta_t\big)u \qquad (7.11)\\ &\quad = \chi''u + 2\chi'u_t + 2a\chi'u =: g_1. \qquad (7.12)\end{aligned}$$

With g_1 supported in $X \times (0, 1)$ and $u_1 \equiv 0$ for $t \leqslant 0$, we have

$$\|g_1\|^2_{L^2((0,\infty);H^\epsilon)} \leqslant C\big(\|u\|^2_{L^2((0,1);H^\epsilon)} + \|\partial_t u\|^2_{L^2((0,1);H^\epsilon)}\big). \qquad (7.13)$$

Now

$$\begin{aligned}\partial_t\langle u, u\rangle_{H^\epsilon(X)} &= 2\langle\partial_t u, u\rangle_{H^\epsilon(X)} \\ &\leqslant \|\partial_t u\|^2_{H^\epsilon(X)} + \|u\|^2_{H^\epsilon(X)} \\ &\leqslant CE^\epsilon(t) + \|u\|^2_{H^\epsilon(X)},\end{aligned}$$

so by Gronwall's inequality,

$$\begin{aligned}\|u(t,\cdot)\|_{H^\epsilon(X)}^2 &\leqslant Ce^t\left(\|u(0,\cdot)\|_{H^\epsilon(X)}^2+\int_0^t E^\epsilon(s)\,ds\right)\\ &\leqslant Cte^t\|f\|_{H^\epsilon(X)}^2.\end{aligned}$$

Thus (7.13) is bounded by $C\|f\|_{H^\epsilon(X)}^2$.

We now apply the Fourier transform to (7.11), (7.12) to write $\hat{u}_1=P(\tau)^{-1}\hat{g}_1$. By Proposition 7.3, we have for $\operatorname{Im}\tau=C^{-1}>0$

$$\begin{aligned}\|e^{t/C}u_1\|_{L^2((0,\infty);H^1)} &= \|\hat{u}_1(\cdot+iC^{-1})\|_{L^2((-\infty,\infty);H^1)}\\ &= \|P(\cdot+iC^{-1})\hat{g}_1(\cdot+iC^{-1})\|_{L^2((-\infty,\infty);H^1)}\\ &\leqslant C\|\hat{g}_1\|_{L^2((-\infty,\infty);H^\epsilon)}\\ &\leqslant C\|g_1\|_{L^2((0,\infty);H^\epsilon)}\\ &\leqslant C\|f\|_{H^\epsilon(X)}.\end{aligned}$$

Thus

$$\|e^{t/C}u\|_{L^2((1,\infty);H^1)}\leqslant C\|f\|_{H^\epsilon(X)}.$$

Now for $T>2$, choose $\chi_2\in\mathcal{C}^\infty(\mathbb{R})$, $0\leqslant\chi_2\leqslant 1$, such that $\chi_2\equiv 0$ for $t\leqslant T-1$, and $\chi_2\equiv 1$ for $t\geqslant T$. Set $u_2(x,t)=\chi_2(t)u(x,t)$. We have

$$\left(\partial_t^2-\Delta+2a\partial_t\right)u_2=g_2$$

for $g_2=\chi_2''u_t+2\chi_2'u_t+2a\chi_2'u$, and $\operatorname{supp} g_2\subset X\times[T-1,T]$. Define

$$E_2(t)=\frac{1}{2}\int_X(\partial_t u_2)^2+\left|\sqrt{-\Delta}u_2\right|^2dx,$$

and observe

$$\begin{aligned}E_2'(t) &= \left\langle\partial_t^2u_2,\partial_t u_2\right\rangle_X-\langle\Delta u_2,\partial_t u_2\rangle_X\\ &= -\left\langle 2a(x)\partial_t u_2,\partial_t u_2\right\rangle_X+\langle g_2,\partial_t u_2\rangle_X\\ &\leqslant C\int_X|\partial_t u_2|\left(|\partial_t u|+|u|\right)dx\\ &\leqslant C\left(E_2(t)+\int_X\left(|\partial_t u|^2+|u|^2\right)dx\right).\end{aligned}$$

Now since $E_2(T-1)=0$ and $E_2(T)=E(T)$, Gronwall's inequality gives

$$E(T)\leqslant C\big(\|\partial_t u\|^2_{L^2((T-1,T);L^2)}+\|u\|^2_{L^2((T-1,T);L^2)}\big). \tag{7.14}$$

We need to bound the first term on the right-hand side of (7.14). Choose $\chi_3\in\mathcal{C}^\infty(\mathbb{R})$ such that $\chi_3\equiv 0$ for $t\leqslant T-2$ and $t\geqslant T+1$, $\chi_3\equiv 1$ for $T-1\leqslant t\leqslant T$. Then

$$\begin{aligned}0&=\int_{T-2}^{T+1}\int_X \chi_3^2 u\big(\partial_t^2 u-\Delta u+2a\partial_t u\big)\,dx\,dt\\&=\int_{T-2}^{T+1}\int_X \chi_3^2(\partial_t u)^2-2\chi_3\chi_3' u\partial_t u+2\chi_3^2 a\partial_t u+\chi_3^2\big|\sqrt{-\Delta}u\big|^2\,dx\,dt\end{aligned}$$

whence

$$\|\partial_t u\|_{L^2((T-1,T);L^2)}\leqslant C\|u\|_{L^2((T-2,T+1);H^1)},$$

giving

$$E(T)\leqslant C\|u\|^2_{L^2((T-2,T+1);H^1)}\leqslant Ce^{-T/C}\|f\|^2_{H^\epsilon(X)}$$

as claimed. □

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