Fricke Spaces

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INTRODUCTION

This paper is semi-expository; except for the existence of a natural metric (Theorem I in Section 12) most other results are either in the literature or may be considered folk-theorems. Since the subject is important and no full exposition can be found in the literature we decided to prove every result for which we know no good reference. The authors want to thank D. Sullivan for suggesting a simplification of a proof.

The name Fricke space for the object defined in Section 1 seems justified since the same object occurs in Fricke and Klein [FK], where it is defined in terms of Fuchsian groups. Fricke was primarily after a proof of the uniformization theorem (by the continuity method). Today, there are easier ways to reach this goal, but Fricke spaces are of interest in their own right. In particular, they have applications in the theory of spaces of Riemann surfaces with nodes; these will be discussed in a forthcoming paper by one of the authors.

The main results of this paper are formulated as Theorems I, II, III, and IV in Sections 12, 13, 14, and 15.

Conventions. Throughout this paper all surfaces are assumed to be oriented and all homeomorphisms between surfaces are assumed to be orientation preserving. On the other hand, closed curves on surfaces are, in general, assumed not to have a direction. Two closed curves on a surfaces

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are called freely homotopic if they become so after being given proper orientation.

By a Fuchsian group we mean, in this paper, a discrete subgroup of $PSL(2, \mathbb{R})$, acting on the upper half-plane $\mathbb{H}$ as a group of non-Euclidean motions.

1. Fricke Spaces and Fricke Modular Groups

In what follows $p$ and $n$ are fixed non-negative integers, and the numbers $a$ and $d$ are defined by

$$a = 2p - 2 + n, \quad d = 3p - 3 + n. \quad (1.1)$$

We always assume that

$$a > 0, \quad (1.2)$$

so that $d \geq 0$.

Throughout the paper $X$ denotes a topological (oriented) surface of topological type $(p, n)$, i.e., a surface homeomorphic to one obtained by removing $n$ distinct points from a closed surface of genus $p$. We shall be concerned with (orientation preserving) homeomorphisms $f$ of $X$ onto Riemann surfaces. Two such homeomorphisms, $f_1: X \to S_1$ and $f_2: X \to S_2$ will be called equivalent if there is a conformal map $h$ of $S_1$ onto $S_2$ such that the map

$$f_2^{-1} \circ h \circ f_1$$

is homotopic (or, which is here the same, isotopic) to the identity. The equivalence class of a map $f: X \to f(X)$ will be denoted by $[f]$. The set of all such equivalence classes is the Fricke space $\mathcal{F}(X)$ of $X$. We proceed to topologize this space.

Condition (1.2) insures that a Riemann surface $S$ of topological type $(p, n)$ carries a Poincaré metric (unique complete Riemannian metric of Gaussian curvature $-1$) consistent with the conformal structure of $S$. If $C$ is a closed (non-oriented) curve on $S$ we denote by $l_p(C)$ the infimum of the Poincaré lengths of all closed curves on $S$ freely homotopic to $C$. This number depends only on the free homotopy class of $C$; if it is positive, then it is the length of the unique closed (Poincaré) geodesic freely homotopic to $C$.

Now let $C$ be a closed curve on $X$. One sees easily that the number

$$l_{f(X)}(f(C))$$
depends only on the equivalence class $[f]$ of $f$ and on the free homotopy class of $C$. We write

$$l_{(f)}(C) = l_{f_0}(f(C)).$$

For a fixed $C$, $l_{(f)}(C)$ is a function from $\mathcal{F}(X)$ to the set $\mathbb{R}_+ \cup \{0\}$ of non-negative reals. We give to $\mathcal{F}(X)$ the weakest topology which makes all these functions continuous.

Now let $Y$ be another surface of type $(p, n)$. Every homeomorphism $\omega$ of $X$ onto $Y$ induces a bijection $\omega_*$, called an allowable mapping, of $\mathcal{F}(X)$ onto $\mathcal{F}(Y)$; it is defined by

$$\omega_*([f]) = [f \circ \omega^{-1}].$$

One verifies easily that this definition is legitimate, that $\omega_*$ depends only on the homotopy class of $\omega$, that $\id_* = \id$, and that $(\omega')_* = (\omega')_* \circ \omega_*$ (where $\omega'$ is a homeomorphism of $Y$ on some surface $Z$). Also, every allowable mapping $\omega_*$ is a homeomorphism of $\mathcal{F}(X)$ onto $\mathcal{F}(\omega(X))$.

The allowable self-mappings of $\mathcal{F}(X)$ form a group which we call the Fricke modular group of $X$ and denote by $\mathcal{M}(X)$. An allowable mapping $\omega$ of $\mathcal{F}(X)$ onto $\mathcal{F}(Y)$ conjugates $\mathcal{F}_\omega(X)$ onto $\mathcal{F}_\omega(Y)$.

(Two points, $[f_1]$ and $[f_2]$, of $\mathcal{F}(X)$ are equivalent under the Fricke modular group if and only if the Riemann surfaces $f_1(X)$ and $f_2(X)$ are conformally equivalent.)

The above considerations show that the Fricke space $\mathcal{F}(X)$ and the Fricke modular group $\mathcal{M}(X)$ actually depend only on the integers $p, n$. Therefore we denote them by $\mathcal{F}_{p,n}$ and $\mathcal{M}_{p,n}$ whenever convenient.

## 2. A Lemma on Absolute Traces

Throughout this paper we consider the upper half-plane $\mathbb{H} = \{ y > 0 \}$ as endowed with the Poincaré metric $ds = y^{-1}|dz|$, $z = x + iy$, i.e., as the non-Euclidean plane, and $\text{PSL}(2, \mathbb{R})$ as the group of conformal self-mapping of $\mathbb{H}$, i.e., of non-Euclidean motions. The absolute trace, $|\text{trace } g|$, of a real Möbius transformation $g \in \text{PSL}(2, \mathbb{R})$ is the absolute value of the trace of a real unimodular $2 \times 2$ matrix representing $g$.

**Lemma 2.1.** Let $g_1$ and $g_2$ be two hyperbolic real Möbius transformations such that the axis of $g_2$ intersects that of $g_1$ from left to right. Let $g_3$ be any real Möbius transformation. There are finitely many words in $g_1$, $g_2$, $g_3$, and three real Möbius transformations $h_1$, $h_2$, $h_3$ which are continuous functions of the absolute traces of these words, such that

$$g_v = \theta \cdot h_v \cdot \theta^{-1}$$

with $\theta \in \text{PSL}(2, \mathbb{R})$. 
If \( g_1 \) and \( g_2 \) are normalized so that the repelling and attracting fixed points of \( g_1 \) are 0 and \( \infty \), those of \( g_2 \) are a negative number and 1, then \( \theta = \text{id} \).

This is a slight extension of a result by Teichmüller \([T]\) and certainly not new (cf. Keen \([K]\) and Wolpert \([W]\)). A proof follows.

The condition on \( g_1 \) and \( g_2 \) means precisely that the normalization described in the lemma can be accomplished by a conjugation in \( \text{PSL}(2, \mathbb{R}) \); we assume this has been done. Then \( g_1 \) and \( g_2 \) can be represented by matrices

\[
A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda > 1
\]

and

\[
B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha, \beta, \gamma, \delta > 0, \quad \alpha \delta - \beta \gamma = 1, \quad \alpha + \beta = \gamma + \delta.
\]

Now \( \lambda \) can be computed from the relation \( \lambda + \lambda^{-1} = |\text{trace } g_1| \) and \( \alpha \) and \( \delta \) from the relations \( \alpha + \delta = |\text{trace } g_2|, \quad \lambda \alpha + \lambda^{-1} \delta = |\text{trace } (AB)| \). A simple calculation yields that \( 2\beta = \delta - \alpha + \sigma, \ 2\gamma = \alpha - \delta + \sigma, \ \sigma = \sqrt{(\alpha + \delta)^2 - 4} \).

Next, let \( g_3 \) be represented by the matrix

\[
C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1, \ a + d \geq 0.
\]

To find \( C \) we note the identity

\[
(\text{trace } P)(\text{trace } Q) = \text{trace } (PQ) + \text{trace } (P^{-1}Q)
\]

valid for any two \( 2 \times 2 \) unimodular matrices. If \( \text{trace } P \neq 0 \) and \( \text{trace } Q \neq 0 \), this identity permits us to compute \( \text{trace } (PQ) \) and \( \text{trace } (P^{-1}Q) \) as a continuous function of \( \text{trace } P \), \( \text{trace } Q \), \( |\text{trace } PQ| \), and \( |\text{trace } P^{-1}Q| \).

If \( \text{trace } C \neq 0 \) we can (noting that \( \text{trace } AB \neq 0 \)) compute \( \text{trace } AC \), \( \text{trace } BC \), and \( \text{trace } ABC \) as continuous functions of the traces of \( A, B, C \) (which are already known) and of the absolute traces of \( AC, A^{-1}C, BC, B^{-1}C, ABC, \) and \( B^{-1}A^{-1}C \).

Now we compute the numbers \( a \) and \( d \) from the relations

\[
a + d = \text{trace } C, \quad \lambda a + \lambda^{-1} d = \text{trace } AC
\]

and then the numbers \( b \) and \( c \) from the relations

\[
\beta c + \gamma b = \text{trace } (BC) = \alpha a - \delta d, \\
\lambda \beta c + \lambda^{-1} \gamma b = \text{trace } (ABC) = \lambda \alpha a - \lambda^{-1} \delta d.
\]
If trace $C = 0$ but trace $(AC) \neq 0$ (or trace $(BC) \neq 0$) we compute as before $AC$ or $BC$ and obtain $C$.

If, finally, trace $C = \text{trace } (AC) = \text{trace } (BC) = 0$, then an easy calculation shows that

$$C = \pm \begin{pmatrix} 0 & \sqrt{\beta/\gamma} \\ -\sqrt{\gamma/\beta} & 0 \end{pmatrix}.$$ 

Actually the case trace $C = 0$ is included only for the sake of completeness; it is not needed in what follows.

3. Fricke Spaces and Fuchsian Groups

In this section we construct a homeomorphism of $\mathcal{F}_{p,n} = \mathcal{F}(X)$ into $\mathbb{R}^N$, for $N$ sufficiently large; this will imply that $\mathcal{F}_{p,n}$ is a Hausdorff space and will have other consequences.

Before constructing the mapping we recall some properties of a (torsion free) Fuchsian group $\Gamma$ (operating on $\mathcal{U}$). Let $S = \mathcal{U}/\Gamma$. There is a canonical bijection between free homotopy classes of closed curves $C$ on $S$ and conjugacy classes, in $\Gamma$, of (not ordered) pairs $(\gamma, \gamma^{-1}), \gamma \in \Gamma$. This bijection is defined as follows. Given $\gamma \in \Gamma$, let $\zeta$ be any point in $\mathcal{U}$ and $\tilde{C}$ any curve in $\mathcal{U}$ joining $\zeta$ and $\gamma(\zeta)$. The canonical projection $\mathcal{U} \to S$ takes $\tilde{C}$ into a closed curve $C$. Conversely, given a closed curve $C$ on $S$, let $\tilde{C}$ be any lift of $C$ to $\mathcal{U}$, via the canonical projection $\mathcal{U} \to S$, and let $\gamma \in \Gamma$ identify the endpoints of $C$. The free homotopy class of $C$ depends only on the conjugacy class of the pair $(\gamma, \gamma^{-1})$ and vice versa. Permitting ourselves a slight abuse of language we shall say that $\gamma$ and $C$ correspond to each other.

To every pair $(\gamma, \gamma^{-1}), \gamma \in \Gamma$, there corresponds a unique closed geodesic $C$ on $S$, and vice versa. It is known, and easy to see, that the absolute trace $|\tau|$ of $\gamma$ and the length $l$ of $C$ are connected by the relation

$$\cosh \frac{l}{2} = \frac{|\tau|}{2}. \quad (3.1)$$

The construction of the desired mapping will depend on several choices, the most important being the choice of a conformal structure for $X$.

Remark. Choosing a conformal structure for $X$ amounts to choosing a point in $\mathcal{F}(X)$. Indeed, pick a point $[f_0]$ in $\mathcal{F}(X)$ and choose some representative, say $f_0$, of $[f_0]$. Now give $X$ the unique conformal structure which makes the map $f_0$ conformal. Every $[f]$ in $\mathcal{F}(X)$ can be written as $[f \circ f_0^{-1} \circ f_0]$ and the equivalence class of the map

$$f \circ f_0^{-1} : f_0(X) \to f(X)$$
depends only on $[f]$, and vice versa. Furthermore, the map $[f] \mapsto [f \cdot f_0^{-1}]$ is a homeomorphism of $\mathcal{F}(X)$ onto $\mathcal{F}(f(X_0))$.

Once $X$ is given a conformal structure, we choose a Fuchsian group $G$ such that $X$ can be identified with $\mathcal{U}/G$. In view of condition (1.2), $G$ may be chosen so that it contains two hyperbolic elements $g_1$ and $g_2$ represented by matrices of the form (2.1) and (2.2), respectively. Let $f$ be a homeomorphism of $X$ onto a Riemann surface. Since there is a holomorphic universal covering of $f(X)$ by $\mathcal{U}$, determined but for a premultiplication by an element of $PSL(2, \mathbb{R})$, and since any homeomorphism $f$ between Riemann surfaces can be lifted to a homeomorphism $\tilde{f}$ between their universal covering surfaces, there is a commutative diagram

\[ \begin{array}{ccc}
\mathcal{U} & \xrightarrow{f} & \mathcal{U} \\
\downarrow & & \downarrow \\
X = \mathcal{U}/G & \xrightarrow{\tilde{f}} & \mathcal{U}/\tilde{G}\tilde{f}^{-1} = f(X)
\end{array} \]  

(3.2)

where the vertical arrows represent holomorphic universal coverings with covering groups $G$ and $\tilde{G}\tilde{f}^{-1}$. Also, there is an isomorphism

\[ G \ni g \mapsto \tilde{f} \circ g \circ \tilde{f}^{-1} = \chi_f(g) \in \tilde{G}\tilde{f}^{-1}. \]

The map $\tilde{f}$ need not extend by continuity to $\hat{\tilde{\mathbb{R}}} = \mathbb{R} \cup \{ \infty \}$, as it would if $f$ and therefore $\tilde{f}$ were quasiconformal; neither must $\chi_f(g)$ be hyperbolic or parabolic if $g \in G$ is.

**Lemma 3.1.** In the situation described above the Möbius transformations $x_f(g_1)$ and $x_f(g_2)$ are hyperbolic, with four distinct fixed points, and the repelling fixed point of $x_f(g_2)$, the repelling fixed point of $x_f(g_1)$, the attracting fixed point of $x_f(g_2)$, and the attracting fixed point of $x_f(g_1)$ follow each other in this order.

**Proof.** Let $\alpha_1$ and $\alpha_2$ be the $\tilde{f}$-images of the axes of $g_1$ and $g_2$, respectively. Then $x_f(g_v)$ stabilizes $\alpha_v$, $v = 1, 2$. Each curve $\alpha_v$ must have two not necessarily distinct endpoints on $\mathbb{R} \cup \{ \infty \}$, namely, the fixed points of $x_f(g_v)$. If one of the Möbius transformations $x_f(g_v)$, $v = 1, 2$, were parabolic, $x_f(g_1)$ and $x_f(g_2)$ would have had a fixed point in common and would either commute (if both were parabolic) or generate a non-discrete group (if precisely one were). Neither can happen.

The statement about the relative position of the fixed points follows from the observation that if we assign to $\alpha_v$ the direction by which it is moved by $x_f(g_v)$, then $\alpha_2$ intersects $\alpha_1$ from left to right.
In view of Lemma 3.1, one may require that \( x_f(g_1) \) have the attracting fixed point \( \infty \) and the repelling fixed point 0, and that the attracting fixed point of \( x_f(g_2) \) be at 1 and the repelling one at a negative real number.

This can be achieved by replacing, if need be, \( f \) by \( \gamma \circ f \) (where \( \gamma \) is the unique real Möbius transformation which puts the fixed points of \( x_f(g_1) \) and \( x_f(g_2) \) in the desired position), \( G \) by \( \gamma G \gamma^{-1} \), and \( f \) by \( h \circ f \) (where \( h \) is a conformal map defined by the commutativity of the right square in the diagram (3.3)):

\[
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{f} & \mathcal{U} \\
\downarrow & & \downarrow \\
\mathcal{U}/G & \xrightarrow{G} & \mathcal{U}/G \\
\end{array}
\]

Of course, \( f \) and \( h \circ f \) belong to the same equivalence class \([f]\).

On the other hand, if \( \psi \) is a homeomorphism of \( \mathcal{U}/G \) onto itself which is homotopic to the identity, then there is a homeomorphism \( \tilde{\psi} \) of \( \mathcal{U} \) onto itself commuting with every \( g \) in \( G \) and such that the diagram

\[
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{\tilde{\psi}} & \mathcal{U} \\
\downarrow & & \downarrow \\
\mathcal{U}/G & \xrightarrow{\psi} & \mathcal{U}/G \\
\end{array}
\]

commutes. This square can be attached to the left side of diagram (3.3) and this shows that the isomorphism \( x_{f, f} \) depends only on the equivalence class \([f]\).

**Proposition 3.1.** Let \( X = \mathcal{U}/G \) where \( G \) is a (torsion free) Fuchsian group containing two hyperbolic elements \( \gamma_1, \gamma_2 \) with repelling fixed points 0 and \( \eta < 0 \) and attracting fixed points \( \infty \) and 1. For every \([f]\) in \( \mathcal{F}(X) \) there exists a canonical isomorphism

\[
x_{[f]}: G \to PSL(2, \mathbb{R})
\]

such that (i) the repelling (attracting) fixed points of \( x_{[f]}(g_1) \) and \( x_{[f]}(g_2) \) are at 0 and at \( \eta' < 0 \) (at \( \infty \) and at 1) and (ii) \( x_{[f]}(G) \) is a Fuchsian group and \( x_{[f]} \) is induced by a lift to \( \mathcal{U} \) of a homeomorphism of \( X = \mathcal{U}/G \) onto \( f(X) = \mathcal{U}/x_{[f]}(G) \) which belongs to \([f]\).

**Proof.** Using the notation introduced above set \( x_{[f]} = x_{h, f} \).
PROPOSITION 3.2. Under the hypothesis of Proposition 3.1 there is a finite set of elements \((g_1, g_2, \ldots, g_N)\) in \(G\) such that the map

\[
[f] \mapsto (|\text{trace } x_{[f]}(g_1)|, \ldots, |\text{trace } x_{[f]}(g_N)|)
\]

is a homeomorphism of \(\mathcal{F}(X)\) into \(\mathbb{R}^N_+\).

Proof. There are finitely many elements \(g_3, \ldots, g_n\) of \(G\) such that \((g_1, \ldots, g_n)\) is a set of generators for this group, and, by Lemma 2.1, there are finitely many elements \(g_{n+1}, \ldots, g_N\) of \(G\) such that \(g_1, \ldots, g_n\) are continuous functions of the absolute traces of \(g_1, \ldots, g_N\). Clearly, the elements

\[
x_{[f]}(g_1), \ldots, x_{[f]}(g_N)
\]

generate \(x_{[f]}(G)\) and, as one convinces oneself by reviewing the proof of Lemma 2.1, depend continuously on the numbers

\[
|\text{trace } x_{[f]}(g_1)|, \ldots, |\text{trace } x_{[f]}(g_N)|.
\]

Knowing the numbers (3.6) we know the isomorphism \(x_{[f]}\) and therefore, by a known topological theorem (cf., for instance, the reasoning in \([B_1, \text{pp. } 98-100]\)), the homotopy class of \(f\) which determines the equivalence class \([f]\). Hence (3.4) is an injection.

The map (3.4) is continuous since, for \(j = 1, \ldots, N\),

\[
|\text{trace } x_{[f]}(g_j)| = 2 \cosh \frac{1}{2} l_{[f]}(C_j)
\]

where \(C_j\) is a closed curve on \(X\) corresponding to \(g_j\) (cf. relation (3.1)). The number \(l_{[f]}(C_j)\) is a continuous function of \([f] \in \mathcal{F}(X)\) by the definition of the topology of \(\mathcal{F}(X)\).

The map inverse to (3.4) is also continuous since, for every closed curve \(C\) on \(X\) corresponding to an element \(g\) in \(G\), the number

\[
l_{[f]}(C) = 2 \text{ arcosh } \frac{1}{2} |\text{trace } x_{[f]}(g)|
\]

depends continuously on the Möbius transformations (3.5), since \(x_{[f]}(g)\) is a word in (3.5). On the other hand, the generators (3.5) depend continuously on the \(N\) numbers (3.6), by construction.

PROPOSITION 3.3. There is a finite set of closed curves \(C_1, \ldots, C_N\) on \(X\) such that the map

\[
[f] \mapsto (l_{[f]}(C_1), \ldots, l_{[f]}(C_N))
\]

is a homeomorphism of \(\mathcal{F}(X)\) into \((\mathbb{R}_+ \cup \{0\})^N\).
Without loss of generality we may assume the hypothesis of Proposition 3.1. For \( j = 1, \ldots, N \) let \( C_j \) be a closed curve on \( X \) which corresponds to the element \( g_j \) (in \( G \)) of Proposition 3.2. Note relation (3.1) and apply that proposition.

Remark. By the method used in [FLP, pp. 131–137] this proposition can be improved in two ways. The curves \( C_1, \ldots, C_n \) may be taken to be simple closed curves. Moreover, the element \([f']\) in \( F(X) \) is determined uniquely by the projective class of the vector

\[
(l_{[f]}(C_1), \ldots, l_{[f]}(C_n)).
\]

However, we will not need either of these results in what follows.

### 4. Punctures, Holes, and Collars

In this section we recall some known facts about Riemann surfaces. Throughout this section \([f']\in \mathcal{P}(X)\) and \( f(X) = \mathcal{U}/G = S \) where \( G \) is a Fuchsian group.

A Jordan curve \( C \) on \( X \) defines an end \( B \) of \( X \) if \( X \setminus C \) has two components precisely one of which is doubly connected. If so, \( f(C) \) defines an end \( f(B) \) of \( f(X) \). If the Jordan curve \( C' \) is freely homotopic to \( C \), \( C' \) and \( C \) define the same end, and vice versa.

Let \( C \) be as above. The radius of the end \( f(B) \) is the number

\[
r = \frac{1}{2\pi} l_{[f]}(C)
\]

and \( f(B) \) is called a hole or a puncture according to whether \( r > 0 \) or \( r = 0 \). Every hole on \( f(X) \) is defined by a unique geodesic Jordan curve \( C_0 \) (of length \( 2\pi r \)), we call \( C_0 \) an outer loop belonging to the end, and we call the doubly connected component of the complement \( S \setminus C_0 \) the funnel adjacent to \( C_0 \). (The other component has topological type \((p, n)\).) Two distinct outer loops (or two distinct funnels) are disjoint. The complement in \( S \) of all funnels is called the Nielsen core of \( S \) and will be denoted by \( K(S) \). The set \( K(S) \) is convex (in the Poincaré metric on \( S \)); it is compact if and only if \( S \) has no punctures.

The above statements are best understood (and can be proved) by considering the limit set \( A \) of \( G \), i.e., the set of accumulation points of orbits \( \{g(z)\}, z \) fixed, \( g \in G \). This set either coincides with \( \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \) or is a closed, nowhere dense subset of \( \hat{\mathbb{R}} \). (By assumption (1.2) the set \( A \) is infinite.) The first case happens if and only if \( S \) has no holes. In the second case, let \( I \) be a component of \( \hat{\mathbb{R}} \setminus A \). Then \( I \) is an open interval and the
stabilizer $G_I$ of $I$ in $G$ is a cyclic group $I_\gamma$ generated by a hyperbolic $\gamma \in G$ which fixes the endpoints of $I$. Let $O_I$ denote the subdomain of $\mathcal{U}$ bounded by $I$ and the axis $z_\gamma$ of $\gamma$, i.e., the Poincaré line in $\mathcal{U}$ with the same endpoints as $I$. The canonical projection $\pi: \mathcal{U} \rightarrow S = \mathcal{U}/G$ maps $z_\gamma$ onto an outer loop and $O_I$ onto the funnel adjacent to this loop. All funnels are obtained this way, and

$$\pi^{-1}(\mathcal{K}(S)) = \mathcal{U} \setminus \Sigma O_I$$

where $\Sigma$ denotes, here and hereafter, disjoint union, and $I$ runs over the components of $\mathcal{R} \setminus A$.

A funnel $F$ adjacent to an outer loop of length $l$, together with the restriction to $F$ of the Poincaré metric on $S$, is uniquely determined by the number $l$. Indeed, $C \cup F$ is isometric to the quotient of the subdomain $0 < \arg z < \pi/2$ of $\mathcal{U}$ by the group generated by $z \mapsto e^l z$.

It is not difficult to verify (cf, [B3]) that every Riemann surface $S$ of topological type $(p, n)$. $2p - 2 + 2n > 0$, is the Nielsen core of a uniquely determined Riemann surface $S'$; $S'$ has the same type as $S$ and the same number of punctures. We call $S'$ the Nielsen extension of $S$ and denote it by $N(S)$. If $S$ has no holes, $\mathcal{K}(S) = N(S) = S$.

The punctures of $S$ are in a canonical bijection with conjugacy classes (in $G$) of maximal parabolic subgroups of $G$. More precisely, let $\gamma$ be a generator of such a subgroup $\Gamma_\gamma$ and let $Q_\gamma$ be the fixed point of $\gamma$. Then the puncture on $S$ corresponding to the subgroup $\Gamma_\gamma$ can be defined by $\pi(C_0)$ where $C_0$ is a suitably chosen horocycle at $Q_\gamma$, i.e., a circle tangent to $\mathbb{R}$ at $Q_\gamma$ and lying, except for the point $Q_\gamma$, in $\mathcal{U}$. (If $Q_\gamma = \infty$, the circle is to be interpreted as a horizontal line.) If $\gamma$ is the map $z \mapsto z + 1$, as can always be achieved by conjugation, every “circle” $\Im z = y_0 > 1$ is suitable. This result is due to Shimizu [S] and to Leutbecher [L] and is a special case of an important proposition which we are about to state. First some definitions.

A (non-oriented) geodesic Jordan curve on $S$ will be called a loop, an inner loop if it is not outer, i.e., if it does not define a hole. Let $C$ be a loop (inner or outer) and $\varepsilon$ a positive number. An $\varepsilon$-collar about a loop $C$ is a doubly connected subdomain of $S$, of area $2\varepsilon$ (sic), bounded by two distinct Jordan curves $C'$ and $C''$ which have the same constant distance from $C$.

If $C$ has length $l$ and is the image under $\pi$ of the positive imaginary axis (which can be achieved by a conjugation), then an $\varepsilon$-collar about $C$, if it exists, is the image under $\pi$ of the sector $|\pi/2 - \arg z| < \theta < \pi/2$ where

$$\theta = \arctan(\varepsilon/l).$$

The length $m$ of the Jordan curves $C'$ and $C''$ (the images under $\pi$ of the rays $\arg z = \pm \theta$) is given by

$$m = l/\cos \theta.$$
so that

$$m^2 = l^2 + \varepsilon^2,$$

and the distance \( \delta \) from \( C' \) (or \( C'' \)) to \( C \) is

$$\delta = \log \left( \frac{1 + \sin \theta}{\cos \theta} \right),$$

so that

$$\delta = \log \left( \sqrt{1 + \varepsilon^2/l^2} + \varepsilon/l \right).$$  \hspace{1cm} (4.3)

The formulas (4.2) and (4.3) are, of course, valid independently of the normalization, i.e., of the condition that \( \pi^{-1}(C) \) be the positive imaginary axis.

For future reference we note that for \( \varepsilon_1 > \varepsilon_2 > 0 \) the distance between the boundaries of the \( \varepsilon_1 \)-collar and the \( \varepsilon_2 \)-collar about \( C \) (if both exist) is given by

$$\delta_{\varepsilon_1, \varepsilon_2} = \log \left( \frac{\sqrt{l^2 + \varepsilon_1^2} + \varepsilon_1}{\sqrt{l^2 + \varepsilon_2^2} + \varepsilon_2} \right).$$  \hspace{1cm} (4.4)

as follows at once from (4.3).

An \( \varepsilon \)-collar about a puncture on \( S \) is a doubly connected domain in \( S \), of area \( \varepsilon \) (sic), bounded by the puncture and by a horocycle (i.e., a Jordan curve \( C \) orthogonal to the pencil of geodesics leading toward the puncture).

If the puncture is defined by the \( \pi \)-image of the line \( \text{Im } z = y_0 > 0 \) and the corresponding maximal parabolic subgroup of \( G \) is generated by \( z \mapsto z + 1 \) (both conditions can be achieved by a conjugation), then the \( \varepsilon \)-collar about the puncture, if it exists, is the \( \pi \)-image of the half-plane \( \text{Im } z > 1/e \). We conclude that the horocycle bounding the \( \varepsilon \)-collar has length \( \varepsilon \), and that for \( \varepsilon_1 > \varepsilon_2 > 0 \) the distance between an \( \varepsilon_1 \)-collar and an \( \varepsilon_2 \)-collar about the same puncture is

$$\delta_{\varepsilon_1, \varepsilon_2} = \log(\varepsilon_1/\varepsilon_2).$$  \hspace{1cm} (4.5)

Thus formula (4.4) remains valid for \( l = 0 \) and \( \delta = \infty \).

**Proposition 4.1 (Collar Lemma).** About every loop of length \( l \) and about every puncture (of length \( l = 0 \), by definition) on a Riemann surface \( S \) of type \((p, n)\) there is an \( \varepsilon \)-collar, for every \( \varepsilon \) with

$$0 < \varepsilon < \frac{l}{2 \csc h \frac{l}{2}},$$

(\textit{where } \csc h \textit{ is the hyperbolic cosecant function}).
Two such collars, about two distinct loops or about two distinct punctures, or about a loop and a puncture, do not intersect.

The existence of collars has been proved by Keen for small $l$ and by Halpern [H] for all $l$; the sharp condition (4.6) is due to Matelski [Ma] (cf. also Buser [Bu]).

5. De Rham's Lemma

In the following sections we shall have to recognize a non-Euclidean polygon in $\mathcal{U}$ as the fundamental polygon of a Fuchsian group. The relevant theorem goes back to Poincaré; correct proof will be found, for instance, in [Si, M, D] (the literature contains some incorrect ones). Actually we need a weaker form of Poincaré's theorem which occurs in de Rham's elegant argument.

PROPOSITION 5.1 (de Rham's Lemma). Hypothesis: $D \subset \mathcal{U}$ is a Jordan domain whose boundary $\partial D$ consists of $2r$ Jordan arcs $\alpha_1, \beta_1, \ldots, \alpha_r, \beta_r$, lying in $\mathcal{U}$ ($r \geq 1$), $s$ subarcs of $\mathbb{R} = \mathbb{R} \cup \{ \infty \}$ ($s \geq 0$), and of their endpoints. Each $\alpha_j$ is either a non-Euclidean segment, or a non-Euclidean ray or a non-Euclidean line. There are given $r$ elements $g_1, \ldots, g_r$ of $PSL(2, \mathbb{R})$ such that $g_j(\alpha_j) = \beta_j$. The space $S$ obtained from $D \cup (\partial D \cap \mathcal{U})$ by identifying points of $\alpha_j$ (including the endpoints of $\alpha_j$ which lie in $\mathcal{U}$) with their images under $g_j$ ($j = 1, \ldots, s$) is an orientable $C^\infty$ surface carrying a complete Riemann metric $ds$ of Gaussian curvature $-1$; the restriction of this metric to $D \subset S$ coincides with the Poincaré metric $(\Im z)^{-1}|dz|$. 

Conclusion: The group $G$ generated by $g_1, \ldots, g_r$ is discrete (and thus acts properly discontinuously on $\mathcal{U}$) and torsion free, the quotient $\mathcal{U}/G$ is isometric to $S$, and $D$ is a fundamental polygon for $G$ in $\mathcal{U}$ (i.e., $g(D) \cap D = \emptyset$ for $id \neq g \in G$ and each $z \in \mathcal{U}$ belongs to $g(D \cup (\partial D \cap \mathcal{U}))$ for some $g \in G$).

For the convenience of the reader we present a proof.

There is a universal isometric covering $\pi: \mathcal{U} \to S$, since $ds$ lifts to a complete Riemannian metric with Gaussian curvature $(-1)$ on the (simply connected) universal covering surface $S$. Let $\Gamma \subset PSL(2, \mathbb{R})$ be the covering group of $\pi$. It is clear that every component $A$ of $\pi^{-1}(D)$ is a fundamental domain for $\Gamma$ in $\mathcal{U}$. We fix a $A$ and note that $\pi|A$ is an isometry of $A$ onto $D$, i.e., the restriction of an element $\sigma \in PSL(2, \mathbb{R})$ to $A$. Replacing if need be $\pi$ by $\pi \circ \sigma^{-1}$ we may and do assume that $\pi(D) = D$ and $\pi|D = id$.

We must show that

$$G = \Gamma.$$  

(5.1)
Let $z_j$ be an (inner) point of $x_j$. There must be an element $g_j \in \Gamma$ such that for $\zeta \in z_j$ and close to $z_j$, we have $\hat{g}_j(\zeta) = g_j(\zeta)$. This implies that $\hat{g}_j = g_j$ and $g_j \in \Gamma$. Since this holds for $j = 1, \ldots, r$ we conclude that $G \subset \Gamma$.

Next $g_j(D)$ is a fundamental polygon for $\Gamma$ adjacent to $D$, with $g_j(z_j) = \beta_j$ being a common side of $D$ and $g_j(D)$. Similarly $g_j^{-1}(D)$ is a fundamental polygon for $\Gamma$ with $z_j$ being a common side of $D$ and $g_j^{-1}(D)$. Since a point in every fundamental domain $g(D)$, $\hat{g} \in \Gamma$, can be joined to a point in $D$ by a curve in $\mathcal{M}$ which avoids all points $\Gamma$-equivalent to endpoints of $x_1, \ldots, x_r$, we conclude that for every $\hat{g} \in \Gamma$ there is a $g \in G$ with $\hat{g}(D) = g(D)$. The last equation implies that $\hat{g} \circ g^{-1} = \text{id}$. Thus $\Gamma \subset G$ and (5.1) holds.

**Remarks.** (1) This proof does not depend on the uniformization theorem and can be used to establish this theorem by the classical continuity method or by the method of quasiconformal mappings (cf. [B2]).

(2) Statement and proof of the lemma can be extended to infinitely generated Fuchsian groups and groups with torsion. There is no need to do it here.

6. **Triply Connected Domains**

In this section (and in this section only) $S$ denotes a Riemann surface of topological type $(0, 3)$ and $q$ the number of punctures of $S$.

**Lemma 6.1.** (a) $S$ is conformally equivalent to its mirror image.

(b) There is a canonical anticonformal involution $J$ on $S$ which fixes the ends of $S$.

**Proof.** We consider only the case $q = 0$ (the cases $q = 1, 2, 3$ are similar but simpler). Assume (since this can be achieved by a conformal map) that $S$ is a plane domain bounded by the circles $|z| = r < 1$ and $|z| = R > 1$ and by an arc of the unit circle which is symmetric about the imaginary axis. The maps $z \mapsto \bar{z}$ and $z \mapsto -z$ take $S$ into the mirror image of $S$. The desired involution $J$ is $z \mapsto -\bar{z}$. The uniqueness of $J$ is trivial.

From now on we assume (since this can be achieved by a conformal map) that $S \subset \mathbb{C}$ and that each of the three boundary components of $S$ is either a point or an analytic Jordan curve. We order the boundary components (which are the ends of $S$) and denote them by $B_1$, $B_2$, $B_3$. If $B_i$ is not a point, let $C_i$ denote the outer loop of $S$ about $B_i$; if $B_i$ is a point, set $C_i = B_i$. Also, let $E_{ik} = E_{ki}$ (where $\{i, k\} \subset \{1, 2, 3\}$ and $i \neq k$) denote the simple Poincaré geodesic on $S$ which joins $B_i$ to $B_k$ and is orthogonal to $C_i$ ($r = i, k$) if $C_i$ is not a point. (The existence and uniqueness of $E_{ik}$ is easily established.) If $C_i$ is not a point, we denote by $C_i \cap E_{jk}$ the intersection point of $C_i$ and $E_{jk}$, for $\{s\} \in \{1, 2, 3\} \setminus \{i\}$. We call the curves $E_{jk}$ the con-
necting geodesic on $S$ and the two points $C_j \cap E_{j\ell}$ and $C_j \cap E_{j'}$ (where \( \{r, s\} = \{1, 2, 3\} \setminus \{j\} \) the distinguished points on $C_j$. Also $C_j \cap E_{j\ell}$ will be called the first distinguished point on $C_j$ if $j = 2$ or $3$ and $i = j - 1$ or if $j = 1$ and $i = 3$.

**Lemma 6.2.** (a) Two connecting geodesics on $S$ do not intersect in $S$.

(b) The two distinguished points on an outer loop $C$ on $S$ bisect this loop.

Proof. (a) If they would, there would be a non-Euclidean triangle with angles adding up to more than $\pi$.

(b) Since the anticonformal involution $J$ is an isometry of $S$ it fixes the two distinguished points on $C$. Hence $J$ maps the two arcs of $C$ with distinguished points as endpoints onto each other. Since $J$ is an isometry of $C$, the arcs have the same lengths.

Let $a_1, a_2, a_3$ be three non-negative numbers. It is not difficult to verify, by a direct calculation, that there are in the non-Euclidean plane (which we represent as the upper half-plane $\mathbb{H}$ with its Poincaré metric) three lines $L_1$, $L_2$, $L_3$ which follow each other in this order (in the counterclockwise direction) such that the distances between $L_2$ and $L_3$, $L_3$ and $L_1$, and $L_1$ and $L_2$ are $a_1$, $a_2$, $a_3$, respectively. By a slight abuse of language we call the domain in $\mathbb{H}(a_1, a_2, a_3) \subset \mathbb{H}$ whose boundary in $\mathbb{H}$ is $L_1 \cup L_2 \cup L_3$ the large (non-Euclidean) hexagon determined by $(a_1, a_2, a_3)$. Clearly $\mathbb{H}$ is determined by $(a_1, a_2, a_3)$ but for an (orientation preserving) non-Euclidean motion. Also, $\mathbb{H}$ is convex, is a true hexagon if $q = 0$, and has $6 - q$ vertices if $q > 0$.

**Lemma 6.3.** Let $S$ and $C_1, C_2, C_3$ be as above, and let $2a_1, 2a_2, 2a_3$ denote the lengths of the curves $C_1, C_2, C_3$ (with $a_i = 0$ when $C_i$ is a point). Then $S$ is divided by the connecting geodesics into two domains conformally equivalent to a large hexagon $\mathbb{H}$ determined by $(a_1, a_2, a_3)$ and to its mirror image.

This follows from Lemma 6.2.

Let $L_1, L_2,$ and $L_3$ be as above and let $A_{ij}, j=1, 2, 3,$ be the line orthogonal to $L_i$ and $L_k$, where \( \{i, k\} = \{1, 2, 3\} \setminus \{j\} \) if $a_j > 0$, the common endpoints of $L_i$ and $L_k$ if $a_j = 0$. Let $H_j (j = 1, 2, 3)$ be the line joining $L_i$ and $A_{ij}$ which is orthogonal to $L_i$ and, if $a_j > 0$, also to $A_{ij}$. We call $H_1, H_2, H_3$ the axes of $\mathbb{H}$.

**Lemma 6.4.** The axes of $\mathbb{H}$ intersect at one point $Z \in \mathbb{H}$.

This is a known theorem in elementary non-Euclidean geometry (cf. [Bu]). We call $Z$ the centroid of $\mathbb{H}$.
The centroid $Z$ divides each axis $H_j$ into two rays; let $H'_j$ be the ray which does not intersect $L_j$. We denote by $\sigma_j$ the reflection about the line $L_j$ and we set

$$F_1 = \sigma_1(H'_2 \cup H'_3), \quad F_2 = \sigma_2(H'_3 \cup H'_1), \quad F_3 = \sigma_3(H'_1 \cup H'_2),$$

(6.1)

$$\gamma_1 = \sigma_3 \circ \sigma_2, \quad \gamma_2 = \sigma_1 \circ \sigma_3, \quad \gamma_3 = \sigma_2 \circ \sigma_1.$$  

(6.2)

The $\gamma_j$ are, of course, non-Euclidean motions (i.e., elements of $PSL(2, \mathbb{R})$) and

$$\gamma_3 \circ \gamma_2 \circ \gamma_1 = 1.$$

(6.3)

We also denote by $\Lambda = \Lambda(2a_1, 2a_2, 2a_3)$ the subdomain of $\mathcal{U}$ whose boundary in $\mathcal{U}$ is $F_1 \cup F_2 \cup F_3$. This is a convex polygon with three vertices in $\mathcal{U}$ and $6-q$ vertices on $\mathbb{R} \cup \{\infty\}$.

**Lemma 6.5.** For any ordered triple of non-negative numbers $(a_1, a_2, a_3)$, and with the notations introduced above, the group $\Gamma$ generated by $\gamma_1, \gamma_2, \gamma_3$ is a torsion-free Fuchsian group with $\mathcal{U}/\Gamma$ conformally equivalent to a triply connected plane domain with ordered ends, $a_j$ being half the Poincaré length of the outer loop belonging to the $j$th end if this end is not a puncture, and being 0 if it is. The polygon $\Lambda$ is a fundamental domain for $\Gamma$ in $\mathcal{U}$.

The proof follows at once from de Rham's lemma (Proposition 5.1).

**Addition 1 to Lemma 6.5.** Every Riemann surface of topological type $(0, 3)$ can be so represented.

This follows from Lemma 6.3. A second addition to this lemma will be found in Section 9.

Let us now normalize the construction of $\mathcal{U}$ by the following requirement:

$$\begin{cases}
\text{the endpoints of } L_1 \text{ are } 0 \text{ and } r, \quad 0 < r \leq 1 \\
\text{the endpoints of } L_2 \text{ are } 1 \text{ and } s, \quad 1 < s \leq \infty \\
\text{the endpoints of } L_3 \text{ are } \infty \text{ and } t, \quad t \leq 0.
\end{cases}$$

(6.4)

Then the generators $\gamma_1, \gamma_2, \gamma_3$ can be written down explicitly as real analytic functions of the lengths $2a_1, 2a_2, 2a_3$ of the outer loops of $S$. On the other hand, these lengths are, by definition, continuous functions on $F_{0,3}$. Combining this remark with Lemma 6.5 and the results of Section 3 we obtain

**Proposition 6.1.** Let $X$ be a surface of type $(0, 3)$, let $\beta_1, \beta_2, \beta_3$ be the ends of $X$, and let $\Phi$ be the map which assigns to $[f] \in \mathcal{F}(X)$ the triple...
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\((2a_1, 2a_2, 2a_3)\) defined as follows: if \(f(\beta)\) is a puncture on \(f(X)\), \(a_j = 0\); if it is not, then \(2a_j\) is the length of the outer loop on \(f(X)\) defined by \(f(\beta)\).

Then \(\Phi\) is a homeomorphism of \(\mathcal{F}(X) = \mathcal{F}_{0, 3}\) onto \((\mathbb{R}_+ \cup \{0\})^3\).

7. REDUCED FENCHEL–NIELSEN MAPS

Let \(X\) be as in Section 1. A maximal partition of \(X\) (also called a pant decomposition) is a set of \(d = 3p - 3 + n\) disjoint, homotopically non-trivial Jordan curves (partition curves) on \(X\) none of which is freely homotopic to another and none of which defines an end of \(X\). A component of the complement of the union of all partition curves is called a region of the partition; in our case there are \(a = 2p - 2 + n\) regions, each a triply connected plane domain. Each end of a region \(\sigma\) is a bank of a partition curve or of an end of \(X\); the reader will have no difficulty in defining the term “bank” precisely should she or he so desire. If \((p, n) \neq (0, 3)\), at least one bank of a region is the bank of a partition curve.

A maximal partition is ordered by ordering the ends \(B_1, \ldots, B_n\) of \(X\), the partition curves \(C_1, \ldots, C_d\), the regions \(\sigma_1, \ldots, \sigma_a\), and the three banks \(b_{11}, b_{12}, b_{13}\) of each region \(\sigma_j\). Nothing would be gained or lost had we required that the ordering of the region be derived (lexicographically) from that of the ends and partition curves, or had we given only a cyclic ordering of the banks of a region.

We do require, only for the sake of convenience, that the ordering of the partition be consistent in the following sense. Let the banks of the \(j\)th partition curve \(C_j (j = 1, \ldots, d)\) be \(b_{k_j \mu_j}\) and \(b_{l_j \nu_j}\), with either \(k_j < l_j\) or \(k_j = l_j\) and \(\mu_j < \nu_j\). (Recall that \(b_{pq}\) is the \(q\)th bank of the \(p\)th region \(\sigma_p\).) Then

\[
\begin{align*}
k_1 &= 1, \\
l_1 &= 2, \\
k_2 &\in \{1, 2, \ldots, r\}, \\
l_r &= r + 1 \quad (r = 2, \ldots, a - 1).
\end{align*}
\]

Condition (7.1) is vacuous if \(a = 0\). Condition (7.2) is vacuous if \(a = 2\).

We assume from now on that \(X\) is equipped with a consistently ordered maximal partition. (If \(p = 0, n = 3\), there are no partition curves. An ordered partition is simply an ordering of the three ends of \(X\).)

A homeomorphism \(f\) of \(X\) onto a Riemann surface will be called standardized, with respect to the given partition, if \(f\) maps every partition curve onto a geodesic.

**Proposition 7.1.** Every \([f] \in \mathcal{F}(X)\) contains a standardized homeomorphism.

**Proof.** For every partition curve \(C_j, f(C_j)\) is freely homotopic to a unique geodesic Jordan curve \(\Gamma_j, j = 1, \ldots, d\), and \(\Gamma_i \cap \Gamma_j = \emptyset\) for \(i \neq j\). By a
known topological theorem (cf. Epstein [E]) there is a homeomorphism \( \omega \) of \( S \) onto itself which maps \( f(C_j) \) onto \( \Gamma_j, j = 1, \ldots, d \), and is homotopic to the identity. The map \( \omega \circ f : X \to S \) is standardized; it is equivalent to \( f \) since

\[
f^{-1} \circ (\omega \circ f) : X \to X
\]

is homotopic to the identity.

Let \([f] \in \mathcal{F}(X)\), with \( f \) standardized. This condition does not determine \( f \) uniquely. But the geodesics \( f(C_j) \), the ends \( f(B_i) \), the regions \( f(\sigma_k) \), and the banks \( f(b_{kr}) \)—the notations are self-explanatory—depend only on \([f]\), so that \( f(X) \) is equipped with a consistently ordered maximal partition with geodesic partition curves.

Let \( \sigma \) be a partition region \( s \) of whose banks are banks of partition curves (so that \( s = 1, 2 \) or 3); we call them the bounding partition curves of \( \sigma \). It is convenient to introduce the domain \( f(\sigma)_{\text{ext}} \), which is the union of \( f(\sigma) \), the images \( f(C) \) of the \( s \) bounding partition curves \( C \) of \( \sigma \) and of \( s \) funnels attached to the geodesic Jordan curves \( f(C) \). If \( s = 3 \), then \( f(\sigma)_{\text{ext}} \) is the Nielsen extension of \( f(\sigma) \).

On every partition curve \( f(C_j), j = 1, \ldots, d \), there are two (not necessarily distinct) distinguished points "belonging" to the two banks of \( f(C_j) \) and defined as follows.

Let \( h \) be a bank of \( C_j, \sigma_i \), the unique partition region on \( X \) containing \( b \), and \( f(\sigma_i)_{\text{ext}} \) the domain defined above. The ordering of the banks of \( \sigma_i \) induces an ordering of the ends of \( f(\sigma_i)_{\text{ext}} \); \( f(C_j) \) is an outer loop on this domain. The distinguished point on \( f(C_j) \) belonging to the bank \( f(h) \) is the first distinguished point on \( f(C_j) \), as defined in Section 6 (viewing \( f(C_j) \) as a loop on \( f(\sigma_i)_{\text{ext}} \)).

Remark 7.1. It is important to remember that the restriction of the Poincaré metrics on \( f(X) \) and on \( f(\sigma_i)_{\text{ext}} \) to \( f(\sigma_i)_{\text{ext}} \) coincide. This is an immediate consequence of the definition in Section 4.

Let \([f'] \in \mathcal{F}(X)\), with \( f \) standardized. For every end \( B_k \) of \( X \), let \( 2\pi \rho_k \) be the length of the outer loop \( f(X) \) defining \( f(B_k) \), with \( \rho_k = 0 \) if \( f(B_k) \) is a puncture. For every partition curve \( C_j \), let \( 2\pi r_j \) be the length of the loop \( f(C_j) \), and let \( \theta_j \) denote any real number with the following property: if we approach a distinguished point \( P \) on \( f(C_j) \) along a geodesic segment in the bank belonging to \( P \), then turn to the right or to the left according to whether \( \theta_j \geq 0 \) or \( \theta_j \leq 0 \), and proceed the distance \( r_j |\theta_j| \) along \( f(C_j) \), we reach the other distinguished point \( Q \) on \( f(C_j) \). The number \( \theta_j \) is, of course, determined only modulo \( 2\pi \), with \( \theta_j \equiv 0 \) (mod \( 2\pi \)) if \( Q = P \).

It is clear that the numbers \( r_j e^{i\theta_j}, j = 1, \ldots, d \), and \( \rho_k \), \( k = 1, \ldots, n \), depend only on \([f]\), so that we can define a map

\[
\Phi : \mathcal{F}(X) \to (\mathbb{C}^*)^d \times (\mathbb{R} \cup \{0\})^n
\]

(7.3)
by setting
\[ \Phi([f]) = (r_1 e^{\theta_1}, \ldots, r_d e^{\theta_d}; \rho_1, \ldots, \rho_n). \]
(7.4)

We call \( \Phi \) the reduced Fenchel–Nielsen map.

**Lemma 7.1.** The reduced Fenchel–Nielsen map is a continuous surjection.

**Proof.** The case \( a = 1 \) (i.e., \( (p, n) = (0, 3) \)) is taken care of by Proposition 6.1. In the general case the continuity of \( \Phi \) follows from the results of Section 3 since the numbers \( r_j e^{\theta_j} \) and \( \rho_k \) \( (j = 1, \ldots, d; k = 1, \ldots, n) \) are continuous functions of the generators \( x_{[r]}(g_1), \ldots, x_{[r]}(g_n) \) defined there (cf. Propositions 3.1, 3.2, and 3.3).

To prove surjectivity, let
\[ v = (z_1, \ldots, z_d, \rho_1, \ldots, \rho_n) \in (\mathbb{C}^*)^d \times (\mathbb{R}_+ \cup \{0\})^n \]
be given. Using the numbers \( r_i = |z_j| \) and \( \rho_k \) we construct the triply connected domains \( S_i; \ i = 1, \ldots, a \), which would have to be the domains \( f(\sigma_i) \) if there were a homeomorphism \( f \) of \( X \) onto a Riemann surface \( S = f(X) \) with
\[ \Phi([f]) = v. \]
(7.5)

Using the arguments of the numbers \( z_1, \ldots, z_d \) we can construct the surface \( S \) and a map \( f: X \to S \) satisfying (7.3).

**Remark 7.2.** The construction of \( S \) consists of identifying \( d \) pairs of ideal boundary curves of the \( S_i \); the curves to be identified may belong to the same or to different \( S_i \). The identification is by an isometry in the relevant Poincaré metrics (cf. Remark 7.1) and is uniquely determined by the argument of the relevant \( z_j \).

The map \( f \), or even its equivalence class \( [f] \), is not uniquely determined by \( \Phi([f]) \) (cf. Section 8 below).

### 8. DEHN TWISTS

We call a Jordan curve on \( X \) *essential* if it is not homotopic to a point of \( X \) and does not define an end of \( X \).

Recall (cf. [Bi]) that a homeomorphism of \( X \) onto itself is called a *Dehn twist* about an essential Jordan curve \( C \) on \( X \) if it is homotopic to a homeomorphism which fixes all points in the complement of a neighborhood of \( C \). The homotopy classes of Dehn twists about \( C \) form an infinite cyclic group \( \Delta_C \).

(Let \( m \) be an orientation preserving map of the closed annulus
\[ A: \frac{1}{2} \leq |z| \leq 1 \])
into $X$ which takes the circle $|z| = 1$ onto an essential curve $C$. Let $\varphi$ be the self-map of $A$ described in polar coordinates $(z = re^{i\theta})$ by

$$r \rightarrow r, \quad \theta \rightarrow \theta + (4r - 2)\varepsilon\pi \quad (1/2 < r < 1, \ 0 < \theta < 2\pi)$$

where $\varepsilon = 1$ or $\varepsilon = -1$. Let $M$ be a self-map of $X$ which fixes every point outside $m(A)$ and equals $m \circ \varphi \circ m^{-1}$ on $m(A)$. Then $M$ is a Dehn twist about $C$. Such an $M$ is called a primitive Dehn twist about $C$, a left (right) twist if $\varepsilon = -1$ (if $\varepsilon = 1$). The homotopy class of $M$ generates $A_{C_e}$.

If $C_1$ and $C_2$ are essential Jordan curves and $C_1 \cap C_2 = \emptyset$, then every element of $A_{C_1}$ commutes with every element of $A_{C_2}$. In all cases, $A_{C_1} = A_{C_2}$ if and only if $C_1$ and $C_2$ are freely homotopic.

It is also known that if $C_1, \ldots, C_k$ are disjoint essential Jordan curves, none freely homotopic to another, then the group $A_{C_1 \ldots C_k}$ generated by $A_{C_1}, \ldots, A_{C_k}$ is a free abelian group of rank $k$. We use this fact to establish

**Lemma 8.1.** Let $C_1, \ldots, C_k$ be as above. Assume that a topological self-map $\omega$ of $X$ is homotopic to a product of Dehn twists about $C_1, \ldots, C_k$, but not homotopic to the identity. Then the element $\omega_*$ of $\mathcal{F}M(X)$ induced by $\omega$ (cf. Section 1) fixes no point of $\mathcal{F}(X)$.

**Proof.** Assume there is a $[f] \in \mathcal{F}(X)$ with $\omega_*([f]) = f$. Then $f \circ \omega^{-1}$ is equivalent to $f$, so that there is a conformal self-map $\varphi$ of $f(X)$ and a topological self-map $\psi$ of $X$, homotopic to the identity, such that

$$\varphi \circ f \circ \omega^{-1} = f \circ \psi$$

or

$$\varphi = f \circ \psi \circ \omega \circ f^{-1}.$$ 

Since $f(X)$ is a Riemann surface of topological type $(p, n)$, with $a = 2p - 2 + n > 0$, there is an integer $N$ such that $\varphi^N = \text{id}$. This means that $(f \circ \psi \circ \omega \circ f^{-1})^N = \text{id}$ or $(\psi \circ \omega)^N = \text{id}$. Since the homotopy class of $\psi \circ \omega$ belongs to $A_{C_1 \ldots C_k}$, a free Abelian group, we conclude that $\psi \circ \omega$ is homotopic to the identity. So therefore is $\omega$, which contradicts the hypothesis.

**Lemma 8.2.** Two elements $[f_1]$ and $[f_2]$ of $\mathcal{F}(X)$ have the same image under the reduced Fenchel-Nielsen map $\Phi$ if and only if $[f_2] = \omega_*([f_1])$ where $\omega$ is a product of Dehn twists about the partition curves $C_1, \ldots, C_n$ on $X$ which define the map $\Phi$.

**Proof.** The if part is obvious since preceding the map $f_1$ by a Dehn twist about a partition curve does not change the geodesics freely homotopic to the $f_i(C_j)$, $j = 1, \ldots, d$, nor the distinguished points on these geodesics, nor the outer loops on $f(X)$. 


Assume next that $\Phi(\{f_1\}) = \Phi(\{f_2\})$ and also, which involves no loss of generality, that both $f_1$ and $f_2$ are standardized. Since $\Phi(\{f_1\})$ determines the Riemann surface $f_1(X)$ together with the geodesic $f_1(C_1), \ldots, f_1(C_d)$ on $f_1(X)$ and an analogous statement holds for $f_2$, there is a conformal map $\varphi$ of $f_1(X)$ onto $f_2(X)$ which takes $f_1(C_j)$ onto $f_2(C_j)$, for all $d$ partition curves $C_j$, and also "takes," in an obvious way, an end $f_1(B_i)$ of $f_1(X)$ onto the end $f_2(B_i)$ of $f_2(X)$, for all $i$. The homeomorphism

$$\omega = f_2^{-1} \circ \varphi \circ f_1$$

of $X$ onto itself fixes each $C_j$, each $B_i$, and each partition region $\sigma_k$. Since each $\sigma_k$ is a triply connected plane domain, $\omega$ is homotopic to a homeomorphism which keeps all points on $X$ outside a neighborhood of $C_1 \cup \cdots \cup C_d$ fixed. Thus $\omega$ is a product of Dehn twists about $C_1, \ldots, C_d$.

We have that $\varphi^{-1} \circ f_2 = f_1 \circ \omega^{-1}$ so that $\{f_2\} = \{\varphi^{-1} \circ f_2\} = \{f_1 \circ \omega^{-1}\} = \omega_*([f_1])$ as asserted.

9. CONSTRUCTION OF A SPECIAL FUNDAMENTAL POLYGON

Let $[f] \in \mathcal{P}(X)$, with $f$ standardized, and set

$$v = \Phi([f]) = (z_1, \ldots, z_d; \rho_1, \ldots, \rho_n)$$

We shall describe a construction (which goes back, in principle, to Fricke and Klein) of a fundamental polygon $\mathcal{P}$ and of a sequence of generators for a Fuchsian group $\mathcal{G}$ representing $f(X)$.

We recall the construction of the polygon $\mathcal{A}$ and of the generators $\gamma_1, \gamma_2, \gamma_3$ (satisfying relation (6.3)) of the group $\Gamma$ in Section 6, and retain the notation introduced there.

Under the circumstances described in Section 6, let $a_j$ (for some $j=1, 2, 3$) be positive. We denote by $Q^j$ the quadrilateral bounded by the line $A_j$, an arc of $\mathbb{R}_+ \cup \{0\}$ and arcs of the curves $F_i$ and $F_k$, where $\{i, k\} = \{1, 2, 3\} \setminus \{j\}$, which does not contain the point $Z$ (this point is defined in Section 6 after Lemma 6.4). If $a_j = 0$, we set $Q^j = \emptyset$. We consider $\mathcal{A}$ and $Q^1, Q^2, Q^3$ as closed in $\mathcal{H}$ and denote by $\mathcal{A}^0$ the closure in $\mathcal{H}$ of $\mathcal{A}\setminus\{Q^1 \cup Q^2 \cup Q^3\}$. Then

$$\mathcal{A} = \mathcal{A}^0 \cup Q^1 \cup Q^2 \cup Q^3$$

and the interior of any two terms on the right do not intersect. The polygon $\mathcal{A}^0$ has $9 - 2q$ vertices in $\mathcal{H}$ and $q$ vertices on $\mathbb{R} \cup \{\infty\}$, $q$ being the number of zeros among $a_1, a_2, a_3$.

If $a_j > 0$, the edge of $\mathcal{A}^0$ lying on $A_j$ will be called the $j$th principal edge of $\mathcal{A}^0$ (and also of $\mathcal{A}^0 \cup Q^i$ and of $\mathcal{A}^0 \cup Q^i \cup Q^k$ where $\{i, k\} = \{1, 2, 3\} \setminus \{j\}$).
The point at which the \( j \)th principal edge intersects the line \( L_i \) (where \( i = j - 1 \) for \( j = 2, 3 \) and \( i = 3 \) for \( j = 1 \)) will be called the distinguished point on that principal edge.

**Addition 2 to Lemma 6.5.** The canonical map \( \mathcal{U} \to \mathcal{W}/\Gamma \) takes \( \Delta^0 \) onto the Nielsen core \( \mathbf{K}(\mathcal{W}/\Gamma) \) of the triply connected domain \( \mathcal{W}/\Gamma \). If \( a > 0 \) it takes the distinguished point on the \( j \)th principal edge of \( \Delta^0 \) onto the first distinguished point on the outer loop defining the \( j \)th end of \( \mathcal{W}/\Gamma \).

Both statements follow by construction (cf. Sections and 6).

We now begin the definition of \( \mathscr{Q} \) and \( \mathscr{G} \). For \( i = 1, \ldots, a \) we consider the banks \( h_{i1}, h_{i2}, \) and \( h_{i3} \) of the partition region \( \sigma_i \) and set \( a_{iv} = \pi|z_j| \) if \( h_{iv} \) is a bank of the partition curve \( C_i \) and \( a_{iv} = \pi\rho_k \) if \( h_{iv} \) is a bank of the \( k \)th end \( f(B_k) \). Let \( \Delta_i, Q_i^e, t = 0, 1, 2, 3, \) and \( \gamma_i, v = 1, 2, 3, \) and \( \Gamma_i \) denote the polygons \( \Delta_i, Q_i^e \), the Möbius transformations \( \gamma_i \), and the group \( \Gamma \) formed for

\[
(a_1, a_2, a_3) = (a_{i1}, a_{i2}, a_{i3}). \tag{9.1}
\]

We denote by \( D_i \) the union of \( \Delta_i^0 \) and quadrilaterals \( Q_i^e \) with \( h_{iv} \) the bank of an end of \( X \). It is easy to see that the canonical map \( \mathcal{W} \to \mathcal{W}/\Gamma \) takes the interior of the polygon \( D_i \) onto a domain conformal to \( f(\sigma_i) \).

The domain \( D_i \) (and the real Möbius transformations \( \gamma_i, v = 1, 2, 3, \) which generate the groups \( \Gamma_i \)) are determined by condition (9.1) only up to a real Möbius transformation (and up to a conjugation by this Möbius transformation). We proceed to remove this indeterminacy.

Recalling the consistency conditions (7.1), (7.2), we agree to denote the \( v \)th principal edge of \( D_i \) (which exists if and only if \( h_{iv} \) is the bank of a partition curve) by \( e(i, v) \), and the distinguished point on this edge by \( P(i, v) \). It should be remembered that \( P(i, v) \) is never an endpoint of \( e(i, v) \).

The construction of \( \mathscr{Q} \) will involve certain choices, but we always position \( D_1 \) so as to satisfy the normalization condition (6.4).

If \( a > 1 \) we choose, successively, numbers \( \theta_1, \ldots, \theta_a \) with

\[
z_j = |z_j| e^{i\theta_j}, \quad |\theta_j| \leq 2\pi, \tag{9.2}
\]

and positions for the domains \( D_2, \ldots, D_a \) so that the following conditions hold for \( r = 2, \ldots, a \). The edge \( e(r + 1, v_r) \) and the edge \( e(k_r, u_r) \) have a line \( A_r \) in common and the regions \( D_{r+1} \) and \( D_{k_r} \) lie on different sides of \( A_r \) (this is plainly possible). The point \( P(r + 1, v_r) \) belongs to the edge \( e(k_r, u_r) \) and the distance between \( P(r + 1, v_r) \) and \( P(k_r, u_r) \) is \( |z_r\theta_r| \). If \( \theta_r \neq 0 \), and if one approaches \( P(k_r, u_r) \) from \( D_{k_r} \), one has to turn to the left or to the right onto \( A_r \) in order to reach \( P(r + 1, v_r) \), according to whether \( \theta < 0 \) or \( \theta > 0 \).
It is not difficult to verify that no two of the polygons $D_1, \ldots, D_a$ so positioned have an interior point in common.

If $a - 1 < d$, i.e., if $p > 0$, one can choose $p$ more numbers $\theta_i$ satisfying (9.2) and $p$ real Möbius transformations $g_a, g_{a+1}, \ldots, g_d$ so that the following holds, for $r = a, \ldots, d$. The segments $e(k_r, \mu_r)$ and $g_r(e(l_r, v_r))$ lie on a line $A_r$ and the regions $g_r(A_r)$ and $A_r$ lie on opposite sides of $A_r$. The point $g_r(P(l_r, v_r))$ belongs to the segment $e(k_r, \mu_r)$ and the distance between $g_r(P(l_r, v_r))$ and $P(k_r, \mu_r)$ is $|z, \theta_r|$. If $\theta_r \neq 0$, and if one approaches $P(k_r, \mu_r)$ in $D_k$, one has to turn to the left or to the right onto $A_r$ in order to reach $g_r(P(l_r, v_r))$, according to whether $\theta_r < 0$ or $\theta_r > 0$.

Now we proceed to modify the polygons $D_2, \ldots, D_a$. If for some $r = 1, \ldots, a - 1$ the endpoints of $\varepsilon(r+1, v_r)$ and of $e(k_r, \mu_r)$ do not coincide, we first modify the two edges, call them $e$ and $e'$, of $D_{r+1}$ which meet $e(r+1, v_r)$ and are identified by the real Möbius transformation $\gamma_{r+1, v_r}$. More precisely let the endpoints of $e$ be $e \cap e(r+1, v_r)$ and a point $Q$. We replace $e$ by a line segment leading from $Q$ to the endpoint of $\varepsilon(k_r, v_r)$ closest to $e \cap e(r+1, v_r)$. The edge $e'$ is replaced analogously. Then we replace $e(r+1, v_r)$ by $\varepsilon(k_r, \mu_r)$.

If $p > 0$ and if for some $r = a, \ldots, d$ the endpoints of $g_r(e(l_r, v_r))$ and of $\varepsilon(k_r, \mu_r)$ do not coincide, we first modify the edges, call them $e$ and $e'$, of $D_r$, which meet $e(l_r, v_r)$ and are identified by the real Möbius transformation $\gamma_{l_r, v_r}$. More precisely, let the endpoints of $e$ be $e \cap e(l_r, v_r)$ and a point $Q$. We replace $e$ by a line segment leading from $Q$ to the endpoint of $\varepsilon(k_r, v_r)$ closest to $e \cap e(l_r, v_r)$. The edge $e'$ is replaced analogously. Then we replace $e(l_r, v_r)$ by $g^{-1}(\varepsilon(k_r, \mu_r))$.

We carry out all such modifications and denote the polygon resulting from modifying $D_i$ by $D_i^*$. (It may happen that $D_i^* = D_i$; this is always so for $i = 1$.)

Each Möbius transformation $g_n$ still identifies two sides of $D_i^*$ and the image of $D_i^*$ under $\mathcal{U} \to \mathcal{U}/\Gamma_i$ coincides with that of $D_i$.

Next we set
\[ \mathcal{Q} = D_1^* \cup \cdots \cup D_a^* \] (9.3)
and denote by $\mathcal{Q}$ the group generated by
\[ \gamma_{11}, \gamma_{12}, \ldots, \gamma_{a3}, g_a, \ldots, g_d. \] (9.4)

No two terms on the right in (9.3) have a common interior point, and $\mathcal{Q}$ is a polygon; indeed one can verify that $\mathcal{Q}$ is a Jordan polygon. The boundary of $\mathcal{Q}$ in $\mathcal{U}$ consists of $3a$ non-principal edges of $D_1^*, \ldots, D_a^*$ and of the $2p$ principal edges $e(r+1, v_r)$ and $e(k_r, \mu_r)$, $r = 1, \ldots, d$. The former are pairwise identified by $3a$ real Möbius transformation $\gamma_{\mu_r}, i = 1, \ldots, a$, $v = 1, 2, 3$, and the latter are pairwise identified by the $p$ real Möbius trans-
formations \( g_a, \ldots, g_d \) (if \( p > 0 \)). These identifications satisfy the hypotheses of de Rham's lemma (Proposition 5.1). We conclude that \( \mathcal{G} \) is a (torsion-free) Fuchsian group and \( \mathcal{B} \) is a fundamental polygon for \( \mathcal{G} \) in \( \mathcal{U} \). Also, by construction, \( f(X) \) may be identified with \( \mathcal{U}/\mathcal{G} \). Furthermore, assuming (without loss of generality) that \( f \) is standardized, we may identify the partition curves \( f(C_r) \) with the images under \( \mathcal{U} \to \mathcal{U}/\mathcal{G} \) of \( \varepsilon(k_r, \mu_r) \), \( r = 1, \ldots, d \), and the closures of the partition regions \( f(\sigma_r) \) with the images under \( \mathcal{U} \to \mathcal{U}/\mathcal{G} \) of the closures in \( \mathcal{U} \) of the polygons \( D_r, r = 1, \ldots, a \).

Remark. The generators (9.4) of \( \mathcal{G} \) satisfy a certain finite set of relations. Therefore, one can replace (9.4) by a suitably chosen subset.

We call \( \mathcal{B} \), equipped with the described identification of sides, a special fundamental polygon associated with \( v \).

10. DEFORMATION OF THE SPECIAL FUNDAMENTAL POLYGON

In Section 9 we associated to a point \( v \in \Phi([f]), [f] \in \mathcal{F}(X) \), a fundamental polygon \( \mathcal{B} \) for a Fuchsian group \( \mathcal{G} \), with \( \mathcal{U}/\mathcal{G} \) conformal to \( f(X) \), and a set (9.4) of real Möbius transformations which identify pairs of edges of \( \partial \mathcal{B} \cap \mathcal{U} \) and generate \( \mathcal{G} \). Reviewing the construction we see that it only used the ordered maximal partition of \( X \) and the components \( z_1, \ldots, z_d, \rho_1, \ldots, \rho_a \) of \( v \). On the other hand, the construction involved certain choices, namely those of the numbers \( \theta_1, \ldots, \theta_d \) satisfying (9.1) and of the positions of the polygons \( D_2, \ldots, D_a \). Each choice involved only a finite number of possibilities. We conclude from this that we can repeat the construction for every vector

\[
(\hat{z}_1, \ldots, \hat{z}_d; \hat{\rho}_1, \ldots, \hat{\rho}_a) = \hat{v} \in (\mathbb{C}^*)^d \times (\mathbb{R}^+ \cup \{0\})^n
\]

sufficiently close to \( v \). This time we eliminate all choices by requiring that the numbers \( \theta_1, \ldots, \theta_d \) and the polygons \( \hat{D}_2, \ldots, \hat{D}_a \)—the notation is self-explanatory—be close to the number \( \theta_1, \ldots, \theta_d \) and to the polygons \( D_2, \ldots, D_a \), respectively. We will obtain a fundamental polygon \( \hat{\mathcal{B}} \) and a sequence of generators

\[
\hat{g}_{11}, \ldots, \hat{g}_a
\]

(10.1) for a torsion-free Fuchsian group \( \hat{\mathcal{G}} \) which are close to \( \mathcal{G} \) and to \( (g_{11}, \ldots, g_a) \), respectively, in fact, arbitrarily close if \( \hat{v} \) is sufficiently close to \( v \). (Note, however, that \( \hat{\mathcal{G}} \) may have more edges on \( \mathbb{R} \cup \{\infty\} \) than \( \mathcal{G} \) has. This will happen if \( \rho_k < 0 \) and \( \hat{\rho}_k > 0 \) for some \( k = 1, \ldots, n \).) We observe that the real Möbius transformations \( \hat{g}_{11}, \ldots, \hat{g}_a \) and the polygon \( \hat{\mathcal{B}} \) depend continuously on \( \hat{v} \) in a neighborhood of \( v \).
We already noted in Section 9 that we may assume \( f \) to be standardized and we may identify \( f(X) \) with \( \mathcal{U} / \mathcal{G}, f(C_i) \) with the image of \( \varepsilon(k_r, \mu_r) \) under the map \( \mathcal{U} \rightarrow \mathcal{U} / \mathcal{G} \), and \( f(\sigma_r) \) with the image of \( D_{k_r} \). We can find a homeomorphic bijection

\[
h: \partial \mathcal{D} \cap \mathcal{U} \rightarrow \partial \mathcal{D} \cap \mathcal{U}
\]

which maps each edge of \( \partial \mathcal{D} \cup \mathcal{U} \) onto the "corresponding" edge of \( \partial \mathcal{D} \cup \mathcal{U} \) and conjugates the identification of two edges of \( \partial \mathcal{D} \cup \mathcal{U} \) by \( \gamma_{i,v} \) (or by a \( g_r \)) into the identification of the corresponding two edges of \( \partial \mathcal{D} \cup \mathcal{U} \) by \( \tilde{\gamma}_{i,v} \) (or by \( \tilde{g}_r \)). There is a bijective homeomorphism \( \tilde{H}: \mathcal{U} \rightarrow \mathcal{U} \) which restricts to \( h \) on \( \partial \mathcal{D} \cup \mathcal{U} \) and satisfies \( \tilde{\gamma}_{i,v} \circ \tilde{H} = \tilde{H} \circ \gamma_{i,v} \) and \( \tilde{g}_r \circ \tilde{H} = \tilde{H} \circ g_r \) (\( i = 1, \ldots, a; v = 1, 2, 3; r = 1, \ldots, n \)) so that \( \tilde{\mathcal{D}} = \tilde{H} \mathcal{D} \tilde{H}^{-1} \). Also there is a bijective homeomorphism \( H: \mathcal{U} / \mathcal{G} \rightarrow \mathcal{U} / \tilde{\mathcal{D}} \) such that the diagram

\[
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{f} & \mathcal{U} \\
\downarrow & & \downarrow \\
\mathcal{U} / \mathcal{G} & \xrightarrow{H} & \mathcal{U} / \tilde{\mathcal{D}}
\end{array}
\]

commutes. Set \( \hat{f} = H \circ f \). Then \( \hat{f} \) is a (standardized) homeomorphism of \( X \) onto the Riemann surface \( \mathcal{U} / \mathcal{G} \) and, by construction,

\[
\Phi([\hat{f}]) = \hat{v}.
\]

The point \([\hat{f}]\) reduces to \([f]\) for \( \hat{v} = v \) and depends continuously on \((\hat{\gamma}_{11}, \ldots, \hat{\gamma}_{ai}, \hat{g}_a)\), by the result of Section 3, and therefore also on \( \hat{v} \). Thus we have established the following

**Lemma 10.1.** The reduced Fenchel-Nielsen map is a local homeomorphism.

The isomorphism of \( \mathcal{G} \) onto \( \tilde{\mathcal{G}} \) induced by \( \tilde{H} \) (i.e., the conjugation by \( \tilde{H} \)) which takes \( \gamma_{11}, \ldots, g_a \) into \( \hat{\gamma}_{11}, \ldots, \hat{g}_a \) will be denoted by \( \hat{x} \); it depends continuously on \( v \) and reduces to the identity for \( \hat{v} = v \).

**Remark 10.1.** The continuous dependence of \( \tilde{\mathcal{D}} \) on \( \hat{v} \) is to be understood as follows: the vertices of the Jordan polygon \( \tilde{\mathcal{D}} \) (whose edges are either non-Euclidean segments or rays in \( \mathcal{U} \) or arcs of \( \mathbb{R} \cup \{\infty\} \)) are continuous functions of \( v \). Two vertices on \( \mathbb{R} \cup \{\infty\} \) may coalesce into one vertex; a vertex on \( \mathbb{R} \cup \{\infty\} \) may split into a pair of vertices on \( \mathbb{R} \cup \{\infty\} \). Vertices in \( \mathcal{U} \) may neither coalesce nor split, but the interior angle at a vertex in \( \mathcal{U} \) may be or become \( \pi \).

**Remark 10.2.** The construction of \( \tilde{\mathcal{D}} \) and of the isomorphism \( \hat{x} \) has the following feature. If \( g \in \mathcal{G} \) and \( \mathcal{D} \cap g(\mathcal{D}) \) is empty, or a vertex of \( \mathcal{D} \), or an
edge of $\mathcal{D}$, then, for $\hat{g} = \hat{x}(g)$, we have that $\mathcal{D} \cap \hat{g}(\mathcal{D})$ is empty, or the corresponding vertex of $\mathcal{D}$, or the corresponding edge of $\mathcal{D}$. ("Corresponding" refers to the map $h$ above. Recall that no point on $\mathbb{R} \cup \{\infty\}$ is considered as belonging to $\mathcal{D}$.)

Both remarks are true by construction.

11. Locally Uniform Discreteness

A closed curve on a topological surface is called primitive if it is not freely homotopic to another closed curve traversed more than once. (We recall that closed curves on surfaces are considered as being without orientation.) A curve freely homotopic to a primitive curve is primitive. On a Riemann surface a closed curve is primitive if and only if it is freely homotopic to a primitive geodesic.

The purpose of this section is to prove the following.

**Proposition 11.1.** Let $[f] \in \mathcal{F}(X)$ and let $L > 0$ be given. There is a neighborhood $\mathcal{N}$ of $[f]$ and a finite set $\Sigma$ of primitive closed curves on $X$ such that for every primitive closed curve $C$ on $X$, either $C$ is freely homotopic to an element of $\Sigma$, or

$$l_{[f]}(C) > L.$$

for every $[\hat{f}]$ in $\mathcal{N}$.

**Corollary.** For a given $L > 0$ and a given Riemann surface $S$ of topological type $(p, n)$ (with $2p - 2 + n > 0$), there are only finitely many closed geodesics of length not exceeding $L$.

The corollary is known. The proposition says that the discreteness asserted by the corollary holds locally uniformly.

**Lemma 11.1** Let $S$ be as in the above corollary. Let $C$ be an outer loop on $S$ and $F$ the funnel adjacent to $C$ (cf. Section 4). No closed geodesic on $S$ penetrates $F$.

**Proof.** No closed curve in $F$ can be a geodesic since every such curve is freely homotopic to $C$ traversed some number of times, and $C$ is not in $F$.

On the other hand a closed geodesic cannot intersect $C$, for if it did it would contain an arc in $F$ with endpoints on $C$; such an arc cannot be geodesic.

**Lemma 11.2.** Let $S$ be as before. Let

$$0 < \varepsilon < 1 \quad \text{and} \quad L = -2 \log \varepsilon. \quad (11.1)$$
No closed geodesic on $S$ of length not exceeding $L$ can penetrate an $\varepsilon$-collar about a puncture (cf. Section 4).

Proof. No closed geodesic can lie in any collar about the puncture, for otherwise it could be deformed in $S$ into one of arbitrarily small length. We conclude that a closed geodesic penetrating the $\varepsilon$-collar must contain two arcs joining the horocyclic boundary of the 1-collar to that of the $\varepsilon$-collar. By (4.5) the distance between those two horocycles is $\log(1/\varepsilon) = L/2$.

Lemma 11.3. Let $L > 0$ be given. There are numbers

$$\lambda_0 > 0 \text{ and } \varepsilon, \quad 0 < \varepsilon < \frac{1}{2}, \quad (11.2)$$

such that, for every $\lambda$, $0 \leq \lambda < \lambda_0$, we have

$$\frac{\lambda}{2} \cosh \frac{\lambda}{2} \geq 1 - \varepsilon$$

(11.3)

and

$$\frac{\sqrt{\lambda^2 + (1 - \varepsilon)^2} + (1 - \varepsilon)}{\sqrt{\lambda^2 + \varepsilon^2 + \varepsilon}} > e^{L/2}.$$  

(11.4)

Proof. The left side of (11.3) approaches 1 as $\lambda$ approaches 0. The left side of (11.4) exceeds $(\lambda + 2\varepsilon)^{-1}$ so that (11.4) holds for $2\lambda_0 < e^{-L/2}$ and $4\varepsilon < e^{-L/2}$.

Lemma 11.4. Let $S$ be as in the corollary to Proposition 11.1, let $L > 0$ be given, and let $\lambda_0$ and $\varepsilon$ be chosen as in Lemma 11.3. No primitive closed geodesic on $S$ of length not exceeding $L$, and not an outer loop, can penetrate an $\varepsilon$-collar about an outer loop on $S$ of length $\lambda \leq \lambda_0$, or about a puncture.

Proof. A primitive closed curve lying in a collar about an outer loop $C$ is freely homotopic to $C$. If the length $\lambda$ of $C$ does not exceed $\lambda_0$, (11.3) and Proposition 4.1 imply the existence of an $(1 - \varepsilon)$-collar about $C$. If a closed geodesic, distinct from $C$, penetrates the $\varepsilon$-collar about $C$, this geodesic must contain two arcs joining the boundaries of the $\varepsilon$-collar and the $(1 - \varepsilon)$-collar. Comparing formula (4.4), for $\varepsilon_1 = 1 - \varepsilon$, $\varepsilon_2 = \varepsilon$, with inequality (11.4) we conclude that both arcs have length exceeding $L/2$.

This proves the assertion of the lemma concerning loops. The one concerning punctures is proved similarly, cf. the proof of Lemma 11.1.

Lemma 11.5. Let $G$ be a Fuchsian group and $D$ a finitely many sided fundamental polygon for $G$ in $\mathbb{H}$. (We assume that $D$ is closed in $\mathbb{H}$ but that the boundary points of $D$ on $\mathbb{R} \cup \{\infty\}$ do not belong to $D$.) Let $K \subset \mathbb{H}$ be com-
pact. Then there is a finite set \( H \subset G \) such that \( K \subset H(D) \). (Here \( H(D) \) denotes the union of all sets \( h(D) \) with \( h \in H \).)

We prove this known lemma, for the sake of completeness. Let \( A \) be the finite set of all \( a \in G \) with \( D \cap a(D) \neq \emptyset \). Then \( D \subset \text{Int} \ A(D) \). Since \( G(D) = \mathbb{U} \) we have that \( G(A(D)) = \mathbb{U} \) and the compactness of \( K \) implies the existence of a finite \( B \subset G \) with

\[
K \subset B(\text{Int}(A(D))) \subset B(A(D)).
\]

Now let \( H \) be the finite set of all \( b \circ a \) with \( a \in A, b \in B \).

Proof of Proposition 11.1. We are given the point \([f]\) in \( \mathcal{F}(X) \) and a number \( L > 0 \). We determine a neighborhood \( \mathcal{V} \) of \([f]\) such that \( \Phi|_{\mathcal{V}} \) has an inverse, call it \( \Phi_{(f)}^{1} \), and we choose numbers \( \lambda_{0} \) and \( \varepsilon \) satisfying the requirements of Lemma 11.3 and two additional conditions which refer to the vector

\[
\hat{v} = \Phi([f]) = (\hat{z}_{1}, \ldots, \hat{z}_{d}; \hat{\rho}_{1}, \ldots, \hat{\rho}_{n}), \quad [f] \in \mathcal{V},
\]

and to the special value

\[
v = \Phi([f]) = (z_{1}, \ldots, z_{d}; \rho_{1}, \ldots, \rho_{n}).
\]

The additional conditions are:

\[
\begin{aligned}
\text{If, for some } k \in \{1, \ldots, n\}, & \rho_{k} = 0, \text{ then } \hat{\rho}_{k} < \lambda_{0}/2\pi, \\
\text{if, for some } k \in \{1, \ldots, n\}, & \rho_{k} \neq 0, \text{ then } \hat{\rho}_{k} > \lambda_{0}/2\pi.
\end{aligned}
\] (11.5)

This can be achieved by diminishing first \( \lambda_{0} \) and then \( \mathcal{V} \).

For every \([\hat{f}] \in \mathcal{V} \), let \( S_{0}([\hat{f}]) \) denote the complement in \( f(X) \) of all funnels and of all (open) \( \varepsilon \)-collars about punctures and about outer loops of length \( \lambda < \lambda_{0} \). We construct, as in Sections 9 and 10, the fundamental polygons \( \mathcal{G} \) and \( \hat{\mathcal{G}} \) and the groups \( \mathcal{G} \) and \( \hat{\mathcal{G}} \) associated with \( v = \Phi([f]) \) and \( \hat{v} = \Phi([\hat{f}]) \), respectively, and the isomorphism \( \hat{x} \) of \( \mathcal{G} \) onto \( \hat{\mathcal{G}} \). Let \( \hat{\mathcal{G}}_{0} \) denote the component of the inverse image of \( S_{0}([\hat{f}]) \) under the canonical map \( \mathcal{U} \to \mathcal{U}/\hat{\mathcal{G}} \) which lies in \( \hat{\mathcal{G}} \).

Claim 1. We may assume, diminishing \( \mathcal{V} \) if need be, that there is a (non-Euclidean) disk \( A_{0} \) in \( \mathcal{U} \), with radius \( R \) and center \( i \), containing all \( \hat{\mathcal{G}}_{0} \).

Proof. \( \hat{\mathcal{G}}_{0} \) is obtained from \( \hat{\mathcal{G}} \) by removing pieces bounded by axes of certain hyperbolic elements of \( \hat{\mathcal{G}} \), or by equidistants of such axes, or by horocycles of certain parabolic elements of \( \hat{\mathcal{G}} \). It is easy to verify that the closure of \( \hat{\mathcal{G}}_{0} \) is compact and depends continuously on \( v \). This implies the assertion.
Let $A_1$ denote the non-Euclidean disk with center at $i$ and radius $R + L$.

**Claim 2.** We may assume, diminishing $\mathcal{N}$ if need be, that there is a finite set $\Gamma \subset \mathcal{G}$ with

$$A_1 \subset \bigcup_{A \in \Gamma} \tilde{x}(A)(\mathcal{D}). \quad (11.6)$$

**Proof.** Let $A_2$ be the (closed) non-Euclidean disk with center at $i$ and radius $R + L + 1$. By Lemma 11.5 there is a finite $\Gamma \subset \mathcal{G}$ with $A_2 \subset \Gamma(\mathcal{D})$. Hence, in particular, $\partial \Gamma(\mathcal{D}) \cap \mathcal{N}$ lies outside $A_2$. Since $\partial \Gamma(\mathcal{D})$ consists of edges of the polygons $\gamma(\mathcal{D})$, $\gamma \in \Gamma$, we conclude from Remarks 10.1 and 10.2 that (11.6) will hold for $\mathcal{N}$ sufficiently close to $v$.

**Claim 3.** Every closed primitive geodesic curve $\hat{C}$ in $\hat{f}(X)$ which is not an outer loop and has length not exceeding $L$ has a lifting $\hat{C}$ under $\mathcal{U} \to \mathcal{U}/\mathcal{G}$ which is a Jordan arc beginning in $A_{1}$.

This follows by comparing the definition of $\mathcal{D}_0$ with Lemmas 11.1 and 11.3.

**Claim 4.** Let $\hat{C}$ be as in Claim 3. Then $\hat{C}$ is a geodesic arc beginning at a point $\zeta$ in $\mathcal{D}_0$ and ending at a point $\tilde{x}(g)(\zeta)$ where $g \in \Gamma$.

The first assertion about $\hat{C}$ is obvious. The second follows from Claim 2 by noting that the length of the lifting $\hat{C}$ is less than the distance from $A_1$ to the boundary of $A_1$.

We can now complete the proof. Claim 4 associates to each closed primitive geodesic on $\hat{f}(X)$, not an outer loop, and of length not exceeding $L$, a $g \in \Gamma \subset \mathcal{G}$. For a fixed $\mathcal{N}$ this association is injective. Indeed, two curves in $\mathcal{U}$ leading from 2 points, $\zeta_1$ and $\zeta_2$, in a fundamental domain, to $\tilde{g}(\zeta_1)$ and to $\tilde{g}(\zeta_2)$, with $\tilde{g} \in \mathcal{G}$, project under $\mathcal{U} \to \mathcal{U}/\mathcal{G}$ onto freely homotopic curves on $\hat{f}(X)$.

The assertion of Proposition 11.1 follows if we take for $\Sigma$ the union of the set of $n$ Jordan curves defining the $n$ ends of $X$ and of the set of curves $f^{-1}(A_\gamma)$, $\gamma \in \Gamma$, where $A_\gamma$ is the image under $\mathcal{U} \to \mathcal{U}/\mathcal{G}$ of the non-Euclidean segment joining some fixed point $\zeta_0$ in $\mathcal{D}_0$ to $\gamma(\mathcal{D}_0)$.

**Remark.** If $X$ is closed or, more generally, if $f(X)$ has no punctures, Proposition 11.1 remains true for nonprimitive geodesics.

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**12. A Metric for Fricke Space**

We are now in position to prove

**Theorem 1.** The Fricke space $\mathcal{F}_{p,n} = \mathcal{F}(X)$ has a metric $\delta = \delta_X$ which is consistent with the topology of $\mathcal{F}(X)$ and which is invariant under the Fricke
modular group \( \mathcal{F} \mathcal{M}(X) \), as well as under any allowable map of \( \mathcal{F}(X) \). The \( \delta \)-distance between two points \([f]\) and \([g]\) of \( \mathcal{F}(X) \) is

\[
\delta([f], [g]) = \sup \left| \frac{1}{1 + l_{[f]}(C)} - \frac{1}{1 + l_{[g]}(C)} \right| \quad (12.1)
\]

where \( C \) runs over all primitive closed curves on \( X \).

Proof. Since \( l_{[f]}(C) \) and \( l_{[g]}(C) \) depend only on the free homotopy class of \( C \) rather than on \( C \) itself, the supremum in (12.1) is unchanged if we let \( C \) run over a maximal sequence \( C_1, C_2, \ldots \) of primitive closed curves on \( X \), no two of which are freely homotopic.

Obviously, the supremum in (12.1) is finite and, by the corollary to Proposition 11.1, it is a maximum. Thus we may write

\[
\delta([f], [g]) = \max \left| \frac{1}{1 + l_{[f]}(C_i)} - \frac{1}{1 + l_{[g]}(C_i)} \right| \quad (12.2)
\]

where \( C_1, C_2, \ldots \) is as before.

It is trivial that \( \delta \) is symmetric and satisfies the triangle inequality, and that \( \delta([f], [g]) = 0 \) if \([f] = [g]\). If \( \delta([f], [g]) = 0 \), then

\[
l_{[f]}(C) = l_{[g]}(C) \quad (12.3)
\]

for every primitive \( C \). But if a closed curve \( C \) on \( X \) is not primitive, it is freely homotopic to a primitive curve \( C_0 \) traversed \( n > 0 \) times, so that

\[
l_{[f]}(C) = nl_{[f]}(C_0) = nl_{[g]}(C_0) = l_{[g]}(C) \quad (12.4)
\]

Thus (12.3) holds for every closed curve \( C \) on \( X \). Hence \([f] = [g]\), by Proposition 3.3 (or, indeed, by Lemma 2.1). Thus \( \delta \) is indeed a metric.

This metric is consistent with the topology in \( \mathcal{F}(X) \) defined in Section 1. Indeed, if

\[
\lim \delta([f_i], [f]) = 0, \quad (12.5)
\]

then

\[
\lim |l_{[f_i]}(C) - l_{[f]}(C)| = 0 \quad (12.6)
\]

for every primitive \( C \), and, by an argument analogous to the one used above (cf. (12.4)), also for every nonprimitive \( C \). This means that

\[
\lim [f_i] = [f] \quad (12.7)
\]

in the topology of Section 1.
Assume now that (12.7) holds. This means that (12.6) is valid for every closed curve $C$ on $X$. Using Proposition 11.1 we choose an $L > 0$ and then a neighborhood $\mathcal{V}$ (in the topology of Section 1) of $[f]$ and finitely many curves $\Gamma_1, \ldots, \Gamma_k$ on $X$ such that, for $C$ primitive and not freely homotopic to one of the $\Gamma_i$, $l_{\Gamma_i}(C) \geq L$ for $[g] \in \mathcal{V}$ and in particular for $[g] = [f]$. For $j$ large enough, $[f_j]$ belongs to $\mathcal{V}$ and we have, for such $C$,

$$\frac{1}{1 + l_{\Gamma_j}(C)} \leq \frac{2}{1 + L}. \tag{12.8}$$

For $j$ large enough, and for $C$ freely homotopic to one of the $\Gamma_i$, inequality (12.8) holds in view of (12.6). Since $L$ is arbitrary, (12.5) is valid.

Let $\omega$ be a topological self-map of $X$. Then

$$l_{\omega \cdot [f]}(C) = l_{[f \cdot \omega^{-1}]}(C) = l_{[f]}(\omega^{-1}(C))$$

and similarly for $g$. Also, $\omega^{-1}(C)$ is primitive if and only if $C$ is. This implies that $\delta(\omega([f]), \omega([g]))$ and $\delta([f], [g])$ are the suprema (or, rather, maxima) of the same set of numbers. Thus elements of $\mathcal{F}.\mathcal{H}(X)$ are isometries. The corresponding statement for allowable maps is proved in the same way.

13. PROPER DISCONTINUITY OF THE FRICKE MODULAR GROUP

THEOREM II. The Fricke modular group $\mathcal{F}.\mathcal{H}_{p,n} = \mathcal{F}.\mathcal{H}(X)$ acts properly discontinuously on the Fricke space $\mathcal{F}_{p,n} = \mathcal{F}(X)$.

The result was stated by Fricke; his proof is hard to follow.

LEMMA 13.1. Every isotropy subgroup of $\mathcal{F}.\mathcal{H}(X)$ is finite.

Proof. Let $[f] \in \mathcal{F}(X)$. The isotropy group of $[f]$ in $\mathcal{F}.\mathcal{H}(X)$ is the subgroup of $\mathcal{F}.\mathcal{H}(X)$ which fixes $[f]$. Let $\omega$ be a topological self-map of $X$ which induces an element $\omega_*$ of the subgroup. Then

$$\omega_*([f]) = [f \cdot \omega^{-1}] = [f]$$

or

$$f \cdot \omega^{-1} = \varphi \cdot f \cdot \psi \tag{13.1}$$

where $\varphi: f(X) \to f(X)$ is conformal and $\psi: X \to X$ is homotopic to the identity. Since $\psi \cdot \omega$ is homotopic to $\omega$ so that $(\psi \cdot \omega)_* = \omega_*$, relation (13.1) yields

$$\psi \cdot \omega = f \cdot \varphi^{-1} \cdot f^{-1}. $$
We conclude that the order of the isotropy subgroup considered is at most the order of the group of conformal self-maps of $f(X)$. Since $f(X)$ is of topological type $(p, n)$, the latter is finite and has a bound depending only on $(p, n)$. (This is classical if $f(X)$ has $n$ punctures, in particular if $n = 0$. If $f(X)$ has less than $n$ punctures, apply the classical result to the Schottky double of $f(X)$.)

**Lemma 13.2.** Every orbit of $\mathcal{F}\cdot\mathcal{H}(X)$ is discrete.

**Proof.** Let $[f] \in \mathcal{F}(X)$. We must find a neighborhood of $[f]$ which contains only finitely many points equivalent to $[f]$ under $\mathcal{F}\cdot\mathcal{H}(X)$.

Let $C_1, \ldots, C_N$ be a set of closed curves on $X$ satisfying the conclusion of Proposition 3.3. By the corollary of Proposition 11.1 there is a number $\varepsilon > 0$ such that, for every $j = 1, \ldots, N$ and every closed curve $C$ on $X$, either

$$|l_{f_1}(C_j) - l_{f_1}(C_i)| < \varepsilon,$$

or

$$|l_{f_1}(C_j) - l_{f_1}(C_i)| > \varepsilon.$$

Let $a \in \mathcal{F}(X)$ be a neighborhood of $[f]$ defined by the inequalities

$$|l_{f_1}(C_j) - l_{f_1}(C_i)| < \varepsilon, \quad j = 1, \ldots, N. \tag{13.2}$$

We claim that it contains no points equivalent to $[f]$ under $\mathcal{F}\cdot\mathcal{H}(X)$ except $[f]$ itself.

Indeed, let $\omega \in \mathcal{F}\cdot\mathcal{H}(X)$ and assume that $[\hat{f}] = \omega([f]) = [f \circ \omega^{-1}]$ lies in $a$. Since

$$l_{f_1}(C_j) = l_{f \circ \omega^{-1}}(C_i) = l_{f_1}(\omega^{-1}(C_i)),$$

and (13.2) holds, we conclude that

$$l_{f_1}(C_j) = l_{f_1}(C_j), \quad j = 1, \ldots, N.$$

By Proposition 3.3, $[\hat{f}] = [f]$, as asserted.

**Lemma 13.3.** Let $\Gamma$ be a group of isometries of a metric space $M$. If $\Gamma$ has only finite isotropy groups and only discrete orbits, then $\Gamma$ acts properly discontinuously on $M$.

This is certainly known; we include a proof for the sake of completeness.

Let $\delta$ be the distance function in $M$. The hypotheses mean that for every $x$ in $M$ there is an $\varepsilon > 0$ such that

$$\delta(x, \gamma(x)) < \varepsilon \tag{13.3}$$
for only finitely many $\gamma \in \mathcal{F}$. The conclusion means that for every compact $K \subset \mathcal{M}$,
\[ K \cap \gamma(K) \neq \emptyset \] (13.4)
for only finitely many $\gamma \in \mathcal{F}$.
Assume that for some $K$, (13.4) holds for infinitely many $\gamma \in \mathcal{F}$. Then there exist infinite sequences of distinct elements $\gamma_s$ of $\mathcal{F}$ and of points $x_s$ of $K$ with
\[ y_s = \gamma_s(x_s) \in K, \quad s = 1, 2, \ldots \] (13.5)
Selecting if need be a subsequence we may assume that the limits
\[ x = \lim x_n, \quad y = \lim y_n \] (13.6)
exist. Now
\[
\delta(\gamma_m \circ \gamma_s^{-1}(y_k), y_k) \\
\leq \delta(\gamma_m \circ \gamma_s^{-1}(y_k), \gamma_m \circ \gamma_s^{-1}(y_s)) + \delta(\gamma_m \circ \gamma_s^{-1}(y_s), y_s) + \delta(y_s, y_k) \\
= 2\delta(y_s, y_k) + \delta(\gamma_m(x_s), y_s) \\
\leq 2\delta(y_s, y_k) + \delta(\gamma_m(x_s), \gamma_m(x_m)) + \delta(\gamma_m(x_m), y_s) \\
= 2\delta(y_s, y_k) + \delta(x_s, x_m) + \delta(y_m, y_s).
\]
In view of (13.6) this can be made arbitrarily small by making $s$, $m$, and $k$ sufficiently large. Thus, for a given $\varepsilon > 0$ there is a $v(\varepsilon)$ such that, for $s$, $m$, $k > v(\varepsilon)$,
\[ \delta(\gamma_m \circ \gamma_s^{-1}(y_k), y_k) < \varepsilon. \]
Keeping $m$ and $s$ fixed, let $k \to \infty$. We obtain that
\[ \delta(\gamma_m \circ \gamma_s^{-1}(y), y) \leq \varepsilon \]
for infinitely many distinct elements $\gamma_m \circ \gamma_s^{-1}$ of $\mathcal{F}$, a contradiction.
Theorem II follows from Theorem I in Section 12 and Lemmas 13.1, 13.2, and 13.3.

14. Fenchel–Nielsen Coordinates

Proposition 14.1. The reduced Fenchel–Nielsen map (cf. Section 7)
\[ \Phi: \mathcal{F}(X) \to (\mathbb{C}^*)^d \times (\mathbb{R}_+ \cup \{0\})^n \]
is a universal covering.
Proof. We know that \( \Phi \) is surjective, continuous, and open (Lemmas 7.1 and 10.1). We also know that the subgroup \( \mathcal{H} \) of \( \mathcal{F}(X) \) generated by Dehn twists about the partition curves (of the ordered maximal partition defining \( \Phi \)) acts freely (by Lemma 8.1) and properly discontinuously on \( \mathcal{F}(X) \) (by Theorem II). Hence the natural map

\[ \pi: \mathcal{F}(X) \to \mathcal{F}(X)/\mathcal{H} \]

is a Galois covering. By Lemma 8.2 there is a bijection \( q \) such that \( \Phi = q \circ \pi \). Since \( \pi \) is continuous and open, \( q \) is a homeomorphism and \( \Phi \) is a Galois covering (with covering group \( \mathcal{H} \)). The fundamental group \( \pi_1 \) of

\[ \Phi(\mathcal{F}(X)) = (\mathbb{C}^\times)^d \times (\mathbb{R}_+ \cup \{0\})^n \]

is isomorphic to \( \mathbb{Z}^d \) and so is the covering group \( \mathcal{H} \) of \( \Phi \). Hence, there is an exact sequence of covering groups

\[ 0 \to A \to \pi_1 \to \mathcal{H} \to 0 \]

or

\[ 0 \to A \to \mathbb{Z}^d \to \mathbb{Z}^d \to 0 \]

where \( A \) is (or is isomorphic to) the subgroup of \( \pi_1 \) defining \( \Phi \). It follows that \( A = 0 \) so that \( \Phi \) is universal.

Observe that the map

\[ \exp: \mathbb{C}^d \times (\mathbb{R}_+ \cup \{0\})^n \to (\mathbb{C}^\times)^d \times (\mathbb{R}_+ \cup \{0\})^n \]

defined by

\[ (\zeta_1, \ldots, \zeta_d; \rho_1, \ldots, \rho_n) \mapsto (e^{\zeta_1}, \ldots, e^{\zeta_d}; \rho_1, \ldots, \rho_n) \]

is a universal covering. Therefore, there exists a homeomorphism \( \Psi \) such that

\[ \Phi = \exp \circ \Psi. \]

We say that \( \Psi \) is consistent (with the ordered partition of \( X \) used in defining the reduced Fenchel-Nielsen map \( \Phi \)) and we record our result as

**Theorem III.** The Fricke space \( \mathcal{F}_{p,n} \) is homeomorphic to the product of \( 2d \) open and \( n \) half-open intervals. More precisely, an ordered maximal partition of the reference surface \( X \) induces a consistent homeomorphism \( \Psi \) of \( \mathcal{F}_{p,n} = \mathcal{F}(X) \) onto \( \mathbb{C}^d \times (\mathbb{R}_+ \cup \{0\})^n \).

We call \( \Psi \) the Fenchel-Nielsen map and the numbers \( \zeta_1, \ldots, \zeta_d, \rho_1, \ldots, \rho_n \) such that

\[ (\zeta_1, \ldots, \zeta_d; \rho_1, \ldots, \rho_n) = \Psi^{-1}([f]), \quad [f] \in \mathcal{F}(X) , \]

the Fenchel-Nielsen coordinates of \([f]\).
The first statement of Theorem III appears in Fricke and Klein [FK] as a theorem on Fuchsian groups (with a different proof). These groups are not assumed to be torsion free. The Fenchel–Nielsen coordinates appear in the celebrated unpublished manuscript.

15. TEICHMÜLLER SPACES IN FRICKE SPACE

Let $I$ be a subset of the set of ends of $X$ containing $v$ elements, $0 < v < n$. The Teichmüller space $\mathcal{T}_{p,v,n-v}$ is, by definition, the space of equivalence classes $[f]$, in the sense of Section 1, of homeomorphisms $f$ of $X$ onto Riemann surfaces $f(X)$ subject to the provision that an end of $X$ is taken by $f$ (in an obvious sense of this term) onto a puncture (rather than into a hole) if and only if the end belongs to $I$.

(In the general theory of Teichmüller spaces (cf. [B4]), $\mathcal{T}_{p,v,n-v}$ as defined above is called a "reduced" Teichmüller space; we do not have to use this terminology here.)

The space $\mathcal{T}_{p,v,n-v}$ is a subspace of $\mathcal{F}_{p,n}$ and therefore inherits the topology defined in Section 1 or derivable from the metric $\delta$ defined in Section 12. It also has another topology derived from the Teichmüller distance function

$$<[f_1], [f_2]> = \inf \log K(g), \quad g \text{ homotopic to } f_2 \circ f_1^{-1}.$$  

Here $K(g)$ is the global dilatation of the mapping $g$ if $g$ is quasiconformal, $+\infty$ if $g$ is not. It is known that $\mathcal{T}_{p,v,n-v}$ is a complete metric space under the Teichmüller metric, homeomorphic to a cell of dimension

$$6p - 6 + 2v + 3(n - v). \quad (15.1)$$

At the same time, $\mathcal{T}_{p,v,n-v}$ as a subset of $\mathcal{F}_{p,n}$ is a manifold of dimension (15.1). To see this note that the reduced Fenchel–Nielsen map $\Phi$ (cf. Section 7) maps $\mathcal{T}_{p,v,n-v}$ onto

$$(\mathbb{C}^*)^d \times \mathbb{R}_+^n \times \mathbb{R}_+^v,$$

a manifold of (real) dimension

$$2d + n - v = 2(3p - 3 + n) + n - v = 6p - 6 + 3n - v,$$

and that this map is a local homeomorphism (by Lemma 10.1).

The identity map of $\mathcal{T}_{p,v,n-v}$ (equipped with the Teichmüller metric) to $\mathcal{T}_{p,v,n-v}$ considered as a subspace of $\mathcal{F}_{p,n}$ is continuous.
This follows from Proposition 3.3 and from the known fact that for $[f]$ in $\mathcal{T}_{p,v,n}$, the number $l_{[f]}(C)$ is a continuous function on $\mathcal{T}_{p,v,n}$. By Brouwer’s theorem the identity map considered is a homeomorphism.

It is important to note that the subgroup of $\mathcal{G}$ induced by topological self-maps of $X$ which fix the set $I$ may be identified with the Teichmüller modular group acting on $\mathcal{T}_{p,v,n}$.

Observe now that for a given $v$, $0 \leq v \leq n$, the set $I$ can be selected in $\binom{n}{v}$ distinct ways and that every point of $\mathcal{T}_{p,n}$ belongs to a copy of some $\mathcal{T}_{p,v,n}$. This implies

**Theorem IV.** The Fricke space $\mathcal{F}_{p,n}$ is canonically a disjoint union of $2^n$ Teichmüller spaces, a copy of $\mathcal{T}_{p,v,n}$ ($v = 0, \ldots, n$) occurring $\binom{n}{v}$ times. In symbols,

$$\mathcal{F}_{p,n} = \sum_{v = 0}^{n} \binom{n}{v} \mathcal{T}_{p,v,n}.$$

(15.2)

In particular, $\mathcal{F}_{p,n,0}$ is open and dense in $\mathcal{F}_{p,n}$. For each copy of $\mathcal{T}_{p,v,n}$ the embedding into $\mathcal{F}_{p,n}$ is a homeomorphism.

**References**


