Multivariate Analogues of Gould–Hsu Inversions with Applications to Convolution-Type Formulas

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A pair of general reciprocal formulas is established which is the multifold analog of Gould–Hsu inversion. Its special cases are used to give a new derivation for two kinds of convolution identities of Carlitz. The application to the multivariate interpolation process is also presented. © 1991 Academic Press, Inc.

1. INTRODUCTION

Suppose that \( N_0 \) and \( C \) are the sets of non-negative integers and complex numbers respectively. For \( x = (x_1, x_2, \ldots, x_k) \), \( y = (y_1, y_2, \ldots, y_k) \in C^k \) and \( n = (n_1, n_2, \ldots, n_k) \in N_0^k \), we put the notations of coordinate sum \( |x| = \sum_{i=1}^{k} x_i \), scalar product \( \langle x, y \rangle = \sum_{i=1}^{k} x_i y_i \), and the vectorial form of the binomial coefficient \( \binom{n}{i} = \prod_{j=1}^{i} \frac{n_j}{j} \). Denote by \( A \) the difference operator with unit increment. As usual, we define

\[
\left( A^x \right)^n f(x) = \left( A \right)^n \{f(x)\}_{x=i}
\]

and its multivariate form

\[
\left( A^x \right)^n = \prod_{i=1}^{k} \left( A^{x_i} \right)^{n_i}.
\]

Based on these notations, we are ready to state our main result as follows.

**Theorem 1.** Let \( \{c_{i,j}(n) : 0 \leq j \leq k, 1 \leq i \leq k\}_n \) be a matrix sequence such that the polynomials defined by

\[
\phi(x; n) = \prod_{i=1}^{k} \prod_{j=1}^{n_i} (c_{i,0}(j) + \langle c_i(j), x \rangle)
\]

342
differ from zero for $x, n \in \mathbb{N}_0^k$ with the convention that the empty product equals one, where $c_i(n) = (c_{i,1}(n), c_{i,2}(n), \ldots, c_{i,k}(n))$. If we put

$$
\varepsilon(x; n) := \det((c_{i,0}(n_i + 1) + \langle c_i(n_i + 1), x \rangle) \delta_{ij} + (n_j - x_j) c_{i,j}(n_i + 1))
$$

(1.4)

then we have a pair of reciprocal formulae

$$
\begin{align*}
\begin{cases}
g(n) = \sum_{0 \leq r \leq n} (-1)^{|r|} \binom{n}{r} \varepsilon(r; n) \phi(r; n) f(r) \\
f(n) = \sum_{0 \leq r \leq n} (-1)^{|r|} \binom{n}{r} \phi^{-1}(n; r+1) g(r),
\end{cases}
\end{align*}
$$

(1.5)

(1.6)

where $I \in \mathbb{N}_0^k$ with each coordinate being 1 and

$$
\sum_{0 \leq r \leq n} = \sum_{r_1 = 0}^{n_1} \sum_{r_2 = 0}^{n_2} \cdots \sum_{r_k = 0}^{n_k}.
$$

When $k = 1$, (1.5) and (1.6) become the famous inverse relations of Gould and Hsu [3]. As is known, a variety of combinatorial identities can be proved by inverse relations. On account of this fact, we shall provide a simple derivation for some combinatorial identities of Carlitz [1] in Sections 3 and 4. The last section will briefly discuss how to construct interpolation formulae (cf. [2]) via (1.5) and (1.6).

### 2. Proof of Theorem 1

It is obvious that the linear transformation defined by (1.6) is non-singular. Hence it is sufficient to show that (1.6) implies (1.5). Now making the substitution of (1.6) into (1.5), using the fact that

$$
\binom{n}{r} \binom{r}{s} = \binom{n}{s} \binom{n-s}{r-s},
$$

and then exchanging the summation order we have

$$
\sum_{0 \leq s \leq n} \binom{n}{s} g(s) \sum_{s \leq r \leq n} (-1)^{|r|+|s|} \binom{n-s}{r-s} \phi(r; s) \varepsilon(r; n)
$$

which reduces to $g(n)$ if we can show that

$$
\left( -A \right)^{n-s} \left\{ \frac{\phi(x; n)}{\phi(x; s+1)} \varepsilon(x; n) \right\} = \delta_{s,n}.
$$

(2.1)
For $s = n$, (2.1) is clearly true. Note that the determinant $\varepsilon(x; n)$ defined by (1.4) possesses the following properties.

1. $\varepsilon(x; n)$ is a polynomial of degree at most $k - 1$ in $x$. If we add the other columns to the first one in $\varepsilon(x; n)$, then the first column becomes the constant vector which is independent of $x$. Thus property (1) follows.

2. For any subset $J$ of $K = \{1, 2, \ldots, k\}$, it can be checked that

$$
\varepsilon(x; n) |_{x_j = n_{(j \in J)}}
= \left\{ \varepsilon'(x; n) \prod_{j \in J} (c_{j,0}(n_j + 1) + \langle c_j(n_j + 1), x \rangle) \right\}_{n_j - n_i},
$$

where $\varepsilon'(x; n) |_{x_j = n_{(j \in J)}}$ is a polynomial of degree at most $k - |J| - 1$ (where $|J|$ denotes the cardinal number of $J$) in $x$ by property (1).

If $s_i < n_i$ $(i \in K)$, $(\phi(x; n)/\phi(x; s + 1)) \varepsilon(x; n)$ is a polynomial of degree $|n| - |s| - 1$ in $x$ by property (1). Hence its $(n - s)$th difference equals zero. For any subset $J$ of $K$ satisfying $s_j = n_j$ $(j \in J)$, that is also a polynomial with the same degree as the previous one in $x$ by property (2). Therefore its $(n - s)$th difference vanishes in this case. Summarizing these facts we conclude that (2.1) is true for $s_i \leq n_i$ $(i \in K)$. This completes the proof of Theorem 1.

3. Abel-Type Convolution Identities

Replacing $f(r)$ by $(-1)^{|r|} \phi^{-1}(r; r + I) F(r)$ in (1.5) and (1.6), we get an alternate form as follows.

**Proposition 2.** Under the conditions of Theorem 1, we have the inversions

$$
\begin{align*}
&f(n) = \sum_{0 \leq r \leq n} \binom{n}{r} \varepsilon(r; n) \frac{\phi(r; n)}{\phi(r; r + I)} F(r) \quad (3.1) \\
&F(n) = \sum_{0 \leq r \leq n} (-1)^{|n - r|} \binom{n}{r} \frac{\phi(n; n + I)}{\phi(n; r + I)} f(r). \quad (3.2)
\end{align*}
$$

Taking

$$
\phi(x; n) = \prod_{i=1}^{k} (a_i - \langle e_i, x \rangle)^n, \quad (3.3)
$$

with $e_i = (c_{i,1}, c_{i,2}, \ldots, c_{i,k})$ for $1 \leq i \leq k$ and

$$
f(r) = \prod_{i=1}^{k} (a_i + b_i)^r,
$$
we can derive from (3.2) that

\[(b_i + \langle \mathbf{c}_i, \mathbf{n} \rangle)_{n} = \sum_{0 \leq r \leq n} (-1)^{|n-r|} \binom{n}{r} \Pi(a_i - \langle \mathbf{c}_i, \mathbf{n} \rangle)^{n-r} \Pi(a_i + b_i)^r,\]

(3.4)

where \( \Pi = \prod_{i=1}^{n}. \)

Corresponding to (3.3), \( e(r; n) \) reduces to

\[e_a(x; n) = \det((a_i - \langle \mathbf{c}_i, x \rangle) \delta_{ij} + (x_j - n_j) c_{i,j}).\]

(3.5)

Accordingly, (3.1) yields the dual result of (3.4):

\[\Pi(a_i + b_i)^n = \sum_{0 \leq r \leq n} \binom{n}{r} e_a(r; n) \times \Pi(a_i - \langle \mathbf{c}_i, r \rangle)^{n-r} \Pi(b_i + \langle \mathbf{c}_i, r \rangle)^r.\]

(3.6)

If we take

\[f(r) = e_{a+b}(0; r) \Pi(a_i + b_i)^{r-1}\]

instead, then (3.2) gives that

\[D_b(n) \Pi(b_i + \langle \mathbf{c}_i, n \rangle)^{n-1} = \sum_{0 \leq r \leq n} (-1)^{|n-r|} \binom{n}{r} \Pi(a_i - \langle \mathbf{c}_i, n \rangle)^{n-r} \times e_{a+b}(0; r) \Pi(a_i + b_i)^{r-1}\]

(3.7)

and

\[D_b(n) = \det((b_i + \langle \mathbf{c}_i, n \rangle) \delta_{ij} - n_j c_{i,j}).\]

(3.8)

Hence the dual result of (3.7) is derived from (3.1):

\[e_{a+b}(0; n) \Pi(a_i + b_i)^{n-1} = \sum_{0 \leq r \leq n} \binom{n}{r} e_a(r; n) \Pi(a_i - \langle \mathbf{c}_i, r \rangle)^{n-r-1} \times D_b(r) \Pi(b_i + \langle \mathbf{c}_i, r \rangle)^{r-1}\]

(3.9)

Replacing \( a_i \) by \( a_i + \langle \mathbf{c}_i, n \rangle \) in (3.6) and (3.8) will give a pair of convolution identities.

**Theorem 3 (Carlitz [1]).**

(i) \[\sum_{0 \leq r \leq n} \binom{n}{r} A_a(n-r) A_b(r) = A_{a+b}(n)\]

(ii) \[\sum_{0 \leq r \leq n} \binom{n}{r} A_a(n-r) A_b(r) = A_{a+b}(n).\]
where
\[ A_d(n) = D_d(n) \prod (a_i + \langle c_i, n \rangle)^{n_i} \]
\[ A_a(n) = \prod (a_i + \langle c_i, n \rangle)^{n_i}. \]

4. **BINOMIAL TYPE CONVOLUTION IDENTITIES**

If we put \( f(n) = n! g(n) \) and \( F(n) = n! G(n) \) (where \( n! = \Pi n_i! \)), then Proposition 2 can be reformulated as follows.

**Proposition 4.** Under the conditions of Theorem 1, there hold the
\[
\begin{align*}
 g(n) &= \sum_{0 \leq r \leq n} \frac{\varepsilon(r; n)}{(n-r)!} \frac{\phi(r; n)}{\phi(r; r+1)} G(r) \\
 G(n) &= \sum_{0 \leq r \leq n} \frac{(-1)^{n-r}}{(n-r)!} \frac{\phi(n; n+I)}{\phi(n; r+1)} g(r).
\end{align*}
\]

(4.1) (4.2)

Suppose that \( \varepsilon_d(x; n) \) and \( D_d(n) \) are defined by (3.5) and (3.8) respectively. Using the binomial convolution theorem we can easily prove that
\[
\Pi \left( b_i + \langle c_i, n \rangle \right)_{n_i} = \sum_{0 \leq r \leq n} (-1)^{n} \prod (a_i - \langle c_i, n \rangle)_{n_i-r_i} \prod (a_i + b_i)_{r_i}.
\]

(4.3)

\[
D_b(n) \Pi \left( \frac{b_i + \langle c_i, n \rangle}{n_i} \right)_{b_i + \langle c_i, n \rangle} = \sum_{0 \leq r \leq n} (-1)^{n} \prod (a_i - c_i, n)_{n_i-r_i} \prod (a_i + b_i)_{r_i} \times \varepsilon_{a+b}(0; r) \frac{r_i}{a_i + b_i}.
\]

(4.4)

Embedding the above relations into (4.2), we get from (4.1) their dual identities. Replacing \( a_i \) by \( a_i + \langle c_i, n \rangle \) in the latter ones, we obtain their reformulations as follows:

**Proposition 5** (Carlitz [1]).
\[
(i) \quad \sum_{0 \leq r \leq n} B_d(n-r) B_b(r) = B_{a+b}(n)
\]
\[
(ii) \quad \sum_{0 \leq r \leq n} B_d(n-r) B_b(r) = B_{a+b}(n),
\]
where

\[ B_d(n) = D_d(n) \Pi (a_i + \langle c_i, n \rangle) \left( a_i + \frac{\langle c_i, n \rangle}{n_i} \right) \]

and

\[ B_u(n) = \Pi \left( a_i + \frac{\langle c_i, n \rangle}{n_i} \right). \]

5. A Kind of Rational Interpolation Formulae

Using the same method as that of proving Theorem 1, we can verify that the factor \( \varepsilon(x, n) \) in (1.5) and (1.6) is transferable; i.e., Theorem 1 can be restated in the following manner.

**Theorem 6.** Under the conditions of Theorem 1, there exists the following pair of reciprocal formulae:

\[
\begin{align*}
g(n) &= \sum_{0 \leq r \leq n} (-1)^{|r|} \binom{n}{r} \phi(r; n) f(r) \quad (5.1) \\
f(n) &= \sum_{0 \leq r \leq n} (-1)^{|r|} \binom{n}{r} \frac{\varepsilon(n; r)}{\phi(n; r + 1)} g(r). \quad (5.2)
\end{align*}
\]

A kind of multivariate rational interpolation formulae can be generated from (5.1) and (5.2). First, substitute (5.1) for \( g(r) \) in the right hand side of (5.2). Next, replace \( n \) by the continuous variables \( x \) in the summand. Then, by means of the difference operator, an interpolation formula can be established, namely

\[
S_n(f; x) = \sum_{0 \leq r \leq n} \binom{x}{r} \frac{\varepsilon(x; r)}{\phi(x; r + 1)} \left\{ \phi(x; r) f(x) \right\} \quad (5.3)
\]

The interpolation conditions

\[
S_n(f; r) = f(r), \quad 0 \leq r, \leq n_i, \quad 1 \leq i \leq k \quad (5.4)
\]

are automatically satisfied in accordance with the reciprocity between (5.1) and (5.2).

For each fixed \( r \) the interpolant function

\[
\psi(x; r) = \frac{\varepsilon(x; r)}{\phi(x; r + 1)} \quad (5.5)
\]
is a proper fraction whose numerator and denominator are polynomials of total degree $|\tau| + k - 1$ and $|\tau| + k$ respectively. Surely such interpolation formulae are applicable to those interpolated functions which have algebraic singularities near or on the lines $c_{i0} + \langle c_i, x \rangle = 0$. The writer conjectures that $S_\mu(f; x)$ can represent the rational function class in the form $P(x)/Q(x)$ in which $P(x)$ and $Q(x)$ are polynomials in $x$ and $Q(x)$ can be factorized into linear functions.

For the most particular polynomials $\phi(x; n) = 1$ (5.3) will reduce to the tensor product form of the classical Newton interpolation formulae.

REFERENCES