Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT

J. Math. Anal. Appl. 289 (2004) 126–142

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

Embeddability of $L_1(\mu)$ in dual spaces, geometry of cones and a characterization of c_0

Ioannis A. Polyrakis

Department of Mathematics, National Technical University of Athens, Zografou 157 80, Athens, Greece

Received 7 December 2001

Submitted by J. Diestel

Abstract

In this article we suppose that (Ω, Σ, μ) is a measure space and T an one-to-one, linear, continuous operator of $L_1(\mu)$ into the dual E' of a Banach space E. For any measurable set A consider the image $T(L_1^+(\mu_A))$ of the positive cone of the space $L_1(\mu_A)$ in E', where μ_A is the restriction of the measure μ on A. We provide geometrical conditions on the cones $T(L_1^+(\mu_A))$ which yield that the measure μ is atomic, i.e., that $L_1(\mu)$ is lattice isometric to $\ell_1(\mathcal{A})$, where $\mathcal{A}$ denotes the set of atoms of μ. This result yields also a new characterization of $c_0(\Gamma)$.

1. Introduction

The study of the isomorphic copies of $L_1(\mu)$ in dual spaces is an old problem of functional analysis. In 1938, Gelfand [6] proved that $L_1[0, 1]$ is not isomorphic to a conjugate Banach space and, in 1959, Dieudonné [4] raised the problem:

Characterize the $L_1(\mu)$ spaces which are isomorphic to a conjugate Banach space.

Motivated by the above problem and by some known results on the geometry of cones, in this article we study the embeddability of $L_1(\mu)$ in dual spaces in connection with the geometry of the images of the positive cone of $L_1(\mu)$ and its subcones. We show that some properties of these cones are not only characteristic for the measure μ but also affect the geometry of the predual space. Especially we suppose that (Ω, Σ, μ) is a measure space

E-mail address: ypoly@math.ntua.gr.

0022-247X/\$ – see front matter
doi:10.1016/j.jmaa.2003.08.033

and T is an one-to-one, linear continuous map of $L_1(\mu)$ into the dual E' of a Banach space E. We use a result of [15] which states that any weak-star closed cone of a dual space cannot have an unbounded, weak-star closed and weak-star dentable base. Recall that a base for a cone is an intersection of the cone with an affine hyperplane defined by a strictly positive linear functional. To this end for any measurable subset A of Ω with $\mu(A) > 0$ which is not the union of a finite number of atoms (we call any such a set infinitely decomposable) we consider the restriction μ_A of μ on A and we study the geometry of the image $T(L_1^+(\mu_A))$ of the positive cone of $L_1(\mu_A)$. The geometry of these cones is very important in our study.

In particular for each infinitely decomposable set A we study the weak-star closure $Q(A)$ of the cone $T(L_1^+(\mu_A))$. Note that any set $Q(A)$ is a wedge because it is the weak-star closure of a cone and also that $Q(A)$ is infinite dimensional because we have assumed that the set A is infinitely decomposable. In this article we suppose that the wedges $Q(A)$ have the following two properties.

At first we suppose that *the set $Q(\Omega)$ is a cone.* Since $Q(\Omega)$ is a wedge, this property is satisfied if and only if $Q(\Omega) \cap (-Q(\Omega)) = \{0\}$. The assumption that $Q(\Omega)$ is a cone implies that any wedge $Q(A)$ is a cone because $Q(A) \subseteq Q(\Omega)$. In the sequel we study the existence of norm unbounded bases in $Q(A)$ and its subcones which are defined (the bases) by elements of the space E. Especially we suppose that *for any infinitely decomposable set A there are a measurable subset D of A and an element y of E such that y, as a linear functional on E', defines an unbounded base for the cone $Q(D)$*. As we will see later, this property of cones is the crucial property of this article. If we assume that the above two properties are satisfied, then we prove (Theorem 12) that

(i) The image $T'(E)$ of E via the adjoint T' of T is contained in $c_0(\mathcal{A})$, where $\mathcal{A}$ is the set of atoms of μ; and
(ii) The measure μ is atomic. Therefore $L_1(\mu)$ is lattice isometric to $\ell_1(\mathcal{A})$.

As a corollary we prove that if the operator T is an into isomorphism, then $T'(E) = c_0(\mathcal{A})$. Moreover if we suppose that the range of T is weak-star dense in E', we show that T' is an isomorphism of E onto $c_0(\mathcal{A})$. Finally we give a new characterization of $c_0(\Gamma)$ based on the above properties of cones.

As we have noted before, the methodology and the proofs of this article are based on the geometry of cones. So Section 2 of this work is an introduction to the geometry of cones and to the properties of their bases. In Section 3 we study the images of the positive cone of $L_1(\mu)$ and its subcones in E' via the operator T. In the sequel we define a subspace of $L_\infty(\mu)$ which is denoted by $c_0(\mu)$ and plays an important role in our study. Specifically we prove that $c_0(\mu)$ is lattice isometric to the space $c_0(\mathcal{A})$. Moreover, as we will see in the proof of the main result, the assumption that $T'(x) \notin c_0(\mu)$ for some element x of E, combined with the definition of $c_0(\mu)$ and our assumptions for the cones $Q(A)$, implies the existence of a weak-star closed cone in a dual space with a norm unbounded, weak-star closed and weak-star dentable base, which contradicts the result of [15]. So we prove that $T'(E) \subseteq c_0(\mu)$. This is the basic step of this work from where it follows everything. In this article we provide also many examples which introduce the reader to the geometry of cones and to the basic ideas of this article.

Recall that the embeddability of $L_1(\mu)$ in dual spaces has been studied by many authors in the past. We refer to the papers of Pelczynski [11,12], Lewis and Stegall [9], Stegall [16], Fonf [5] but we can also refer to many other significant works. For an extensive study of L_1-predual spaces we refer to the book of Lacey [8].

In 1981, Bourgain and Delbaen [2] gave an example of a Banach space E whose dual E' is isomorphic to ℓ_1 but E does not contain any copy of c_0. Moreover E is a separable, $\mathcal{L}_\infty$ space with the Radon–Nikodým property and E is somewhat reflexive. So the Banach spaces whose dual is isomorphic to ℓ_1 seems to be a big class of spaces and the characterization of c_0 among the elements of this class is an interesting problem. Corollary 14 is such a characterization of c_0.

Finally note the following characterization of c_0 which is proved in [14]: *An ordered Banach space E is order isomorphic to c_0 if and only if E is a σ-Dedekind complete vector lattice and its dual E' is order isomorphic to ℓ_1*. The proof of this result is based on the proof that the dual E' of E has a positive Schauder basis. The methods and the results of [14] are quite different from the methodology and the results of the present article.

2. Bases for cones

We start with the basic properties of the bases for cones which we will use in this article.

Let E be a normed space. Denote by E' the norm dual and by E'' the second norm dual of E. Also denote by $\mathbb{R}_+$ the set of real numbers $\lambda \geqslant 0$. For each $x \in E$ denote by $\hat{x}$ the natural image of x in E'' and for each $K \subseteq E$ denote by $\hat{K}$ the set $\hat{K} = \{\hat{x} \in E'' \mid x \in K\}$.

Let P be a *wedge* of E, i.e., P is a convex subset of E such that $\lambda P \subseteq P$ for each $\lambda \in \mathbb{R}_+$. If $E = P - P$ the wedge P is called *generating* and if $P \cap (-P) = \{0\}$ we say that P is a *cone*. Suppose that E is ordered by the wedge P, i.e., for any $x, y \in E$ we have $x \leqslant y$ if and only if $y - x \in P$. If P is a cone the ordering is antisymmetric. A linear functional f on E is *positive* if $f(x) \geqslant 0$ for each $x \in P$ and *strictly positive* if $f(x) > 0$ for each $x \in P$, $x \neq 0$. Also a linear functional f on E is *uniformly monotonic* if a real number $a > 0$ exists such that $f(x) \geqslant a\|x\|$ for each $x \in P$. In the above cases we say also that f is positive, strictly positive and uniformly monotonic on P.

The dual space E' of E is ordered by the wedge

$$P^0 = \{x' \in E' \mid x'(x) \geqslant 0 \text{ for each } x \in P\}$$

which is called *the dual wedge* of P (in E').

Let P be a cone. A subset B of P is a *base for the cone P* if a strictly positive linear functional f of E exists such that B is the intersection of the cone P with the affine hyperplane $\{x \in E \mid f(x) = 1\}$, i.e.,

$$B = \{x \in P \mid f(x) = 1\}.$$

Then we say that *the base B is defined by the functional f*. The base B is convex and it is easy to show that B is bounded if and only if the functional f is uniformly monotonic. Indeed, if we suppose that $\|x\| \leqslant M$ for each $x \in B$, then for each $x \in P$, $x \neq 0$, we have $\|x/f(x)\| \leqslant M$, therefore $\|x\| \leqslant Mf(x)$ for each $x \in P$, hence f is uniformly monotonic.

For the converse suppose that $f(x) \geqslant a\|x\|$ for each $x \in P$. Then for each $x \in B$ we have $1 = f(x) \geqslant a\|x\|$, therefore the base B is bounded.

Note also that if P is a finite-dimensional closed cone then each base B for P is bounded. Indeed if we suppose that B is defined by the linear functional f and $x_n \in B$ with $\|x_n\| \to \infty$, then $f(x_n/\|x_n\|) \to 0$. Since the set $P \cap U_E$ (U_E is the closed unit ball of E) is compact, a subsequence of $\{x_n/\|x_n\|\}$ exists which converges to an element x_0 of P. Then we have that $\|x_0\| = 1$ and $f(x_0) = 0$, contradiction because f is strictly positive on P.

A nonzero element x_0 of P is an *extremal point* of P if for any $x \in E$, $0 \leqslant x \leqslant x_0$ implies that $x = \lambda x_0$ for some real number $\lambda \in \mathbb{R}_+$. A point x_0 of a base B for P is an extreme point of B if and only if x_0 is an extremal point of P. Indeed, if we suppose that x_0 is an extremal point of P and $x_0 = \lambda x + (1-\lambda)y$ with $x, y \in B$, we have that $0 \leqslant \lambda x, (1-\lambda)y \leqslant x_0$. Therefore x, y are positive multiples of x_0 and by the fact that x, y are elements of B we have that $x = y = x_0$. For the converse we suppose that the base B is defined by the linear functional f, x_0 is an extreme point of B and that $0 < x \leqslant x_0$. Then

$$x_0 = f(x)\frac{x}{f(x)} + f(x_0 - x)\frac{x_0 - x}{f(x_0 - x)}$$

and by the fact that x_0 is an extreme point of B we have that x is a positive multiple of x_0. Therefore x_0 is an extremal point of P.

In [13,15] the geometry (dentability, extreme points) of the bases of cones are studied. From these articles we refer some results below which we will use in the present paper.

We start with the notion of the continuous positive projection which is defined in [13] as follows: Let x_0 be an extremal point of P. If there exists a continuous projection Π of E onto the one-dimensional subspace generated by x_0, such that $0 \leqslant \Pi(x) \leqslant x$ for each $x \in P$, then we say that the point x_0 *has* (*admits*) *a continuous, positive projection*. Then it is easy to show that a positive continuous linear functional π of E exists such that $\Pi(x) = \pi(x)x_0$ for each $x \in E$ with $\pi(x_0) = 1$. If x_0 admits a continuous, positive projection Π, then

$$E = [x_0] \oplus Y,$$

where Y is the kernel of Π and for any $x \in E$ we have

$$x \in E_+ \quad \text{if and only if} \quad \Pi(x) \in E_+ \text{ and } x - \Pi(x) \in E_+.$$

As it is proved in [13], if E is a normed lattice (i.e., E is a lattice and for each $x, y \in E$, $|x| \leqslant |y|$ implies that $\|x\| \leqslant \|y\|$) or if E is a Banach space with the Riesz decomposition property and the cone P is closed and generating, then each extremal point of P admits a continuous positive projection. Recall that an ordered space X has the Riesz decomposition property if for any $x, y, z \in X_+$, $x \leqslant y + z$ implies that $x = x_1 + x_2$, where $x_1, x_2 \in X_+$ with $x_1 \leqslant y$, $x_2 \leqslant z$. Also note that every linear lattice has the Riesz decomposition property but the converse is not always true.

A linear functional x' of E *strongly exposes* a point x of a subset D of E if $x'(x) \geqslant x'(y)$ for each $y \in D$ and for any sequence $\{x_n\}$ in D $x'(x_n) \to x'(x)$, implies that $\|x_n - x\| \to 0$.

Theorem 1 [13, Proposition 3.4]. *Let B be a base for a cone P of a normed space E, defined (the base) by a continuous linear functional f on E and let x_0 be an extreme point of B which admits the continuous, positive projection $\Pi(x) = \pi(x)x_0$, $x \in E$. Then*

(i) *x_0 is a strongly exposed point of B if and only if there exists a uniformly monotonic, continuous linear functional of E;*
(ii) *If h is a uniformly monotonic, continuous linear functional on E, then the functional $g = h(x_0)\pi - h$ strongly exposes the point x_0 in B with $g(x_0) = 0$.*

Suppose that K is a convex subset of E'. The set K is *weak-star dentable* if for each real number $\varepsilon > 0$ there exists a point x'_ε of K which does not belong to the weak-star closure of the convex hull of the set $\{x' \in K \mid \|x' - x'_\varepsilon\| \geqslant \varepsilon\}$. An element $x'_0 \in K$ is a *weak-star strongly exposed point* of K if there exists $x \in E$ which, as a linear functional on E', strongly exposes the point x'_0 in K. If a subset K of E' is not weak-star dentable then K does not have weak-star strongly exposed points. This holds because if we suppose that a point x of E strongly exposes a point x'_0 of K then for any real number $\varepsilon > 0$, we have that x separates x'_0 and the set $\{x' \in K \mid \|x' - x'_0\| \geqslant \varepsilon\}$ which is impossible because we have assumed that the set K is not weak-star dentable.

Theorem 2 [15, Corollary 2]. *Let P be a weak-star closed cone of the dual E' of a normed space E and let B be a norm-unbounded base for the cone P. Then each norm-unbounded weak-star closed and convex subset K of B is not weak-star dentable.*

Example 3. (i) Suppose that $E = \ell_1$ and that $P = \ell_1^+$ is the positive cone of ℓ_1. Suppose that $y \in \ell_\infty$ with $y_n = 1/n$ for each n. Then y defines the base

$$B = \{x \in \ell_1^+ \mid y(x) = 1\}$$

for the cone ℓ_1^+, and we remark that B is unbounded because $ne_n \in B$ for each $n \in \mathbb{N}$, where $\{e_n\}$ is the usual Schauder basis of ℓ_1. For each n, the point ne_n of B is an extremal point of P and therefore also an extreme point of B. Also the point ne_n admits the continuous, positive projection

$$\Pi_n(x) = \pi_n(x)ne_n, \quad \text{where } \pi_n(x) = \frac{1}{n}x_n \text{ for each } x = (x_1, x_2, \ldots) \in \ell_1.$$

The element h of ℓ_∞ with $h_i = 1$ for each i is a uniformly monotonic, continuous linear functional on ℓ_1. Therefore by Theorem 1, each extreme point ne_n of B is a strongly exposed point of B and the functional $g_n = h(ne_n)\pi_n - h$ strongly exposes the point ne_n in B with $g_n(ne_n) = 0$. Especially for $n = 1$ we have that the functional $g_1 = (0, -1, -1, -1, \ldots)$ strongly exposes e_1 in B with $g_1(e_1) = 0$.

Now consider ℓ_1 as the dual of c_0. Then the cone ℓ_1^+ is weak-star closed and the base B is unbounded and weak-star closed. By Theorem 2, B is not weak-star dentable, therefore B does not have weak-star strongly exposed points. As we have remarked above, ne_n is a strongly exposed point of B for each n. Since B is not weak-star dentable, any strongly exposing functional of ne_n cannot belong to c_0. Indeed, as we have shown before, the functional $g = (0, -1, -1, -1, \ldots)$ strongly exposes e_1 but g does not belong to c_0.

(ii) Suppose that $E = \ell_p$ with $1 < p < +\infty$, $P = \ell_p^+$ is the positive cone of ℓ_p and $\{e_n\}$ is the usual Schauder basis of E. Suppose that $B = \{x \in \ell_p^+ \mid f(x) = 1\}$ is a base for ℓ_p^+ which is defined by a linear functional $f = (f_1, f_2, \ldots) \in \ell_q$. Since f is strictly positive we have that $f_i > 0$ for each i. Then $e_n/f_n \in B$ for each n, therefore the base B is unbounded. Since the space E is reflexive and the cone P and the base B are weakly closed, by Theorem 2, we have that the base B is not dentable. Therefore for any n, the extreme point e_n/f_n of B cannot be strongly exposed.

We can also show that the base B does not have strongly exposed points as follows. At first we remark that E does not have uniformly monotonic, continuous linear functionals. Indeed if we suppose that $h \in \ell_q$ with $h(x) \geqslant a\|x\|$ for each $x \in E_+$ and for some real number $a > 0$, we have that $h_n = h(e_n) \geqslant a\|e_n\| = a$ for each n, therefore $a = 0$, a contradiction. Since each extreme point e_n/f_n of B admits a continuous positive projection, by Theorem 1 we have that e_n/f_n is not a strongly exposed point of B because E does not have uniformly monotonic, continuous linear functionals.

For a further study of the geometry of convex sets (dentability, extreme points) we refer to the book of Diestel and Uhl [3]. Also for ordered spaces we refer to the books of Jameson [7] and Aliprantis [1].

Let $L : E \to X$ be a continuous linear operator of E into a normed space X. If L is one-to-one and L^{-1} is continuous, we say that L is an *isomorphism* of E into X and also that E is *embeddable* in X. The operator $L' : X' \to E'$ such that

$$(L'x')(x) = x'(Lx) \quad \text{for each } x' \in X' \text{ and } x \in E,$$

is the *adjoint* of L. This operator is continuous with $\|L'\| = \|L\|$. Suppose that E, X are ordered normed spaces and L is an isomorphism of E into X. If for any $x \in E$ it holds $x \in E_+$ if and only if $L(x) \in X_+$, then L is an *order isomorphism* of E into X and if moreover $\|Lx\| = \|x\|$ for each x, we say that L is an *order isometry*. If in the two previous definitions E, X are vector lattices, we say also that L is a *lattice isomorphism* and a *lattice isometry*, respectively.

Recall that an ordered space E is a *vector lattice* if for any two elements $x, y \in E$ the supremum of $\{x, y\}$ in E exists. Then the infimum of $\{x, y\}$ also exists and we denote by $x \vee y$ and by $x \wedge y$ the supremum and the infimum of $\{x, y\}$, respectively. A subspace X of E is a *sublattice* or a *Riesz subspace* of E if for any $x, y \in X$, $x \vee y$ and $x \wedge y$ belong to X.

3. Embeddability of $L_1(\mu)$ in dual spaces

In this section we will denote by (Ω, Σ, μ) a measure space, where the measure μ takes values in the interval $[0, +\infty]$ and Σ is a σ-algebra of measurable subsets of Ω. Also we will denote by T an one-to-one bounded linear operator of $L_1(\mu)$ into the norm dual E' of a Banach space E. We will suppose that the space $L_1(\mu)$ is infinite dimensional, the norm dual of $L_1(\mu)$ is the space $L_\infty(\mu)$ and that $\mu(A) < \infty$ for any atom A of μ.

Recall that $L_1(\mu)$ is the space of absolutely integrable functions $f : \Omega \to \mathbb{R}$ with norm $\|f\|_1 = \int_\Omega |f(t)|\,d\mu$ and $L_\infty(\mu)$ is the space of measurable, essentially bounded functions

$f:\Omega \to \mathbb{R}$ with norm $\|f\|_\infty = \inf\{a \in \mathbb{R}_+ \mid |f(t)| \leqslant a$ for each $t \in \Omega\}$. The above spaces, in the pointwise ordering, are Banach lattices with positive cones $L_1^+(\mu)$ and $L_\infty^+(\mu)$, respectively. Since in these spaces the equality of functions is in the sense of the almost everywhere we will say "for every t" instead of "for almost all t." A measurable subset A of Ω is an *atom* of μ if $\mu(A) > 0$ and for each $B \in \Sigma$ with $B \subseteq A$ we have that $\mu(B) = 0$ or $\mu(B) = \mu(A)$.

We remark here that if the measure space (Ω, Σ, μ) is σ-finite then $L_\infty(\mu)$ is the dual of $L_1(\mu)$ and any atom of μ is of finite measure. If the set Ω is infinite and μ is the counting measure defined on the subsets of Ω, then the above assumptions are also satisfied. For any set Γ denote by $c_0(\Gamma)$ the space of real vectors $a = (a_i)_{i\in\Gamma}$, such that for each real number $\epsilon > 0$, the set $\{i \in \Gamma \mid |a_i| > \epsilon\}$ is finite, with norm $\|a\|_0 = \sup_{i\in\Gamma} |a_i|$ and by $\ell_1(\Gamma)$ the space of real vectors $a = (a_i)_{i\in\Gamma}$, with norm $\|a\|_1 = \sum_{i\in\Gamma} |a_i|$. These spaces, in the pointwise ordering, are Banach lattices with positive cone $c_0^+(\Gamma)$ and $\ell_1^+(\Gamma)$, respectively.

Let $A \in \Sigma$. We will denote by χ_A the *characteristic function* of A and by μ_A the restriction of μ on A, i.e., $\mu_A(B) = \mu(A \cap B)$ for each $B \in \Sigma$. The set A is *infinitely decomposable* if $A = \bigcup_{i=1}^\infty A_i$, where $\{A_i\}$ is a countable family of disjoint measurable subsets of Ω with $\mu(A_i) > 0$ for each i. Therefore the set A is not infinitely decomposable if $\mu(A) = 0$ or if A is the union of a finite number of atoms. Note that $L_1(\mu_A)$ is lattice isometric to the subspace $F_A = \{\xi\chi_A \mid \xi \in L_1(\mu)\}$ of $L_1(\mu)$ and $L_1(\mu_A)$ is infinite dimensional if and only if the set A is infinitely decomposable.

Definition 4. For any measurable subset A of Ω we will denote by $K(A)$ the cone

$$K(A) = \left\{T(\xi\chi_A) \mid \xi \in L_1^+(\mu)\right\}$$

and by $Q(A)$ the weak-star closure of the cone $K(A)$ in E'.

Also we will denote $P(A)$ the dual wedge of $K(A)$ in E and by $P(A)^0$ the dual wedge of $P(A)$ in E'.

Recall that

$$P(A) = \left\{x \in E \mid x'(x) \geqslant 0 \text{ for each } x' \in K(A)\right\}$$

is the dual wedge of $K(A)$ in E and

$$P(A)^0 = \left\{x' \in E' \mid \hat{x}(x') \geqslant 0 \text{ for each } x \in P(A)\right\}.$$

Since $P(A)^0$ is weak-star closed and $K(A) \subseteq P(A)^0$, we have that

$$K(A) \subseteq Q(A) \subseteq P(A)^0.$$

Also the annihilator

$$M = \left\{x \in E\colon x'(x) = 0 \text{ for any } x' \in T\big(L_1(\mu)\big)\right\}$$

of $T(L_1(\mu))$ in E is contained in $P(A)$. Therefore if the range of T is not weak-star dense in E', then $P(A)$ is a wedge of E but not a cone.

We will say that an element y of E *defines an unbounded base for the cone* $Q(A)$, if $\hat{y}$ is strictly positive on $Q(A)$ and the set

$$\{x' \in Q(A) \mid \hat{y}(x') = 1\}$$

is norm unbounded in E'. As we will see in Theorem 12, the existence of certain unbounded bases for cones in E' is crucially related to properties of operators of $L_1(\mu)$ into the dual space E'.

The assumption that $\hat{y}$ is strictly positive on $Q(A)$ implies that $Q(A)$ is a cone. Indeed $x', -x' \in Q(A)$ implies that $\hat{y}(x') = 0$, therefore $x' = 0$ because $\hat{y}$ is strictly positive on $Q(A)$.

In the following proposition we give the properties of $K(A)$ and $Q(A)$. Some of them are known results of the theory of ordered spaces.

Proposition 5. *For any measurable subset A of Ω we have*

(i) *$K(A)$ is a cone*;
(ii) $Q(A) = P(A)^0$;
(iii) *$Q(A)$ is a cone if and only if $P(A) - P(A)$ is dense in E*;
(iv) *An element y of E defines an unbounded base for the cone $Q(A)$ if and only if the functional $\hat{y}$ is strictly positive on $Q(A)$ and $\hat{y}$ defines an unbounded base for the cone $K(A)$*;
(v) *If an element y of E defines an unbounded base for the cone $Q(A)$, the set A is infinitely decomposable.*

Proof. (i) Suppose that $x' \in K(A) \cap (-K(A))$. Then there exist $\xi, \eta \in L_1^+(\mu)$ with $x' = T(\xi\chi_A)$ and $x' = T(-\eta\chi_A)$. Since T is an isomorphism we have that $\xi\chi_A = -\eta\chi_A$, therefore $\xi\chi_A = 0$ because $\xi\chi_A \in L_1^+(\mu) \cap (-L_1^+(\mu))$. So we have that $x' = 0$, therefore $K(A)$ is a cone.

(ii) As we have remarked above $Q(A) \subseteq P(A)^0$. Suppose that $x_0' \in P(A)^0 \setminus Q(A)$. Then there exists $z_0 \in E$ separating x_0' and $Q(A)$, i.e., $\hat{z}_0(x_0') < a \leqslant \hat{z}_0(x')$ for each $x' \in Q(A)$. Since $0 \in Q(A)$, we have that $a \leqslant 0$, therefore $\hat{z}_0(x_0') < 0$. Also for each $x' \in K(A)$ and each $\lambda \in \mathbb{R}_+$, we have that $\lambda x' \in Q(A)$, therefore $\hat{z}_0(\lambda x') \geqslant a$ for each $\lambda \in \mathbb{R}_+$, therefore $\hat{z}_0(x') \geqslant 0$ for each $x' \in K(A)$. By the definition of $P(A)$ we have that $z_0 \in P(A)$. Since $x_0' \in P(A)^0$, we have also that $\hat{z}_0(x_0') \geqslant 0$, a contradiction. Hence $P(A)^0 = Q(A)$.

(iii) Suppose that Z is the closure of $P(A) - P(A)$ in E. Suppose that $Q(A)$ is a cone. If we suppose that $x_0 \in E \setminus Z$, there exists an element x' of E' which is zero on Z and $x'(x_0) > 0$. This implies that $x'(x) = 0$ for each $x \in P(A)$ and $-x'(x) = 0$ for each $x \in P(A)$, therefore $x' \in Q(A) \cap (-Q(A))$ hence $x' = 0$ because we have assumed that $Q(A)$ is a cone. This is a contradiction. Therefore $Z = E$. For the converse suppose that $Z = E$ and that $x' \in Q(A) \cap (-Q(A))$. Then it is easy to show that $x'(x) = 0$ for each $x \in P(A)$ therefore x' is equal to zero on E. Hence $x' = 0$ and $Q(A)$ is a cone.

(iv) Suppose that an element y of E is strictly positive on $Q(A)$ and also that it defines an unbounded base for the cone $K(A)$. Then it is clear that the base B for the cone $Q(A)$ which is defined by y is also unbounded. Suppose now that an element y of E defines an unbounded base for the cone $Q(A)$. Then $\hat{y}$ is strictly positive on $Q(A)$ and the set $B =$

$\{x' \in Q(A) \mid \hat{y}(x') = 1\}$ is unbounded. Suppose that the set $D = \{x' \in K(A) \mid \hat{y}(x') = 1\}$ is bounded. Then each $x' \in B$ is the weak-star limit of a net $(x'_\lambda)_{\lambda \in \Lambda}$ of $K(A)$. Therefore x' is also the weak-star limit of the net $(x'_\lambda/\hat{y}(x'_\lambda))_{\lambda \in \Lambda}$ of D. Therefore the set B, as the weak-star closure of the bounded set D is also bounded, a contradiction. Hence the set D is unbounded, so $\hat{y}$ defines also an unbounded base for the cone $K(A)$.

(v) Suppose that an element y of E defines an unbounded base for the cone $Q(A)$. The cone $Q(A)$ is closed in the norm topology of E' and also $Q(A)$ has an unbounded base, therefore $Q(A)$ is infinite dimensional because as we have remarked in the previous section, a finite-dimensional closed cone of a Banach space cannot have an unbounded base. Since the cone $Q(A)$ is infinite dimensional, we have that the set A is infinitely decomposable. □

In the main theorem below we will assume that

(a) The set $Q(\Omega)$ is a cone; and
(b) For each infinitely decomposable subset A of Ω there are a measurable subset D of A and an element y of E such that y defines an unbounded base for the cone $Q(D)$.

In the following examples we study the above properties for some simple cases of the spaces $L_1(\mu)$, E and the operator T. Examples (i) and (ii) satisfy properties (a) and (b) but in examples (iii) and (iv) property (b) fails. Examples (i) and (ii) are similar but their difference shows the meaning of (b). Especially in (i), for any infinitely decomposable set A an element of E exists which defines an unbounded base for the cone $Q(A)$ but in example (ii) the existence of such an element of E is not guaranteed for any A. However in example (ii), for any infinitely decomposable set A we can find a measurable subset D of A and an element y of E which defines an unbounded base for the cone $Q(D)$. Note also that in (b) the set D is an infinitely decomposable subset of A. This holds because the assumption that the cone $Q(D)$ has an unbounded base implies that $Q(D)$ is infinite dimensional, therefore the set D is infinitely decomposable.

Example 6. (i) Suppose that $E = c_0$, and that $T : \ell_1 \to \ell_1 = c'_0$ is the identity map. (The measure μ is just counting measure on the subsets of $\mathbb{N}$.) Since the positive cone of ℓ_1 is weak-star closed we have that $Q(\Omega) = \ell_1^+$, therefore $Q(\Omega)$ is a cone. Also it is easy to show that a subset of $\mathbb{N}$ is infinitely decomposable if and only if it is infinite. So for any infinite subset A of $\mathbb{N}$ we have that $K(A) = \{\xi \chi_A \mid \xi \in \ell_1^+\}$. The element y of E with $y_i = 0$ if $i \notin A$ and $y_i = 1/i$ for each $i \in A$ defines the base

$$B = \left\{\xi \chi_A \;\middle|\; \xi \in \ell_1^+,\ \sum_{i \in \mathbb{N}} y_i \xi_i = 1\right\}$$

for the cone $K(A)$. The base B is unbounded because $ne_n \in B$ for each $n \in A$, where $\{e_n\}$ is the usual Schauder basis of ℓ_1. Also the cone $K(A)$ is weak-star closed, therefore $K(A) = Q(A)$ and the element y of E defines an unbounded base for the cone $Q(A)$.

(ii) Suppose that $E = c_0(\Gamma)$ and that $T : \ell_1(\Gamma) \to \ell_1(\Gamma)$ is the identity map. (The measure μ is just counting measure on the subsets of Γ.) As in the previous case we have that $Q(\Omega) = \ell_1^+(\Gamma)$, therefore $Q(\Omega)$ is a cone. Suppose that A is an infinitely decomposable subset of Γ. Then the set A is infinite and it is easy to show that $K(A)$ is weak-star closed, so $Q(A) = K(A)$. If the set A is uncountable and $y \in E$, then y cannot be strictly positive on $Q(A)$ because the support of y is at most countable. However, for any countable subset $D = \{\gamma_n \mid n \in \mathbb{N}\}$ of A the element $y = (y_\gamma)$ of E with $y_\gamma = 0$ for each $\gamma \notin D$, and $y_{\gamma_n} = 1/n$, for each n, defines the base

$$B = \left\{ \xi \chi_D \;\middle|\; \xi \in \ell_1^+(\Gamma),\ \sum_{n=1}^{\infty} y_{\gamma_n} \xi_{\gamma_n} = 1 \right\}$$

for the cone $K(D)$. As in the case of ℓ_1, the base B is unbounded because $ne_{\gamma_n} \in B$ for each n. So for each infinitely decomposable subset A of Ω an infinitely decomposable subset D of A and an element $y \in E$ exist such that y defines an unbounded base for the cone $Q(D)$.

(iii) Suppose that T is the natural embedding of $\ell_1(\Gamma)$ in its second dual. Then $K(\Omega) = T(\ell_1^+(\Gamma)) = \ell_1^+(\Gamma)$ and $P(\Omega) = \ell_\infty^+(\Gamma)$ is the dual cone of $K(\Omega)$ in $\ell_\infty(\Gamma)$. Since the dual cone of $\ell_\infty^+(\Gamma)$ in $\ell_\infty'(\Gamma)$ is the positive cone of $\ell_\infty'(\Gamma)$, we have that $Q(\Omega) = (\ell_\infty'(\Gamma))_+$ and $Q(\Omega)$ is a cone.

Suppose that the element f of $\ell_\infty(\Gamma)$ defines an unbounded base for the cone $(\ell_\infty'(\Gamma))_+$. Then f is strictly positive on $(\ell_\infty'(\Gamma))_+$ and by Proposition 5 we have that the set

$$B = \left\{ x \in \ell_1^+(\Gamma) \mid f(x) = 1 \right\}$$

is unbounded. Since f is strictly positive on $\ell_1^+(\Gamma)$ we have also that $f_i = f(e_i) > 0$ for each $i \in \Gamma$. Suppose that $\{x^\nu\}$ is an unbounded sequence in B. If we suppose that $f_i > \rho > 0$ for each i, we have

$$1 = f(x^\nu) = \sum_{i \in \Gamma} f_i x_i^\nu > \rho \sum_{i \in \Gamma} x_i^\nu = \rho \|x^\nu\|_1,$$

a contradiction because the sequence $\{x^\nu\}$ is unbounded. Therefore there exists a sequence $\{i_\nu\}$ in Γ with $\lim_{\nu\to\infty} f_{i_\nu} = 0$. Suppose that L is the set of all $g \in \ell_\infty(\Gamma)$ for which $\lim_{\nu\to\infty} g_{i_\nu}$ exists. Then L is a sublattice of $\ell_\infty(\Gamma)$ and suppose that ϑ is the linear functional on L with $\vartheta(g) = \lim_{\nu\to\infty} g_{i_\nu}$ for each $g \in L$. Then ϑ is positive and continuous, therefore by [7, Proposition 4.2.4], ϑ has a positive and continuous extension on $\ell_\infty(\Gamma)$ which we denote again by ϑ. Then we have that $\vartheta(f) = 0$. This is a contradiction because $\vartheta \neq 0$ and we have assumed that f is strictly positive on $(\ell_\infty'(\Gamma))_+$. Therefore an element $f \in \ell_\infty^+(\Gamma)$ which defines an unbounded base for the cone $Q(\Omega)$ does not exist. Also for any infinite subset A of Γ we have that $K(A) = \ell_1^+(A)$ and $Q(A) = (\ell_\infty'(A))_+$ and as above, we have that an element of $\ell_\infty(A)$ which defines an unbounded base for the cone $Q(A)$ does not exist.

(iv) Let E be the space $C[0,1]$ of continuous real valued functions defined on $[0,1]$. Then by the Kakutani representation theorem, E' as an *AL*-space is lattice isometric to an $L_1(\mu)$ space and suppose that T is a lattice isometry of $L_1(\mu)$ onto E'. Then $T(L_1^+(\mu)) = E'_+$. Since the cone E'_+ is weak-star closed we have that $Q(\Omega) = E'_+$ is a cone. Suppose that an element y of E defines an unbounded base for the cone E'_+. Then $y \in E_+$ and y is strictly positive on E'_+. If we suppose that $y(t) = 0$ for some $t \in [0,1]$, then $\hat{y}$ is equal to zero on the Dirac measure δ_t supported at $\{t\}$, a contradiction. Therefore $y(t) \geqslant \alpha > 0$ for each t. The set

$$B = \{m \in E'_+ \mid \hat{y}(m) = 1\}$$

is the base for E'_+ defined by y. For each $m \in B$ we have

$$1 = \int_{[0,1]} y\,dm \geqslant \alpha m([0,1]),$$

therefore $\|m\| \leqslant 1/\alpha$, a contradiction because we have assumed that the base B is unbounded. Therefore an element $y \in E_+$ which defines an unbounded base for the cone $Q(\Omega)$ does not exist.

Definition 7. Denote by $c_0(\mu)$ the set of functions $f \in L_\infty(\mu)$ with the property: For each infinitely decomposable set A and each real number $\epsilon > 0$ there exists an infinitely decomposable set $B \subseteq A$ such that $\|f\chi_B\|_\infty \leqslant \epsilon$.

Lemma 8. *For each $f \in L_\infty(\mu) \setminus c_0(\mu)$, there exists an infinitely decomposable set A and a real number $\rho > 0$ such that $|f(t)| > \rho$ for each $t \in A$.*

Proof. By the definition of $c_0(\mu)$ there exists an infinitely decomposable set A and a real number $\rho > 0$ such that $\|f\chi_B\|_\infty > \rho$ for each infinitely decomposable subset B of A. Therefore the set $D = \{t \in A \mid |f(t)| \leqslant \rho\}$ is not infinitely decomposable. Hence the set $A \setminus D$ is infinitely decomposable with $|f(t)| > \rho$ for each $t \in A \setminus D$. □

Lemma 9. *Let $f \in c_0(\mu)$ and $A \in \Sigma$ with $\mu(A) > 0$.*

(i) *If A does not contain atoms, then $\|f\chi_A\|_\infty = 0$.*
(ii) *If $f \neq 0$, then the measure μ has at least one atom.*
(iii) *If A is an atom of μ, then $f(t) = \|f\chi_A\|_\infty$ for all $t \in A$.*

Proof. Let a real number $\epsilon > 0$ and let $A_\epsilon = \{t \in A \mid |f(t)| > \epsilon\}$. Suppose that $\mu(A_\epsilon) > 0$. Since A does not contain atoms, each measurable subset B of A_ϵ with $\mu(B) > 0$ is infinitely decomposable with $\|f\chi_B\|_\infty > \epsilon$. This contradicts the definition of $c_0(\mu)$, therefore $\mu(A_\epsilon) = 0$. Hence $\|f\chi_A\|_\infty = 0$ and statement (i) is true. If $f \neq 0$, then by (i) μ has at least one atom, therefore (ii) is also true. (iii) is obvious because A is an atom. □

Two atoms A, B of μ are *equivalent* if $\mu(A \triangle B) = 0$, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$. In this article we will identify the atoms of μ with the corresponding equivalence classes and we will denote by $\mathcal{A}$ *the set of atoms* of μ. The measure μ is *atomic* or purely atomic, if each measurable A with $\mu(A) > 0$, contains at least one atom of μ.

If the measure μ is atomic then $\ell_1(\mathcal{A})$ is lattice isometric (we have assumed that $\mu(A) < \infty$ for any atom A) to $L_1(\mu)$ and the map

$$U(\alpha) = \sum_{A \in \operatorname{supp}(\alpha)} \frac{\alpha_A}{\mu(A)} \chi_A, \quad \alpha \in \ell_1(\mathcal{A}), \tag{1}$$

is a lattice isometry of $\ell_1(\mathcal{A})$ onto $L_1(\mu)$.

Proposition 10. *The set $c_0(\mu)$ is a closed sublattice of $L_\infty(\mu)$. If $\mathcal{A} = \emptyset$, then $c_0(\mu) = \{0\}$ and if $\mathcal{A} \neq \emptyset$, the map*

$$\Psi(\alpha) = \sum_{A \in \operatorname{supp}(a)} \alpha_A \chi_A, \quad \alpha \in c_0(\mathcal{A}),$$

is a lattice isometry of $c_0(\mathcal{A})$ onto $c_0(\mu)$.

Proof. If $\mathcal{A} = \emptyset$, then by Lemma 9 we have that $c_0(\mu) = \{0\}$.

Suppose now that $\mathcal{A} \neq \emptyset$ and $f, g \in c_0(\mu)$. It is clear that $\lambda f \in c_0(\mu)$ for each $\lambda \in \mathbb{R}$. For each infinitely decomposable set A and each $\epsilon > 0$ there exist infinitely decomposable sets $A_1 \subseteq A$ and $A_2 \subseteq A_1$ with $\|f\chi_{A_1}\|_\infty \leqslant \epsilon/2$ and $\|g\chi_{A_2}\|_\infty \leqslant \epsilon/2$. Therefore $\|(f + g)\chi_{A_2}\|_\infty \leqslant \|f\chi_{A_2}\|_\infty + \|g\chi_{A_2}\|_\infty \leqslant \epsilon$. Also it is easy to show that $\|(f \vee g)\chi_{A_2}\|_\infty \leqslant \epsilon$. Therefore $f + g$, $f \vee g \in c_0(\mu)$ and $c_0(\mu)$ is a sublattice of $L_\infty(\mu)$.

For each $\alpha \in c_0(\mathcal{A})$ denote $\Psi(\alpha)$ by f_α. If we suppose that $f_\alpha \notin c_0(\mu)$ then by Lemma 8, there exists an infinitely decomposable subset D of Ω and a real number $\epsilon > 0$ such that $|f_\alpha(t)| > \epsilon$ for each $t \in D$. Since the set $\{A \in \operatorname{supp}(\alpha) \mid |\alpha_A| > \epsilon\}$ is finite, we have that the set D is contained in the union of a finite number of atoms. Therefore D is not infinitely decomposable, a contradiction. Hence $f_\alpha \in c_0(\mu)$ and it is easy to show that $\|\alpha\|_0 = \|f_\alpha\|_\infty$. To prove that the map Ψ is onto, for each $f \in c_0(\mu)$ define the vector $\alpha^f = (\alpha^f_A)_{A \in \mathcal{A}}$ with

$$\alpha^f_A = \|f\chi_A\|_\infty \quad \text{if } f(t) \geqslant 0 \text{ for all } t \in A$$

and

$$\alpha^f_A = -\|f\chi_A\|_\infty \quad \text{if } f(t) < 0 \text{ for all } t \in A.$$

We shall show that $\alpha^f \in c_0(\mathcal{A})$. For each real number $\epsilon > 0$ we put $\mathcal{A}^\epsilon = \{A \in \mathcal{A} \mid |\alpha^f_A| > \epsilon\}$. This set is finite because if we suppose that $\mathcal{A}^\epsilon$ is infinite we take a countable union B of elements of $\mathcal{A}^\epsilon$ and we have that $|f(t)| > \epsilon$ for all $t \in B$, a contradiction because we have assumed that $f \in c_0(\mu)$ and the set B is infinitely decomposable. Hence $\alpha^f \in c_0(\mathcal{A})$ and it is easy to show that $\Psi(\alpha^f) = f$, therefore the map is onto. Also it is easy to show that Ψ is linear and that Ψ, Ψ^{-1} are positive. Therefore Ψ is a lattice isometry. □

Remark 11. Suppose that the measure space (Ω, Σ, μ) is σ-finite. Then we have the following:

The set $\mathcal{A}$ of atoms of μ is at most countable, and the set Ω is decomposed in the sets Ω_1, Ω_2, where $\Omega_1 = \bigcup_{A \in \mathcal{A}} A$, $\Omega_2 = \Omega \setminus \Omega_1$. Suppose that $\mu_1 = \mu_{\Omega_1}$ and $\mu_2 = \mu_{\Omega_2}$. Then $L_1(\mu) = L_1(\mu_1) \oplus L_1(\mu_2)$ and $L_\infty(\mu) = L_\infty(\mu_1) \oplus L_\infty(\mu_2)$, where $L_1(\mu_1)$ and $L_\infty(\mu_1)$ are order isometric to $\ell_1(\mathcal{A})$ and $\ell_\infty(\mathcal{A})$, respectively. Any element f of $c_0^+(\mu)$ is decomposed in the elements f_1, f_2 with $f_1 \in L^+_\infty(\mu_1)$ and $f_2 \in L^+_\infty(\mu_2)$. Then

$$0 \leqslant f_1, f_2 \leqslant f,$$

and by the definition of $c_0(\mu)$ it follows that $f_1, f_2 \in c_0(\mu)$. Since Ω_2 does not contain atoms, by Lemma 9, we have that $f_2 = 0$, therefore $f = f_1$. This implies that $c_0(\mu) \subseteq L_\infty(\mu_1)$ and by Proposition 10, $c_0(\mu)$ is lattice isometric to $c_0(\mathcal{A})$. Therefore

$$L_1(\mu) = \ell_1(\mathcal{A}) \oplus L_1(\mu_2), \qquad L_\infty(\mu) = \ell_\infty(\mathcal{A}) \oplus L_\infty(\mu_2),$$

and

$$c_0(\mu) = c_0(\mathcal{A}) \subseteq \ell_\infty(\mathcal{A}).$$

Note also that if the family $\mathcal{A}$ is infinite, then $\ell_\infty(\mathcal{A}) = \ell_\infty$ and $\ell_1(\mathcal{A}) = \ell_1$.

In the beginning of this section we have assumed that (Ω, Σ, μ) is a measure space, the measure μ takes values in the interval $[0, +\infty]$, $L_1(\mu)$ is infinite dimensional with $L_1'(\mu) = L_\infty(\mu)$ and that $\mu(A) < \infty$ for any atom A of μ. Recall that $\mathcal{A}$ is the set of atoms (equivalence classes) of μ and that for any linear operator $T : L_1(\mu) \to E'$ denote by T' the adjoint of T. Under these assumptions and in according to the notations of Definition 4, we state the following theorem.

Theorem 12. *Let T be an one-to-one bounded linear operator of $L_1(\mu)$ into the norm dual E' of a Banach space E. If*

(a) *$Q(\Omega)$ is a cone*; *and*
(b) *For each infinitely decomposable subset A of Ω there are a measurable subset D of A and an element y of E such that y defines an unbounded base for the cone $Q(D)$,*

then

(i) *$T'(\hat{E}) \subseteq c_0(\mu)$ with $T'(\widehat{P(\Omega)}) \subseteq c_0^+(\mu)$;*
(ii) *The measure μ is atomic. In particular the space $L_1(\mu)$ is lattice isometric to $\ell_1(\mathcal{A})$.*

Proof. Since $Q(\Omega)$ is a cone, $P(\Omega) - P(\Omega)$ is norm dense in E, by statement (iii) of Proposition 5. So to prove that $T'(\hat{E}) \subseteq c_0(\mu)$ it suffices to show that $T'(\widehat{P(\Omega)}) \subseteq c_0^+(\mu)$. We begin with an observation.

For each $x'' \in E''$ with $T'x'' = r \in L_\infty(\mu)$, for each $\xi \in L_1(\mu)$ and each B measurable we have

$$\int_B \xi(t) r(t)\, d\mu = r(\xi \chi_B) = T'x''(\xi \chi_B) = x''(T\xi \chi_B).$$

Suppose that $x_0 \in P(\Omega)$ and that $T'(\hat{x}_0) = r$. Then for each $\xi \in L_1^+(\mu)$ we have

$$0 \leqslant (T\xi)(x_0) = \hat{x}_0(T\xi) = \int_\Omega \xi(t) r(t)\, d\mu$$

and it follows that $r \in L_\infty^+(\mu)$. Now suppose that $r = T'(\hat{x}_0) \notin c_0(\mu)$. We will use the element x_0 and assumption (b) to produce a weak-star closed, weak-star dentable, norm unbounded base K for a weak-star closed cone R in a dual space, which contradicts Theorem 2. This will show that the assumption $T'(\hat{x}_0) \notin c_0(\mu)$ is impossible. Here are the details.

By Lemma 8, there are an infinitely decomposable subset A of Ω and a real number $\rho > 0$ so that $r(t) > \rho$ for each $t \in A$. We may also suppose that $\rho < \|T\|$. So for each $\xi \in L_1^+(\mu)$ we have

$$\hat{x}_0(T\xi\chi_A) = \int_A \xi(t)r(t)\,d\mu > \rho\int_A \xi(t)\,d\mu = \rho\|\xi\chi_A\|_1 \geqslant \frac{\rho}{\|T\|}\big\|T(\xi\chi_A)\big\|. \tag{2}$$

Next we shall show that

$$\hat{x}_0(x') \geqslant \frac{\rho}{2\|T\|}\|x'\| \tag{3}$$

for each $x' \in Q(A)$. To this end suppose that $x' \in Q(A)$, $x' \neq 0$ and that x' is the weak-star limit of the net $T(\xi_\alpha\chi_A)$, where $\xi_\alpha \in L_1^+(\mu)$ for any α. If we suppose that $\hat{x}_0(x') = 0$, then

$$0 = \lim \hat{x}_0(T\xi_\alpha\chi_A) \geqslant \lim \frac{\rho}{\|T\|}\big\|T(\xi_\alpha\chi_A)\big\|$$

by (2) and the net $T(\xi_\alpha\chi_A)$ norm converges to zero. Therefore $x' = 0$, a contradiction. Hence $\hat{x}_0(x') > 0$ and using (2) again we may find a subnet of $T(\xi_\alpha\chi_A)$ (which we do not rename) with

$$2\hat{x}_0(x') > \frac{\rho}{\|T\|}\big\|T(\xi_\alpha\chi_A)\big\|$$

for each α. Since closed bounded balls in E' are weak-star compact and the net $T(\xi_\alpha\chi_A)$ weak-star converges to x' we have that

$$2\hat{x}_0(x') \geqslant \frac{\rho}{\|T\|}\|x'\|,$$

as desired.

By assumption (b) there is a measurable subset D of A and an element $y \in E$ such that y defines an unbounded base for the cone $Q(D)$, i.e., $\hat{y}$ is strictly positive on $Q(D)$ and the set

$$C = \big\{x' \in Q(D) \mid \hat{y}(x') = 1\big\}$$

is unbounded. Now consider the cone

$$R = \mathbb{R}_+ \oplus Q(D) = \big\{(\lambda, x') \mid \lambda \geqslant 0,\ x' \in Q(D)\big\}$$

in $(\mathbb{R} \oplus E)'$ and let

$$K = \big\{(\lambda, x') \in R \mid \lambda + \hat{y}(x') = 1\big\}.$$

Then K is a base for the cone R which is defined by the element $(1, y)$ of $\mathbb{R} \oplus E$. Observe that $(1, 0) \in K$ and that $(0, x') \in K$ for each $x' \in C$. It is clear that the base K is unbounded because C is unbounded. Note that since $Q(D) \subseteq Q(A)$ the linear functional $\hat{x}_0$ satisfies (3) for all $x' \in Q(D)$. So for each $\lambda \in \mathbb{R}_+$ and $x' \in Q(D)$ we have

$$(1, \hat{x}_0)(\lambda, x') = \lambda + \hat{x}_0(x') \geqslant \lambda + \frac{\rho}{2\|T\|}\|x'\| \geqslant \frac{\rho}{2\|T\|}\big(\lambda + \|x'\|\big). \tag{4}$$

Therefore the functional $h = (1, \hat{x}_0)$ is uniformly monotonic on the cone R. Now observe that the point $(1, 0)$ is an extreme point of K with continuous positive projection

$$\Pi(\lambda, x') = \pi(\lambda, x')(1, 0) = \lambda(1, 0),$$

where $\pi(\lambda, x') = \lambda$ for each $(\lambda, x') \in (\mathbb{R} \oplus E)'$. Therefore by Theorem 1, the functional

$$g = h(1, 0)\pi - h = -(0, \hat{x}_0)$$

strongly exposes the point $(1, 0)$ in K. Since $(0, x_0) \in \mathbb{R} \oplus E$ we have that $(1, 0)$ is a weak-star strongly exposed point of K. So the base K is weak-star dentable. On the other hand, the cone R is weak-star closed because $Q(D)$ is weak-star closed. Also the base K is weak-star closed because it is defined by the element $(1, y)$ of $\mathbb{R} \oplus E$. But by Theorem 2, the base K of R is not weak-star dentable and we have arrived at a contradiction. So we have that $T'(\hat{x}_0) = r \in c_0^+(\mu)$ and the proof of (i) is complete.

(ii) Suppose that a measurable subset A contains no atoms and that μ_A is the restriction of μ on A. Then $L_1(\mu_A)$ is lattice isometric to the sublattice $F_A = \{\xi \chi_A \mid \xi \in L_1^+(\mu)\}$ of $L_1(\mu)$ and we identify $L_1(\mu_A)$ with F_A. Also $L_1(\mu_A)$ is infinite dimensional because A is infinitely decomposable. Suppose that T_A is the restriction of T on F_A. Then T_A is an one-to-one, continuous linear operator of $L_1(\mu_A)$ into E' and the assumptions of the theorem are satisfied for T_A. By Proposition 10, $c_0(\mu_A) = \{0\}$ because the measure μ_A is nonatomic and by part (i) of this theorem $T_A'(\hat{E}) \subseteq c_0(\mu_A) = \{0\}$. This implies that $T_A = 0$ and (since T_A is one-to-one) that $L_1(\mu_A) = \{0\}$, a contradiction. Therefore each measurable subset A has at least one atom, therefore the measure μ is atomic and $L_1(\mu)$ is lattice isometric to $\ell_1(\mathcal{A})$, as asserted. □

Corollary 13. *Let T be an isomorphism of $L_1(\mu)$ into E'. Suppose that*

(a) *$Q(\Omega)$ is a cone*; *and*
(b) *For each infinitely decomposable subset A of Ω there are a measurable subset D of A and an element y of E such that y defines an unbounded base for the cone $Q(D)$.*

If U is the lattice isometry of $\ell_1(\mathcal{A})$ onto $L_1(\mu)$ defined in (1) *and $F = ToU$, then*

(i) *$F'(\hat{E}) = c_0(\mathcal{A})$. The quotient space E/M, where M is the annihilator of $T(L_1(\mu))$ in E, is isomorphic to $c_0(\mathcal{A})$.*
(ii) *If moreover $T(L_1(\mu))$ is weak-star dense in E', then $F'|_{\hat{E}}$ is an isomorphism of $\hat{E}$ onto $c_0(\mathcal{A})$.*

Proof. By the theorem, the measure μ is atomic, therefore a lattice isometry U of $\ell_1(\mathcal{A})$ onto $L_1(\mu)$ defined by (1), exists. Consider the measure space $(\mathcal{A}, 2^{\mathcal{A}}, m)$, where m is the counting measure on $\mathcal{A}$. Then the operator $F : \ell_1(\mathcal{A}) \to E'$ is an into isomorphism which satisfies the assumptions of the previous theorem. Indeed $F(\ell_1^+(\mathcal{A})) = T(L_1^+(\mu))$, therefore the assumption (a) holds. Also for any infinite subset $\mathcal{A}'$ of $\mathcal{A}$, we consider a countable subset $\mathcal{B}$ of $\mathcal{A}'$ and we take the union B of the elements of $\mathcal{B}$. Then B is an infinitely decomposable subset of Ω, therefore there are an infinitely decomposable subset D of B and an element y of E such that y defines an unbounded base for the cone $Q(D)$.

Then the subset $\mathcal{D}$ of $\mathcal{B}$ with elements the atoms of D is infinitely decomposable and it is easy to show that y defines an unbounded base for the weak-star closure of the cone $\{F(\xi\chi_{\mathcal{D}}) \mid \xi \in \ell_1^+(\mathcal{A})\}$, hence the assumption (b) is also satisfied.

The adjoint $F' : E'' \to \ell_\infty(\mathcal{A})$ of F is continuous onto and by the theorem, $F'(\hat{E}) \subseteq c_0(\mathcal{A})$.

Suppose that $S = F'|_{\hat{E}}$ is the restriction of F' on $\hat{E}$. Then we may suppose that $S : E \to c_0(\mathcal{A})$ and it is easy to show that F is the adjoint of S, i.e., $S' = F$. By our assertion that F is an isomorphism of $\ell_1(\mathcal{A})$ into E' it follows that S is onto (see, for example, [10, Theorem 3.1.22, p. 293]). Therefore $F'(\hat{E}) = c_0(\mathcal{A})$. Hence the operator F' defines an isomorphism of E/M onto $c_0(\mathcal{A})$. If we suppose moreover that $T(L_1(\mu))$ is weak-star dense in E', then $M = \{0\}$, therefore F' is an isomorphism of $\hat{E}$ onto $c_0(\mathcal{A})$. □

Corollary 14. *For any infinite-dimensional Banach space E the following statements are equivalent*:

(i) *E is isomorphic to $c_0(\Gamma)$*;
(ii) *There exists an isomorphism T of $\ell_1(\Gamma)$ into the dual space E' of E with weak-star dense range in E' such that*: *The weak-star closure of $T(\ell_1^+(\Gamma))$ in E' is a cone and for each infinite subset A of Γ, there are a countable subset D of A and an element y of E such that y defines an unbounded base for the weak-star closure of the cone $\{T(\xi\chi_D) \mid \xi \in \ell_1^+(\Gamma)\}$.*

Proof. Suppose that L is an isomorphism of E onto $c_0(\Gamma)$ and that T is the adjoint of L. Then T is an isomorphism of $\ell_1(\Gamma)$ onto E' and T is also weak-star to weak-star continuous. Therefore $T(\ell_1^+(\Gamma))$ is a weak-star closed cone, hence the weak-star closure of $T(\ell_1^+(\Gamma))$ is a cone. As we have shown in (ii) of Example 6, for each infinite subset A of Γ, a countable subset D of A and an element $\eta \in c_0(\Gamma)$ exist such that η defines an unbounded base for the cone $C_D = \{\xi\chi_D \mid \xi \in \ell_1^+(\Gamma)\}$, i.e., the set

$$B = \left\{\xi\chi_D \mid \xi \in \ell_1^+(\Gamma) \text{with} (\xi\chi_D)(\eta) = 1\right\}$$

is unbounded. Since the cone C_D is weak-star closed its image $K(D) = \{T(\xi\chi_D) \mid \xi \in \ell_1^+(\Gamma)\}$ is also a weak-star closed cone and it is easy to show that $y = L^{-1}(\eta)$, defines the unbounded base

$$T(B) = \left\{x' \in K(D) \mid x'(y) = 1\right\}$$

for the cone $K(D)$.

For the converse, suppose that statement (ii) is true. Then each infinitely decomposable subset A of Γ is infinite, therefore by our assumptions a countable subset D of A and an element $y \in E$ exist such that y defines an unbounded base for cone $Q(D)$. Hence T' is an isomorphism of E onto $c_0(\Gamma)$, therefore the converse is also true. □

Acknowledgment

I would like to thank very much an anonymous referee whose very useful mathematical remarks, comments and suggestions improved significantly this article and made it more accessible to the reader.

References

[1] C.D. Aliprantis, O. Burkinshaw, Positive Operators, Academic Press, New York, 1985.
[2] J. Bourgain, F. Delbaen, A class of $\mathcal{L}_\infty$ spaces, Acta Math. 145 (1981) 155–176.
[3] J. Diestel, J.J. Uhl, Vector Measures, in: American Mathematical Society Surveys, 1977.
[4] J. Dieudonné, Sour the spaces l_1, Arch. Math. 10 (1959) 151–152.
[5] V. Fonf, One property of Lindenstrauss–Phelps spaces, Funct. Anal. Appl. 83 (1979) 66–67.
[6] I.M. Gelfand, Abstracte functionen und lineare operatoren, Mat. Sb. (N.S.) 4 (46) (1938) 235–286.
[7] G.J.O. Jameson, Ordered Linear Spaces, in: Lecture Notes in Mathematics, Vol. 142, Springer-Verlag, Heidelberg, 1970.
[8] H.E. Lacey, The Isometric Theory of Classical Banach Spaces, Springer-Verlag, New York, 1974.
[9] D.R. Lewis, C. Stegall, Banach spaces whose duals are isomorphic to $\ell_1(\Gamma)$, J. Funct. Anal. 16 (1979) 177–187.
[10] R.E. Megginson, An Introduction to Banach Space Theory, Springer-Verlag, Heidelberg, 1998.
[11] A. Pelczynski, On the impossibility of embedding of the space L in certain Banach spaces, Coll. Math. 8 (1961) 199–203.
[12] A. Pelczynski, On Banach spaces containing $L_1(\mu)$, Studia Math. 37 (1968) 231–246.
[13] I.A. Polyrakis, Strongly exposed points in bases for the positive cone of ordered Banach spaces and characterization of $\ell_1(\Gamma)$, Proc. Edinburgh Math. Soc. 29 (1986) 271–282.
[14] I.A. Polyrakis, Spaces with dual order-isomorphic to ℓ_1, Positivity 4 (2000) 197–201.
[15] I.A. Polyrakis, Bases for cones and reflexivity in Banach spaces, Quaestiones Math. 24 (2001) 165–173.
[16] C. Stegall, Banach spaces whose duals contain $\ell_1(\Gamma)$ with applications to the study of dual $L_1(\mu)$ spaces, Trans. Amer. Math. Soc. 176 (1973) 463–477.