Some estimates for the weakly convergent sequence coefficient in Banach spaces

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1. Introduction

Let X be a Banach space without the Schur property, that is, there is a weakly convergent sequence which is not norm convergent. The asymptotic diameter and asymptotic radius of a sequence \{x_n\} in a Banach space X are defined by

\[ \text{diam}_a(\{x_n\}) = \lim_{k \to \infty} \sup \{ \|x_n - x_m\| : n, m \geq k \}, \]

\[ r_a(\{x_n\}) = \inf \left\{ \limsup_{n \to \infty} \|x_n - y\| : y \in \text{conv}(\{x_n\}) \right\}. \]

The weakly convergent sequence coefficient [2] of X is defined by

\[ \text{WCS}(X) = \inf \left\{ \frac{\text{diam}_a(\{x_n\})}{r_a(\{x_n\})} \right\} \]

where the infimum is taken over all weakly convergent sequences \{x_n\} which are not norm convergent. It is clear that

\[ 1 \leq \text{WCS}(X) \leq 2. \]

The definition of WCS(X) above does not make sense if the space X has the Schur property but in that case we may say by convention that WCS(X) = 2. In this paper, we utilize the following equivalent formulation (see also [1, Lemma VI.3.8])

\[ \text{WCS}(X) = \inf \left\{ \lim_{n,m,n \neq m} \|x_n - x_m\| \right\} \]

where the infimum is taken over all weakly null sequences \{x_n\} \subset X with \|x_n\| = 1 for all n and \lim_{n,m,n \neq m} \|x_n - x_m\| exists.

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0022-247X/− see front matter © 2008 Elsevier Inc. All rights reserved.
In this paper, let \( B_X, S_X, X^* \) and \( \hat{X} \) be the closed unit ball, the unit sphere, the dual space, and the ultrapower (over a free ultrafilter on the set of natural numbers \( \mathbb{N} \)) of a Banach space \( X \), respectively. For more details on the ultrapower construction, the reader is directed to [15].

2. Domínguez-Benavides’ coefficient

Domínguez-Benavides [5] defined the coefficient, for \( a \geq 0 \),

\[
R(a, X) = \sup \left\{ \liminf_{n \to \infty} \| x_n + x \| \right\}
\]

where the supremum is taken over all \( x \in X \) with \( \| x \| \leq a \) and all weakly null sequences \( \{x_n\} \) in \( B_X \) such that \( \lim_{n \to m} \| x_n - x_m \| \leq 1 \). We note that \( R(0, X) = 1/WCS(X) \) if \( X \) fails the Schur property. Moreover, the coefficient remains unaltered if in the definition we replace \( \liminf \) by \( \limsup \).

**Theorem 1.** Suppose that a Banach space \( X \) fails the Schur property and \( d = WCS(X) > 1 \). Then, for any \( a \geq 0 \),

\[
d - (d - 1)R \left( \frac{|a - 1|}{d - 1}, X \right) \leq R(a, X) \leq a + \frac{1}{d}. \tag{1}
\]

**Proof.** We prove the first inequality. For \( \varepsilon > 0 \), we choose a weakly null sequence \( \{x_n\} \subset S_X \) such that \( \lim_{n \to m} \| x_n - x_m \| \) exists and

\[
d \leq \lim_{n \not= m} \| x_n - x_m \| \leq d + \varepsilon.
\]

It is easy to see that

\[
\liminf_{n \to \infty} \left\| \frac{x_n}{d + \varepsilon} - ax_m \right\| \leq R(a, X)
\]

for all \( m \in \mathbb{N} \) and \( a \geq 0 \). In particular,

\[
\liminf_{n \to \infty} \| x_n - x_m \| \leq \liminf_{n \to \infty} \left( \left\| \frac{x_n}{d + \varepsilon} - ax_m \right\| + (d + \varepsilon - 1) \left\| \frac{x_n}{d + \varepsilon} - \frac{1 - a}{d + \varepsilon - 1} x_m \right\| \right)
\]

\[
\leq R(a, X) + (d + \varepsilon - 1)R \left( \frac{|a - 1|}{d + \varepsilon - 1}, X \right).
\]

Letting \( m \to \infty \) gives

\[
d \leq R(a, X) + (d + \varepsilon - 1)R \left( \frac{|a - 1|}{d + \varepsilon - 1}, X \right).
\]

By the arbitrariness of \( \varepsilon \), the first inequality is proved.

To prove the latter one, let \( \eta > 0 \) and \( a \geq 0 \). We choose a weakly null sequence \( \{y_n\} \subset B_X \) and \( \| y \| = a \) such that \( \lim_{n \not= m} \| y_n - y_m \| \leq 1 \) and

\[
\liminf_{n \to \infty} \| y_n + y_m \| \geq R(a, X) - \eta.
\]

Then, by the triangle inequality,

\[
\liminf_{n \to \infty} \| y_n + y_m \| \leq \liminf_{n \to \infty} \| y_n \| + \| y_m \| \leq R(0, X) + a = \frac{1}{d} + a.
\]

The proof is finished. \( \square \)

**Corollary 2.** Under the same assumptions done for Theorem 1, the following are true:

(i) \( d - (d - 1)R(0, X) = d - 1 + \frac{1}{d} \leq R(1, X); \)

(ii) \( 1 \leq R \left( \frac{d}{d - 1}, X \right); \)

(iii) \( \frac{d + 1}{d} \leq R \left( \frac{1}{d - 1}, X \right). \)

**Proof.** The assertions are obtained by letting \( a = 1, \frac{1}{d}, 0 \) in (1), respectively. \( \square \)

**Remark 3.** The estimates (i) and (ii) in Corollary 2 remain true also for \( d = 1 \).
3. The James and von Neumann–Jordan constants

The von Neumann–Jordan constant $C_{NJ}(X)$ was defined in 1937 by Clarkson [3] as

$$C_{NJ}(X) = \sup \left\{ \frac{\|x + y\|^2 + \|x - y\|^2}{2\|x\|^2 + \|y\|^2} : x, y \in X \text{ and } \|x\| + \|y\| \neq 0 \right\},$$

and the James constant $J(X)$ was defined by Gao and Lau [11] as

$$J(X) = \sup \left\{ \min \{\|x + y\|, \|x - y\| : x, y \in B_X\} \right\}.$$

It is noted that the James (and also von Neumann–Jordan) constants of a Banach space $X$ and of its Banach space ultrapower $\tilde{X}$ are the same because $X$ can be embedded into $\tilde{X}$ isometrically (see [15]). Moreover, $C_{NJ}(X) = C_{NJ}(X^*)$.

**Lemma 4.** Let $X$ be a super-reflexive Banach space. Suppose that $\text{WCS}(X) = d$ and $X$ does not have Schur property. Then, there exist $\tilde{x}_1, \tilde{x}_2 \in S_X$ and $\tilde{f}_1, \tilde{f}_2 \in S_{X^*}$, such that the following conditions are satisfied:

(a) $\|\tilde{x}_1 - \tilde{x}_2\| = d$ and $\tilde{f}_1(\tilde{x}_i) = 0$ for all $i \neq j$,
(b) $\tilde{f}_1(\tilde{x}_i) = 1$ for $i = 1, 2$,
(c) $\|\tilde{f}_2 - \tilde{x}_i\| \leq R(1, X)$.

**Proof.** For $\varepsilon > 0$, we choose a weakly null sequence $\{x_n\} \subset S_X$ such that $\lim_{n \neq m} \|x_n - x_m\|$ exists and

$$d \leq \lim_{n \neq m} \|x_n - x_m\| < d + \varepsilon.$$

It follows from the definition of Domínguez-Benavides’ coefficient that

$$\liminf_{n \to \infty} \frac{x_n}{d + \varepsilon} - x_n \leq R(1, X)$$

for all $n \in \mathbb{N}$. Passing to a suitable subsequence, we may assume that there exist a sequence $\{f_n\} \subset S_{X^*}$ and $f \in B_{X^*}$ such that

$$f_n(x_n) = \|x_n\| = 1 \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad f_n \overset{w^*}{\to} f.$$

The last convergence follows from the reflexivity of $X$. We first choose an integer $n_1$ so that

$$d - \varepsilon < \lim_{n \to \infty} \|x_n - x_{n_1}\| < d + \varepsilon \quad \text{and} \quad \|x_{n_1}\| < \frac{\varepsilon}{2}.$$

Next, we choose $n_2 > n_1$ so that

$$d - \varepsilon \leq \|x_{n_2} - x_{n_1}\| < d + \varepsilon,$$

$$\left| \frac{x_{n_2}}{d + \varepsilon} - x_{n_1} \right| < R(1, X) + \varepsilon,$$

$$|f_n(x_{n_2})| < \varepsilon, \quad |f(x_{n_2})| < \varepsilon, \quad \text{and} \quad |f_n(x_{n_2}) - f(x_{n_1})| < \varepsilon.$$

This implies that

$$|f_n(x_{n_1})| \leq |(f_n - f)(x_{n_1})| + |f(x_{n_1})| < 2\varepsilon.$$

Next, for $i = 1, 2$, we write

$$g_i = f_n \quad \text{and} \quad z_i = x_{n_i}.$$

Then $\tilde{x}_i = (\{z_i^{(1/n)}\}_{n=1}^\infty)$ and $\tilde{f}_i = (\{g_i^{(1/n)}\}_{n=1}^\infty)$ are our candidates in the Banach space ultrapowers $\tilde{X}$ and $\tilde{X}^* = (\tilde{X})^*$. The latter follows from the super-reflexivity of $X$. $\Box$

**Theorem 5.** Suppose that a Banach space $X$ does not have the Schur property and $\text{WCS}(X) = d$. Then

$$J(X) \geq \frac{1}{d} + \frac{1}{\min(2, \sqrt{4C_{NJ}(X) - d^2})}.$$
Theorem 7. Suppose that a Banach space $X$ fails the Schur property and $d = \text{WCS}(X)$. Then
\[
J(X) \geq \frac{1}{d} + \frac{1}{R(1, X) + (1 - \frac{1}{d})}.
\]

Proof. Let $\tilde{x}_1, \tilde{x}_2 \in S_X$ be elements satisfying the conditions in Lemma 4. It follows that
\[
\|\tilde{x}_1 + \tilde{x}_2\| \leq \frac{1}{d} + \frac{1}{R(1, X) + (1 - \frac{1}{d})}.
\]

Remark 6. If $X$ is a Hilbert space, then the preceding estimate becomes equality. In fact, $J(X) = \text{WCS}(X) = \sqrt{2}$ and $\text{C}_{NJ}(X) = 1$.

Theorem 7. Suppose that a Banach space $X$ fails the Schur property and $d = \text{WCS}(X)$. Then
\[
J(X) \geq \frac{1}{d} + \frac{1}{R(1, X) + (1 - \frac{1}{d})}.
\]

Proof. Let $\tilde{x}_1, \tilde{x}_2 \in S_X$ be elements satisfying the conditions in Lemma 4. It follows that
\[
\|\tilde{x}_2 + \tilde{x}_1\| \leq \frac{1}{d} + \frac{1}{R(1, X) + (1 - \frac{1}{d})}.
\]

Now
\[
J(X) \geq \min \left\{ \frac{\|\tilde{x}_2 - \tilde{x}_1\|}{d} \pm \frac{\|\tilde{x}_2 + \tilde{x}_1\|}{\alpha R(1, X) + (1 - \frac{1}{d})} \right\} \geq \frac{1}{d} + \frac{1}{R(1, X) + (1 - \frac{1}{d})}.
\]

Remark 8. The above estimate becomes equality if $X = l_{2,\infty}$. In fact, $J(X) = 1 + 1/\sqrt{2}$ [13, Theorem 4], $R(1, X) = \sqrt{2}$ [5, Theorem 4.1], and $\text{WCS}(X) = 1$.

Remark 9. Recently Mazcuñán-Navarro proved that [14, Theorem 23]
\[
J(X) \geq \frac{1}{d} \left( 1 + \frac{1}{R(1, X)} \right).
\]

It is easy to see that
\[
\frac{1}{d} + \frac{1}{R(1, X) + (1 - \frac{1}{d})} \geq \frac{1}{d} \left( 1 + \frac{1}{R(1, X)} \right).
\]

The inequality is strict for the case $d > 1$.

Remark 10. For $X = l_{p,q}$, the right value of $J(X)$ is unknown. If $p \leq q$, then $R(1, X) = (1 + (1/2)^{p/q})^{1/p}$ [7, Theorem 3], and $\text{WCS}(X) = 2^{1/q}$ [6, Corollary 3], so we have
\[
J(l_{p,q}) \geq \frac{1}{2^{1/q}} + \frac{1}{(1 + (1/2)^{p/q})^{1/p} + 1 - 2^{-1/q}}.
\]

Corollary 11. Suppose that a Banach space $X$ fails the Schur property. Then
\[
\text{WCS}(X) \geq \frac{2}{2J(X) + 1 - \sqrt{2}}.
\]

In particular, $J(X) < \frac{1 + \sqrt{2}}{2}$ implies $\text{WCS}(X) > 1$.

Proof. It was proved in [10] that
\[
R(1, X) \leq J(X).
\]
Consequently,
\[ J(X) \geq \frac{1}{d} + \frac{1}{R(1, X) + (1 - \frac{1}{d})} \geq \frac{1}{d} + \frac{1}{J(X) + (1 - \frac{1}{d})}, \]
or equivalently \( J(X) - \frac{1}{d} \geq \frac{1 + \sqrt{r}}{2r}. \)

**Remark 12.** The preceding corollary improves [4, Theorem 3.2]. More precisely, it is clear that
\[ \frac{2}{2} \frac{J(X) + 1}{J(X)^2} > \frac{J(X) + 1}{(J(X))^2} \]
provided that \( J(X) < \frac{1 + \sqrt{r}}{2}. \)

**4. The coefficient of weak orthogonality**

Let us mention another interesting coefficient introduced by Sims [16]. As in [12], we prefer to use its inverse, \( \mu(X) \), which is defined as the infimum of the set of real numbers \( r > 0 \) such that
\[ \limsup_{n \to \infty} \|x + x_n\| < r \limsup_{n \to \infty} \|x - x_n\| \]
for all \( x \in X \) and for all weakly null sequences \( \{x_n\} \) in \( X \). The proof of the following lemma is almost the same as that of Lemma 4, so it is omitted.

**Lemma 13.** Let \( X \) be a super-reflexive Banach space. Suppose that \( \text{WCS}(X) = d, \mu(X) = \mu, \) and \( X \) does not have Schur property. Then, there exist \( \tilde{x}_1, \tilde{x}_2 \in \tilde{X} \) and \( \tilde{f}_1, \tilde{f}_2 \in S(\tilde{X}), \) such that the following conditions are satisfied:

(a) \( \|\tilde{x}_1 - \tilde{x}_2\| = d, \|\tilde{x}_1 + \tilde{x}_2\| \leq \mu d, \) and \( \tilde{f}_i(\tilde{x}_j) = 0 \) for all \( i \neq j, \)
(b) \( \tilde{f}_i(\tilde{x}_i) = 1 \) for \( i = 1, 2. \)

We now consider the parameterized James constant \( J(t, X) \), where \( t \geq 0 \), which is defined by
\[ J(t, X) = \sup \{ \min \{ \|x + ty\|, \|x - ty\| \} : x, y \in B_X \}. \]
The following theorem unifies the recent results of Mazcuñán-Navarro.

**Theorem 14.** Suppose that a Banach space \( X \) fails the Schur property and \( d = \text{WCS}(X), \mu = \mu(X). \) Then
\[ J(t, X) \geq \frac{1}{d} \left( 1 + \frac{t}{\mu} \right) \]
for all \( 0 \leq t \leq 1. \)

**Proof.** As before, we have the following estimate
\[ J(t, X) \geq \min \left\{ \frac{1}{d} \left( \|\tilde{x}_1 - \tilde{x}_2\| + \frac{t}{\mu d} (\tilde{x}_1 + \tilde{x}_2) \right) \right\} \geq \frac{1}{d} + \frac{t}{\mu d}. \]

**Remark 15.** It is not hard to see that \( J(t, l_{2, \infty}) = 1 + t/\sqrt{2} \) for all \( 0 \leq t \leq 1 \) and \( \mu(l_{2, \infty}) = \sqrt{2} \) (see [13]). Hence there is a Banach space such that the estimate above becomes equality for all \( 0 \leq t \leq 1. \)

**Corollary 16.** Suppose that a Banach space \( X \) fails the Schur property and \( d = \text{WCS}(X), \mu = \mu(X). \) Then

(i) (see [14, Theorem 31]) \( J(X) \geq \frac{1}{d} (1 + \frac{1}{p}); \)
(ii) (see [14, Proposition 30]) \( C_{\text{NJ}}(X) \geq \frac{1}{d} (1 + \frac{1}{p^2}); \)
(iii) (see [14, Theorem 27]) \( 1 + \rho_X(t) \geq \frac{1}{d} (1 + \frac{1}{p}) \) for all \( t \geq 0. \)

Recall that \( \rho_X \) denotes the modulus of smoothness of \( X \) defined by
\[ \rho_X(t) = \sup \left\{ \frac{\|x + ty\| + \|x - ty\|}{2} - 1 : x, y \in B_X \right\} \]
for \( t \geq 0. \)
Theorem 17. Suppose that a Banach space $X$ fails the Schur property and $d = \text{WCS}(X)$. Then

1. $J(X) \geq \frac{1}{d} + \frac{1}{\rho(X)}$;
2. $C_{NJ}(X) \geq \frac{1}{d^2} + \frac{1}{(\rho(X))^2}$.

Proof. Using García-Falset’s coefficient instead of Domínguez-Benavides’ coefficient in Lemma 4, we have $\|\tilde{x}_2 + \tilde{x}_1\| \leq R(X)$. This implies

$$J(X) \geq \min \frac{\|\tilde{x}_2 - \tilde{x}_1\|}{d} \geq \frac{1}{d} + \frac{1}{R(X)}$$

and

$$C_{NJ}(X) = C_{NJ}(X^*) \geq \frac{1}{4} (\|\tilde{f}_2 - \tilde{f}_1\|^2 + \|\tilde{f}_2 + \tilde{f}_1\|^2) \geq \frac{1}{d^2} + \frac{1}{(\rho(X))^2}.$$  

Remark 18. Both estimates above become equality when $X = \ell_{2,\infty}$ and $X = \ell_p$ where $1 < p \leq 2$ (see [8,13]).

Remark 19. The result above is better than the result involving the coefficient of weak orthogonality of Sims. In fact, our estimate still makes sense when $\mu(X)d > 2$. ((a) of Lemma 13 becomes trivial in this case.)

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