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Some estimates for the weakly convergent sequence coefficient in Banach spaces

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ABSTRACT

We present relations between the weakly convergent sequence coefficient of a Banach space and other coefficients. Some estimates are sharp.

1. Introduction

Let X be a Banach space without the Schur property, that is, there is a weakly convergent sequence which is not norm convergent. The *asymptotic diameter* and *asymptotic radius* of a sequence $\{x_n\}$ in a Banach space X are defined by

$$\operatorname{diam}_a(\{x_n\}) = \lim_{k\to\infty} \sup\{\|x_n - x_m\|:\ n, m \geqslant k\},$$

$$r_a(\{x_n\}) = \inf\Big\{\limsup_{n\to\infty} \|x_n - y\|:\ y \in \overline{\operatorname{conv}}(\{x_n\})\Big\}.$$

The *weakly convergent sequence coefficient* [2] of X is defined by

$$\operatorname{WCS}(X) = \inf\left\{\frac{\operatorname{diam}_a(\{x_n\})}{r_a(\{x_n\})}\right\}$$

where the infimum is taken over all weakly convergent sequences $\{x_n\}$ which are not norm convergent. It is clear that $1 \leqslant \operatorname{WCS}(X) \leqslant 2$. The definition of $\operatorname{WCS}(X)$ above does not make sense if the space X has the Schur property but in that case we may say by convention that $\operatorname{WCS}(X) = 2$. In this paper, we utilize the following equivalent formulation (see also [1, Lemma VI.3.8])

$$\operatorname{WCS}(X) = \inf\Big\{\lim_{n,m;\, n\neq m} \|x_n - x_m\|\Big\}$$

where the infimum is taken over all weakly null sequences $\{x_n\} \subset X$ with $\|x_n\| = 1$ for all n and $\lim_{n,m;\, n\neq m} \|x_n - x_m\|$ exists.

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In this paper, let B_X, S_X, X^* and $\widetilde{X}$ be the closed unit ball, the unit sphere, the dual space, and the ultrapower (over a free ultrafilter on the set of natural numbers $\mathbb{N}$) of a Banach space X, respectively. For more details on the ultrapower construction, the reader is directed to [15].

2. Domínguez-Benavides' coefficient

Domínguez-Benavides [5] defined the coefficient, for $a \geqslant 0$,

$$R(a, X) = \sup\Big\{\liminf_{n\to\infty} \|x_n + x\|\Big\}$$

where the supremum is taken over all $x \in X$ with $\|x\| \leqslant a$ and all weakly null sequences $\{x_n\}$ in B_X such that $\lim_{n\neq m} \|x_n - x_m\| \leqslant 1$. We note that $R(0, X) = 1/\mathrm{WCS}(X)$ if X fails the Schur property. Moreover, the coefficient remains unaltered if in the definition we replace lim inf by lim sup.

Theorem 1. *Suppose that a Banach space X fails the Schur property and $d = \mathrm{WCS}(X) > 1$. Then, for any $a \geqslant 0$,*

$$d - (d-1)R\left(\frac{|a-1|}{d-1}, X\right) \leqslant R(a, X) \leqslant a + \frac{1}{d}. \tag{1}$$

Proof. We prove the first inequality. For $\varepsilon > 0$, we choose a weakly null sequence $\{x_n\} \subset S_X$ such that $\lim_{n\neq m} \|x_n - x_m\|$ exists and

$$d \leqslant \lim_{n\neq m} \|x_n - x_m\| \leqslant d + \varepsilon.$$

It is easy to see that

$$\liminf_{n\to\infty}\left\|\frac{x_n}{d+\varepsilon} - ax_m\right\| \leqslant R(a, X)$$

for all $m \in \mathbb{N}$ and $a \geqslant 0$. In particular,

$$\begin{aligned}\liminf_{n\to\infty} \|x_n - x_m\| &\leqslant \liminf_{n\to\infty}\left(\left\|\frac{x_n}{d+\varepsilon} - ax_m\right\| + (d+\varepsilon-1)\left\|\frac{x_n}{d+\varepsilon} - \frac{1-a}{d+\varepsilon-1}x_m\right\|\right)\\ &\leqslant R(a, X) + (d+\varepsilon-1)R\left(\frac{|a-1|}{d+\varepsilon-1}, X\right).\end{aligned}$$

Letting $m \to \infty$ gives

$$d \leqslant R(a, X) + (d+\varepsilon-1)R\left(\frac{|a-1|}{d+\varepsilon-1}, X\right).$$

By the arbitrariness of ε, the first inequality is proved.

To prove the latter one, let $\eta > 0$ and $a \geqslant 0$. We choose a weakly null sequence $\{y_n\} \subset B_X$ and $\|y\| = a$ such that $\lim_{n\neq m} \|y_n - y_m\| \leqslant 1$ and

$$\liminf_{n\to\infty} \|y_n + y\| \geqslant R(a, X) - \eta.$$

Then, by the triangle inequality,

$$\liminf_{n\to\infty} \|y_n + y\| \leqslant \liminf_{n\to\infty} \|y_n\| + a \leqslant R(0, X) + a = \frac{1}{d} + a.$$

The proof is finished. □

Corollary 2. *Under the same assumptions done for Theorem 1, the following are true:*

(i) $d - (d-1)R(0, X) = d - 1 + \frac{1}{d} \leqslant R(1, X)$;
(ii) $1 \leqslant R(\frac{1}{d}, X)$;
(iii) $\frac{d+1}{d} \leqslant R(\frac{1}{d-1}, X)$.

Proof. The assertions are obtained by letting $a = 1, \frac{1}{d}, 0$ in (1), respectively. □

Remark 3. The estimates (i) and (ii) in Corollary 2 remain true also for $d = 1$.

3. The James and von Neumann–Jordan constants

The *von Neumann–Jordan constant* $C_{NJ}(X)$ was defined in 1937 by Clarkson [3] as

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)}\colon\ x, y \in X \text{ and } \|x\| + \|y\| \neq 0\right\},$$

and the *James constant* $J(X)$ was defined by Gao and Lau [11] as

$$J(X) = \sup\bigl\{\min\bigl\{\|x+y\|, \|x-y\|\bigr\}\colon\ x, y \in B_X\bigr\}.$$

It is noted that the James (and also von Neumann–Jordan) constants of a Banach space X and of its Banach space ultrapower $\widetilde{X}$ are the same because X can be embedded into $\widetilde{X}$ isometrically (see [15]). Moreover, $C_{NJ}(X) = C_{NJ}(X^*)$.

Lemma 4. *Let X be a super-reflexive Banach space. Suppose that* $\mathrm{WCS}(X) = d$ *and X does not have Schur property. Then, there exist $\widetilde{x_1}, \widetilde{x_2} \in S_{\widetilde{X}}$ and $\widetilde{f_1}, \widetilde{f_2} \in S_{(\widetilde{X})^*}$ such that the following conditions are satisfied*:

(a) $\|\widetilde{x_1} - \widetilde{x_2}\| = d$ *and* $\widetilde{f_i}(\widetilde{x_j}) = 0$ *for all* $i \neq j$,
(b) $\widetilde{f_i}(\widetilde{x_i}) = 1$ *for* $i = 1, 2$,
(c) $\|\frac{\widetilde{x_2}}{d} - \widetilde{x_1}\| \leqslant R(1, X)$.

Proof. For $\varepsilon > 0$, we choose a weakly null sequence $\{x_n\} \subset S_X$ such that $\lim_{n\neq m} \|x_n - x_m\|$ exists and

$$d \leqslant \lim_{n\neq m} \|x_n - x_m\| < d + \varepsilon.$$

It follows from the definition of Domínguez-Benavides' coefficient that

$$\liminf_{n\to\infty}\left\|\frac{x_n}{d+\varepsilon} - x_m\right\| \leqslant R(1, X)$$

for all $m \in \mathbb{N}$. Passing to a suitable subsequence, we may assume that there exist a sequence $\{f_n\} \subset S_{X^*}$ and $f \in B_{X^*}$ such that

$$f_n(x_n) = \|x_n\| = 1 \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad f_n \xrightarrow{w^*} f.$$

The last convergence follows from the reflexivity of X. We first choose an integer n_1 so that

$$d - \varepsilon < \lim_{n\to\infty} \|x_n - x_{n_1}\| < d + \varepsilon \quad \text{and} \quad \bigl|f(x_{n_1})\bigr| < \frac{\varepsilon}{2}.$$

Next, we choose $n_2 > n_1$ so that

$$d - \varepsilon \leqslant \|x_{n_2} - x_{n_1}\| < d + \varepsilon,$$
$$\left\|\frac{x_{n_2}}{d+\varepsilon} - x_{n_1}\right\| < R(1, X) + \varepsilon,$$
$$\bigl|f_{n_1}(x_{n_2})\bigr| < \varepsilon, \qquad \bigl|f(x_{n_2})\bigr| < \varepsilon, \quad \text{and} \quad \bigl|(f_{n_2} - f)(x_{n_1})\bigr| < \varepsilon.$$

This implies that

$$\bigl|f_{n_2}(x_{n_1})\bigr| \leqslant \bigl|(f_{n_2} - f)(x_{n_1})\bigr| + \bigl|f(x_{n_1})\bigr| < 2\varepsilon.$$

Next, for $i = 1, 2$, we write

$$g_i^{(\varepsilon)} = f_{n_i} \quad \text{and} \quad z_i^{(\varepsilon)} = x_{n_i}.$$

Then $\widetilde{x_i} = [\{z_i^{(1/n)}\}_{n=1}^\infty]$ and $\widetilde{f_i} = [\{g_i^{(1/n)}\}_{n=1}^\infty]$ are our candidates in the Banach space ultrapowers $\widetilde{X}$ and $\widetilde{X^*} = (\widetilde{X})^*$. The latter follows from the super-reflexivity of X. $\square$

Theorem 5. *Suppose that a Banach space X does not have the Schur property and* $\mathrm{WCS}(X) = d$. *Then*

$$J(X) \geqslant \frac{1}{d} + \frac{1}{\min\{2, \sqrt{4C_{NJ}(X) - d^2}\}}.$$

Proof. We may assume in addition that X is super-reflexive. Otherwise, $J(X) = C_{\mathrm{NJ}}(X) = 2$ and the inequality becomes $2 \geqslant \frac{1}{d} + \frac{1}{2}$ which is trivial. Let $\widetilde{x_1}, \widetilde{x_2} \in S_{\widetilde{X}}$, $\widetilde{f_1}, \widetilde{f_2} \in S_{\widetilde{X}^*}$ be elements satisfying the conditions in Lemma 4. It follows that

$$\|\widetilde{x_1} + \widetilde{x_2}\| \leqslant \min\{2, \sqrt{4C_{\mathrm{NJ}}(X) - d^2}\} := \alpha.$$

Now

$$\begin{aligned} J(X) &\geqslant \min\left\| \frac{\widetilde{x_1} - \widetilde{x_2}}{d} \pm \frac{\widetilde{x_1} + \widetilde{x_2}}{\alpha} \right\| \\ &\geqslant \min\left\{ \widetilde{f_1}\left(\frac{\widetilde{x_1} - \widetilde{x_2}}{d} + \frac{\widetilde{x_1} + \widetilde{x_2}}{\alpha} \right), (-\widetilde{f_2})\left(\frac{\widetilde{x_1} - \widetilde{x_2}}{d} - \frac{x_1 + x_2}{\alpha} \right) \right\} \\ &= \frac{1}{d} + \frac{1}{\alpha}. \qquad \square \end{aligned}$$

Remark 6. If X is a Hilbert space, then the preceding estimate becomes equality. In fact, $J(X) = \mathrm{WCS}(X) = \sqrt{2}$ and $C_{\mathrm{NJ}}(X) = 1$.

Theorem 7. *Suppose that a Banach space X fails the Schur property and $d = \mathrm{WCS}(X)$. Then*

$$J(X) \geqslant \frac{1}{d} + \frac{1}{R(1, X) + (1 - \frac{1}{d})}.$$

Proof. Let $\widetilde{x_1}, \widetilde{x_2} \in S_{\widetilde{X}}$ be elements satisfying the conditions of Lemma 4. It follows that

$$\|\widetilde{x_2} + \widetilde{x_1}\| \leqslant \left\| \frac{\widetilde{x_2}}{d} + x_1 \right\| + \left(1 - \frac{1}{d}\right)\|\widetilde{x_2}\| \leqslant R(1, X) + \left(1 - \frac{1}{d}\right).$$

Now

$$J(X) \geqslant \min\left\| \frac{\widetilde{x_2} - \widetilde{x_1}}{d} \pm \frac{\widetilde{x_2} + \widetilde{x_1}}{R(1, X) + (1 - \frac{1}{d})} \right\| \geqslant \frac{1}{d} + \frac{1}{R(1, X) + (1 - \frac{1}{d})}. \qquad \square$$

Remark 8. The above estimate becomes equality if $X = l_{2,\infty}$. In fact, $J(X) = 1 + 1/\sqrt{2}$ [13, Theorem 4], $R(1, X) = \sqrt{2}$ [5, Theorem 4.1], and $\mathrm{WCS}(X) = 1$.

Remark 9. Recently Mazcuñán-Navarro proved that [14, Theorem 23]

$$J(X) \geqslant \frac{1}{d}\left(1 + \frac{1}{R(1, X)}\right).$$

It is easy to see that

$$\frac{1}{d} + \frac{1}{R(1, X) + (1 - \frac{1}{d})} \geqslant \frac{1}{d}\left(1 + \frac{1}{R(1, X)}\right).$$

The inequality is strict for the case $d > 1$.

Remark 10. For $X = l_{p,q}$, the right value of $J(X)$ is unknown. If $p \leqslant q$, then $R(1, X) = (1 + (1/2)^{p/q})^{1/p}$ [7, Theorem 3], and $\mathrm{WCS}(X) = 2^{1/q}$ [6, Corollary 3], so we have

$$J(l_{p,q}) \geqslant \frac{1}{2^{1/q}} + \frac{1}{(1 + (1/2)^{p/q})^{1/p} + 1 - 2^{-1/q}}.$$

Corollary 11. *Suppose that a Banach space X fails the Schur property. Then*

$$\mathrm{WCS}(X) \geqslant \frac{2}{2J(X) + 1 - \sqrt{5}}.$$

In particular, $J(X) < \frac{1+\sqrt{5}}{2}$ implies $\mathrm{WCS}(X) > 1$.

Proof. It was proved in [10] that

$$R(1, X) \leqslant J(X).$$

Consequently,

$$J(X) \geqslant \frac{1}{d} + \frac{1}{R(1,X) + (1 - \frac{1}{d})} \geqslant \frac{1}{d} + \frac{1}{J(X) + (1 - \frac{1}{d})},$$

or equivalently $J(X) - \frac{1}{d} \geqslant \frac{-1+\sqrt{5}}{2}$. □

Remark 12. The preceding corollary improves [4, Theorem 3.2]. More precisely, it is clear that

$$\frac{2}{2J(X) + 1 - \sqrt{5}} > \frac{J(X)+1}{(J(X))^2}$$

provided that $J(X) < \frac{1+\sqrt{5}}{2}$.

4. The coefficient of weak orthogonality

Let us mention another interesting coefficient introduced by Sims [16]. As in [12], we prefer to use its inverse, $\mu(X)$, which is defined as the infimum of the set of real numbers $r > 0$ such that

$$\limsup_{n\to\infty} \|x + x_n\| \leqslant r \limsup_{n\to\infty} \|x - x_n\|$$

for all $x \in X$ and for all weakly null sequences $\{x_n\}$ in X. The proof of the following lemma is almost the same as that of Lemma 4, so it is omitted.

Lemma 13. *Let X be a super-reflexive Banach space. Suppose that* $\mathrm{WCS}(X) = d$, $\mu(X) = \mu$, *and X does not have Schur property. Then, there exist $\widetilde{x_1}, \widetilde{x_2} \in S_{\widetilde{X}}$ and $\widetilde{f_1}, \widetilde{f_2} \in S_{(\widetilde{X})^*}$ such that the following conditions are satisfied*:

(a) $\|\widetilde{x_1} - \widetilde{x_2}\| = d$, $\|\widetilde{x_1} + \widetilde{x_2}\| \leqslant \mu d$, *and* $\widetilde{f_i}(\widetilde{x_j}) = 0$ *for all* $i \neq j$,
(b) $\widetilde{f_i}(\widetilde{x_i}) = 1$ *for* $i = 1, 2$.

We now consider the *parameterized James constant* $J(t, X)$, where $t \geqslant 0$, which is defined by

$$J(t,X) = \sup\bigl\{\min\bigl\{\|x + ty\|, \|x - ty\|\bigr\}:\ x, y \in B_X\bigr\}.$$

The following theorem unifies the recent results of Mazcuñán-Navarro.

Theorem 14. *Suppose that a Banach space X fails the Schur property and* $d = \mathrm{WCS}(X)$, $\mu = \mu(X)$. *Then*

$$J(t,X) \geqslant \frac{1}{d}\left(1 + \frac{t}{\mu}\right)$$

for all $0 \leqslant t \leqslant 1$.

Proof. As before, we have the following estimate

$$J(t,X) \geqslant \min\left\{\left\|\frac{1}{d}(\widetilde{x_1} - \widetilde{x_2}) \pm \frac{t}{\mu d}(\widetilde{x_1} + \widetilde{x_2})\right\|\right\} \geqslant \frac{1}{d} + \frac{t}{\mu d}. \qquad \square$$

Remark 15. It is not hard to see that $J(t, l_{2,\infty}) = 1 + t/\sqrt{2}$ for all $0 \leqslant t \leqslant 1$ and $\mu(l_{2,\infty}) = \sqrt{2}$ (see [13]). Hence there is a Banach space such that the estimate above becomes equality for all $0 \leqslant t \leqslant 1$.

Corollary 16. *Suppose that a Banach space X fails the Schur property and* $d = \mathrm{WCS}(X)$, $\mu = \mu(X)$. *Then*

(i) *(see [14, Theorem 31])* $J(X) \geqslant \frac{1}{d}(1 + \frac{1}{\mu})$;
(ii) *(see [14, Proposition 30])* $C_{\mathrm{NJ}}(X) \geqslant \frac{1}{d^2}(1 + \frac{1}{\mu^2})$;
(iii) *(see [14, Theorem 27])* $1 + \rho_X(t) \geqslant \frac{1}{d}(1 + \frac{t}{\mu})$ *for all* $t \geqslant 0$.

Recall that ρ_X denotes the modulus of smoothness of X defined by

$$\rho_X(t) = \sup\left\{\frac{\|x + ty\| + \|x - ty\|}{2} - 1:\ x, y \in B_X\right\}$$

for $t \geqslant 0$.

Proof. (i) is obtained by letting $t=1$ while (ii) follows from $C_{\mathrm{NJ}}(X) \geqslant \frac{(J(t,X))^2}{1+t^2}$ where $t=\frac{1}{\mu}$. Finally, it is easy to see that $1+\rho_X(t) \geqslant J(t,X)$ and hence (iii) follows. □

5. García-Falset's coefficient

In 1997, García-Falset proved that every nearly uniformly smooth space has the fixed point property. To prove this, he introduced the following coefficient, the so-called *García-Falset coefficient*,

$$R(X) := \sup\Big\{\liminf_{n\to\infty} \|x_n + x\|\Big\}$$

where the supremum is taken over all weakly null sequences $\{x_n\}$ in B_X and all $x \in S_X$. He proved that a reflexive Banach space X with $R(X) < 2$ enjoys the fixed point property [9].

Theorem 17. *Suppose that a Banach space X fails the Schur property and $d = \mathrm{WCS}(X)$. Then*

(1) $J(X) \geqslant \frac{1}{d} + \frac{1}{R(X)}$;

(2) $C_{\mathrm{NJ}}(X) \geqslant \frac{1}{d^2} + \frac{1}{(R(X))^2}$.

Proof. Using García-Falset's coefficient instead of Domínguez-Benavides' coefficient in Lemma 4, we have $\|\widetilde{x_2} + \widetilde{x_1}\| \leqslant R(X)$. This implies

$$J(X) \geqslant \min\left\|\frac{\widetilde{x_2} - \widetilde{x_1}}{d} \pm \frac{\widetilde{x_2} + \widetilde{x_1}}{R(X)}\right\| \geqslant \frac{1}{d} + \frac{1}{R(X)}$$

and

$$C_{\mathrm{NJ}}(X) = C_{\mathrm{NJ}}(X^*) \geqslant \frac{1}{4}\big(\|\widetilde{f_2} - \widetilde{f_1}\|^2 + \|\widetilde{f_2} + \widetilde{f_1}\|^2\big) \geqslant \frac{1}{d^2} + \frac{1}{(R(X))^2}. \qquad \square$$

Remark 18. Both estimates above become equality when $X = \ell_{2,\infty}$ and $X = \ell_p$ where $1 < p \leqslant 2$ (see [8,13]).

Remark 19. The result above is better than the result involving the coefficient of weak orthogonality of Sims. In fact, our estimate still makes sense when $\mu(X)d > 2$. ((a) of Lemma 13 becomes trivial in this case.)

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