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JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 6, 394-403 (1963)

Asymptotic Expansions of a Class of Hypergeometric Polynomials with Respect to the Order*

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SUMMARY AND INTRODUCTION

In a previous paper [1], a sequence of rational functions was constructed to economize summation of a certain type of asymptotic series. Only in certain special cases could it be shown that the above sequence converged to the desired limit. To facilitate an understanding of the convergence question, it was necessary to study the asymptotic properties with respect to the order of a general class of hypergeometric polynomials of which the well known Jacobi polynomials form a subclass. Such a study is the essence of the present and subsequent papers.

For a discussion of asymptotic properties of the Jacobi polynomials, see [2, 3]. To study the more general class of polynomials, our approach is based on constructing asymptotic solutions of differential equations containing a large parameter, see [4] and the references given there.

Our results are useful to study zeros and extremal values of the polynomials and their confluent forms, e.g., Bessel functions of the first kind.

I. DEFINITIONS AND PRELIMINARY RESULTS

We are concerned with the hypergeometric polynomials[1]

$$F_n(z) = {}_{p+2}F_q\left(\begin{matrix} -n, n+\lambda, \alpha_1, \cdots, \alpha_p \\ 1+\rho_1, \cdots, 1+\rho_q \end{matrix} \middle| z\right), \tag{1.1}$$

* This research is a continuation of work given in Midwest Research Institute Technical Report, July 1959 and was supported by the United States Navy through the Applied Mathematics Laboratory of the David Taylor Model Basin under Contract No. Nonr-2638(00) (X). Reproduction in whole or in part is permitted for any purpose of the United States Government.

[1] Equation (1.1) is a generalization of the classical Jacobi polynomials if $q = p + 1$.

where n is a positive integer and

$$ {}_pF_q\left(\begin{matrix}\alpha_1, \cdots, \alpha_p \\ 1+\rho_1, \cdots, 1+\rho_q\end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_k}{\prod_{j=1}^{q} (1+\rho_j)_k} \frac{z^k}{k!}, \tag{1.2} $$

$$ (\sigma)_\mu = \frac{\Gamma(\sigma+\mu)}{\Gamma(\sigma)}. \tag{1.3} $$

From (1.3) we see that when α_j is a negative integer $(-n)$,

$$ (-1)^k \frac{(-n)_k}{k!} = \frac{\Gamma(n+1)}{\Gamma(n-k+1)\,\Gamma(k+1)} = \binom{n}{k}, \tag{1.4} $$

and the infinite series in (1.2) terminates at $k = n$. However, in general, the α_j's are not negative integers and it is necessary that $p \leq q+1$ for the power series (1.2) to have a nonzero radius of convergence. If $p = q+1$, (1.2) has a radius of convergence unity and if $p < q+1$, it is infinite. Throughout this paper we use a contracted notation and write

$$ {}_pF_q\left(\begin{matrix}\alpha_p \\ 1+\rho_q\end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(\alpha_p)_k}{(1+\rho_q)_k} \frac{z^k}{k!}. \tag{1.5} $$

Thus $(\alpha_p)_k$ is to be interpreted as $\Pi_{j=1}^{p} (\alpha_j)_k$ and a similar interpretation holds for $(1+\rho_q)_k$. For (1.2) to be defined, the ρ_j's cannot be negative integers. Also, it is assumed that no α_j is equal to any $1+\rho_i$. In contracted notation, and incorporating (1.4), we have

$$ F_n(z) = {}_{p+2}F_q\left(\begin{matrix}-n, n+\lambda, \alpha_p \\ 1+\rho_q\end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(n+\lambda)_k\,(\alpha_p)_k}{(1+\rho_q)_k} z^k. \tag{1.6} $$

$F_n(z)$ as given in [3] obeys a differential equation of the form

$$ \{\delta(\delta+\rho_1)\cdots(\delta+\rho_q) - z(\delta-n)(\delta+n+\lambda)(\delta+\alpha_1)\cdots(\delta+\alpha_p)\}\,F_n(z) = 0, $$
$$ \delta = z\mathscr{D} = \frac{zd}{dz}. \tag{1.7} $$

The order of the differential equation is $M = \max\{p+2, q+1\}$. Equation (1.7) can also be written in the form

$$ \{\delta(\delta+\rho_1)\cdots(\delta+\rho_q) + zN^\beta(\delta+\alpha_1)\cdots(\delta+\alpha_p) $$
$$ - z\delta(\delta+\lambda)(\delta+\alpha_1)\cdots(\delta+\alpha_p)\}\,F_n(z) = 0, \tag{1.8} $$

$$ N^\beta = n(n+\lambda), \tag{1.9} $$

where the parameter β is defined later, see the remarks following (1.20).

Some properties of the δ operator are next of interest. Define

$$A_m^K = \sum a_{t_m} a_{t_{m-1}} \cdots a_{t_1},\ t_m > t_{m-1} > \cdots > t_1,\ t_j \in \{1, \cdots, K\},\ j = 1, \cdots, m \quad m > 0$$
$$= 1,\ m = 0. \tag{1.10}$$

Then by induction on K, it can be shown that

$$\prod_{j=1}^{K} (\delta + a_j) = \sum_{\nu=0}^{K} \left\{ \sum_{t=0}^{K-\nu} P_{K-t,K+1-\nu-t} A_t^K \right\} z^\nu \mathscr{D}^\nu, \qquad \mathscr{D}^0 = 1, \tag{1.11}$$

where the $P_{K,L}$'s satisfy the difference equation

$$P_{K+L,L+1} + (K+1)\, P_{K+L,L} = P_{K+L+1,L+1}, \qquad K \geq 0, \qquad L \geq 0 \tag{1.12}$$

with boundary conditions

$$\begin{gathered} P_{K,L} = 0,\ L > K \geq 0; \qquad P_{K,1} = 1,\ K > 0, \\ P_{K,L} = 0,\ L \leq 0,\ K > 0;\ P_{K,K} = 1,\ K \geq 0. \end{gathered} \tag{1.13}$$

From (1.12) and the boundary conditions (1.13), it follows that

$$P_{K+L,L} = \sum_{m=0}^{K} (1+m)\, P_{m+L-1,L-1}, \qquad K \geq 0, \qquad L \geq 1. \tag{1.14}$$

By direct computation from (1.14), we find

$$\begin{aligned}
P_{K+1,1} &= \binom{K+1}{1} \frac{[K+1]^{-1}}{1} = 1, \\
P_{K+2,2} &= \binom{K+2}{2} \frac{[2K^0]}{2} = \frac{(K+2)(K+1)}{2}, \\
P_{K+3,3} &= \binom{K+3}{3} \frac{[3K+4]}{4}, \\
P_{K+4,4} &= \binom{K+4}{4} \frac{[4K^2+12K+8]}{8}, \\
P_{K+5,5} &= \binom{K+5}{5} \frac{\left[5K^3 + 25K^2 + \left(\frac{110}{3}\right) K + 16\right]}{16}, \\
P_{K+6,6} &= \binom{K+6}{6} \frac{[6K^4 + 44K^3 + 106K^2 + 100K + 32]}{32}, \text{ etc.}
\end{aligned} \tag{1.15}$$

Though it is not convenient for our purposes, an explicit expression for the $P_{K,L}$'s is

$$P_{K+L,L} = \frac{1}{K!} \sum_{m=0}^{K} (-1)^{K-m} \binom{K}{m} (1+m)^{K+L-1},$$

$$K \geq 0, \qquad L \geq 1. \tag{1.16}$$

That (1.16) is a solution of (1.12) with boundary conditions (1.13) is readily verified with the aid of a result given in [5].

For our later work, we need to consider a function of the form

$$G(z) = A \exp\left\{ N \int^{z} \tau(N, t)\, dt \right\}, \tag{1.17}$$

where A is a constant. Where no confusion will result, we write

$$\tau(N, t) = \tau. \tag{1.18}$$

The precise form of τ is not pertinent at the moment, but is defined later by (1.20).

By induction on K, it can be shown that

$$\begin{aligned} \frac{\mathcal{D}^K G(z)}{G(z)} &= (N\tau)^K + N^{K-1}\left[\frac{(-K)_2}{2!}\, \tau^{K-2}\tau^{(1)}\right] \\ &+ N^{K-2}\left[\frac{(-1)(-K)_3}{3!}\, \tau^{K-3}\tau^{(2)} + \frac{(-K)_4}{2\cdot 2^2}\, \tau^{K-4}\left(\tau^{(1)}\right)^2\right] \\ &+ N^{K-3}\left[\frac{(-K)_4}{4!}\, \tau^{K-4}\tau^{(3)} \right. \\ &+ (-1)\frac{(-K)_5}{3\cdot 2^2}\, \tau^{K-5}\tau^{(1)}\tau^{(2)} + \left.\frac{(-K)_6}{3\cdot 2^4}\, \tau^{K-6}\left(\tau^{(1)}\right)^3\right] \\ &+ N^{K-4}\left[\frac{(-1)(-K)_5}{5!}\, \tau^{K-5}\tau^{(4)} + \frac{(-K)_6}{3\cdot 2^4}\, \tau^{K-6}\tau^{(1)}\tau^{(3)} \right. \\ &+ \frac{(-K)_6}{3^2 2^3}\, \tau^{K-6}\left(\tau^{(2)}\right)^2 + \frac{(-1)(-K)_7}{3\cdot 2^4}\, \tau^{K-7}\left(\tau^{(1)}\right)^2\tau^{(2)} \\ &+ \left.\frac{(-K)_8}{3\cdot 2^7}\, \tau^{K-8}\left(\tau^{(1)}\right)^4\right] + H_{K-5}(N), \end{aligned} \tag{1.19}$$

where $(-K)_m$ is the notation defined by (1.3), $\tau^{(m)}$ is the mth derivative of τ with respect to z, and $H_{K-5}(N)$ is a polynomial in N of degree $K-5$, whose coefficients depend on τ_j, $j = 0, 1, \cdots, K$.

Assume that

$$\tau(N, t) = \sum_{k=0}^{\infty} \frac{\tau_k(t)}{N^k} = \sum_{k=0}^{\infty} \frac{\tau_k}{N^k}. \tag{1.20}$$

Using Eqs. (1.10)-(1.15), (1.8) can be changed into an equation involving the differential operator $\mathscr{D}$. If we assume that $F_n(z)$ is a function of the form $G(z)$ as given by (1.17) and employ (1.19), formal substitution of these developments in (1.8) gives a power series in N equal to zero. Then, if τ_k and β are chosen so that all coefficients of powers of N are zero, and $\tau_0 \neq 0$, a generally divergent series (1.20) is obtained. With the help of this series and (1.17), formal solutions of (1.8) are obtained which serve as asymptotic representations of certain solutions of (1.8) in appropriate regions of the z-plane. The equation for τ_0 is called the characteristic equation of the differential equation, and its behavior changes radically as $q = p + 1$ (Case I), $q \leq p$ (Case II), or $q \geq p + 2$ (Case III). In this paper we consider Case I only.

II. Case I, $q = p + 1$, $N^2 = n(n + \lambda)$

Since infinity is a regular singular point of (1.8), $F_n(z)$ can be given as a linear combination of solutions around infinity of (1.8). Replacing the solutions around infinity by their asymptotic representations for large n, we arrive at an asymptotic representation for $F_n(z)$, for large n.

A fundamental set of solutions for Case I of (1.8) contains $M = p + 2$ functions. There are p formal, algebraic, descending series solutions of (1.8) of the form

$$\mathscr{L}^{(\alpha_t)}_{p+2,p+1}(z) = \frac{(\alpha_p)_{-\alpha_t}}{(1 + \rho_{p+1})_{-\alpha_t}} (z)^{-\alpha_t}$$

$$\times {}_{p+2}F_{p+1}\left(\begin{matrix}\alpha_t, \alpha_t - \rho_{p+1} \\ 1 + \alpha_t + n, 1 + \alpha_t - n - \lambda, 1 + \alpha_t - \alpha_p\end{matrix}\middle| \frac{1}{z}\right) \tag{2.1}$$

where $t = 1, 2, \cdots, p$ and

$$(\alpha_p)_{-\alpha_t} = \prod_{\substack{j=1 \\ j \neq t}}^{p} (\alpha_j)_{-\alpha_t}, \qquad (1 + \rho_{p+1})_{-\alpha_t} = \prod_{j=1}^{p+1} (1 + \rho_j)_{-\alpha_t}, \tag{2.2}$$

and the denominator parameters of the hypergeometric function are $1 + \alpha_t + n$, $1 + \alpha_t - n - \lambda$, and $1 + \alpha_t - \alpha_j$ ($j = 1, \cdots, p$; $j \neq t$). The $\mathscr{L}^{(\alpha_t)}_{p+2,p+1}(z)$'s are linearly independent if no $\alpha_i - \alpha_j$, $i \neq j$, is equal to an integer or zero. If one of the $\alpha_i - \alpha_j$, $i \neq j$, is equal to an integer, limiting forms of the $\mathscr{L}^{(\alpha_t)}_{p+2,p+1}(z)$ need be taken; see [6]. For $|z| > 1$, the series defining the $\mathscr{L}^{(\alpha_t)}_{p+2,p+1}(z)$'s converge, and are valid solutions of (1.8). Since n enters into two denominator parameters of the ${}_{p+2}F_{p+1}(z)$ in (2.1), $\mathscr{L}^{(\alpha_t)}_{p+2,p+1}(z)$ may be considered as essentially an asymptotic expansion for large n of a valid solution of (1.8). Then for $0 < |z| \leq 1$, these same functions also serve as asymptotic expansions for large n to valid solutions of (1.8), see [4] and the references given there. The $\mathscr{L}^{(\alpha_t)}_{p+2,p+1}(z)$'s then correspond to the p identically vanishing roots of the equation for τ_0, see (1.20).

The lead terms of the exponential asymptotic expansions of the remaining $M - p = 2$ solutions of the fundamental set of solutions are computed by the formal procedure given in Section I, and are denoted by $\mathscr{K}^{(j)}_2(z)$; $j = 1, 2$; see [4] and and the references given there. Thus, denoting asymptotic equivalence by $\sim$, we have

$$F_n(z) \sim \sum_{t=1}^{p} A_t \mathscr{L}^{(\alpha_t)}_{p+2,p+1}(z) + A_{p+1}\mathscr{K}^{(1)}_2(z) + A_{p+2}\mathscr{K}^{(2)}_2(z), \qquad (2.3)$$

where the A_t's are constants. By the principle of confluence,

$$\operatorname*{Lim}_{n\to\infty} F_n\left(\frac{z}{n(n+\lambda)}\right) = \operatorname*{Lim}_{n\to\infty} {}_{p+2}F_{p+1}\left(\begin{matrix} -n, n+\lambda, \alpha_p \\ 1+\rho_{p+1}\end{matrix}\,\middle|\, \frac{z}{n(n+\lambda)}\right)$$

$$= {}_pF_{p+1}\left(\begin{matrix}\alpha_p \\ 1+\rho_{p+1}\end{matrix}\,\middle|\, -z\right). \qquad (2.4)$$

If z in (2.3) is replaced by $z/n(n+\lambda)$, while $0 \ll |z| \ll n(n+\lambda)$, then the A_t's can be determined asymptotically by comparison with the asymptotic representation of ${}_pF_{p+1}\left({\alpha_p \atop 1+\rho_{p+1}} \middle| -z\right)$ for large z; see [6]. Since the real axis is the Stokes line for ${}_pF_{p+1}\left({\alpha_p \atop 1+\rho_{p+1}} \middle| -z\right)$, and the confluence on $F_n(z)$ effectively moves the singularity at unity of (1.8) out to infinity along the positive real axis, we restrict $|\arg z| \leq \pi - \epsilon_1$, $\epsilon_1 > 0$, $|\arg(z-1)| \leq \epsilon_2$, $\epsilon_2 > 0$ and write, through the τ_4 terms (see (1.20)),[2,3]

$${}_{p+2}F_{p+1}\left(\begin{matrix} -n, n+\lambda, \alpha_p \\ 1+\rho_{p+1}\end{matrix}\,\middle|\, z\right) \sim \sum_{t=1}^{p} \frac{(n+\lambda)_{-\alpha_t}}{(n+1)_{\alpha_t}} \mathscr{L}^{(\alpha_t)}_{p+2,p+1}(z)$$

$$+ \frac{\Gamma(1+\rho_{p+1})}{\Gamma(\alpha_p)\,\Gamma(1/2)} N^{2\gamma}[\sin(\theta/2)]^{2\gamma}[\cos(\theta/2)]^{-2\gamma-\lambda}\exp\{N^{-2}\varphi_2(\theta) + 0(N^{-4})\}$$

$$\times \cos\{N\theta + \pi\gamma + N^{-1}\varphi_1(\theta) + N^{-3}\varphi_3(\theta) + 0(N^{-5})\}, \qquad (2.5)$$

$$|\arg z| \leq \pi - \epsilon_1, \qquad |\arg(z-1)| \geq \epsilon_2, \qquad \epsilon_1 > 0, \qquad \epsilon_2 > 0,$$

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[2] The procedure outlined above permits the inference $A_t \sim N^{-2\alpha_t}$ for large n, $t = 1, 2, \cdots, p$. However, $N^{-2\alpha_t} \sim (n+\lambda)_{-\alpha_t}/(n+1)_{\alpha_t}$, and this choice makes (2.5) exact if any of the α_t's are negative integers.

[3] S. O. Rice, see [7], derived the analog of (2.5) for the polynomials ${}_3F_2(-n, n+1, \zeta; 1, p; v)$.

where

$$\cos\theta = 1 - 2z \qquad \text{or} \qquad z = \sin[(\theta/2)]^2$$

$$N^2 = n(n+\lambda) \qquad \gamma = (4)^{-1}(1 + 2B_1 - 2C_1)$$

$$B_1 = \sum_{t=1}^{p} \alpha_t \qquad C_1 = \sum_{t=1}^{p+1} (1+\rho_t)$$

$$B_2 = \sum_{s=2}^{p} \sum_{t=1}^{s-1} (\alpha_s)(\alpha_t) \qquad C_2 = \sum_{s=2}^{p+1} \sum_{t=1}^{s-1} (1+\rho_s)(1+\rho_t)$$

$$B_3 = \sum_{r=3}^{p} \sum_{s=2}^{r-1} \sum_{t=1}^{s-1} (\alpha_r)(\alpha_s)(\alpha_t) \qquad C_3 = \sum_{r=3}^{p+1} \sum_{s=2}^{r-1} \sum_{t=1}^{s-1} (1+\rho_r)(1+\rho_s)(1+\rho_t)$$

etc.

$$\varphi_1(\theta) = (\mu_1 + \mu_2)\tan(\theta/2) - 2\mu_3 \cot\theta - \mu_1\theta/2$$

$$\varphi_2(\theta) = \mu_4[\sec(\theta/2)]^2 + \mu_5[\csc(\theta/2)]^2$$

$$\begin{aligned}\varphi_3(\theta) = {} & (3)^{-1}(\mu_6 + \mu_7 + \mu_8)[\tan(\theta/2)]^3 + (\mu_8 - \mu_6)\tan(\theta/2) + \mu_6\theta/2 \\ & + (-4/3)(\mu_9 + 2\mu_{10})[\cot\theta]^3 + (-4)(\mu_9 + 2\mu_{10})\cot\theta \\ & + (4/3)\mu_9[\csc\theta]^3 \end{aligned} \tag{2.6}$$

$$\mu_1 = -\lambda^2/4$$

$$\mu_2 = (2)^{-1}(C_1 - B_1)(2B_1 + \lambda - 1) + B_2 - C_2 + 1/4$$

$$\mu_3 = (4)^{-1}(B_1 - C_1)(3B_1 + C_1 - 2) + C_2 - B_2 - 3/16$$

$$\mu_4 = (16)^{-1}(2\gamma + \lambda - 1)(2\gamma + \lambda) = -(4)^{-1}(\mu_1 + \mu_2 + \mu_3)$$

$$\begin{aligned}\mu_5 = {} & (16)^{-1}(C_1 - B_1)(8B_2 - 8B_1^2 + 11B_1 + C_1 - 2) \\ & + (4)^{-1}(C_2 - B_2)(2B_1 - 3) - (2)^{-1}(C_3 - B_3) + 3/64\end{aligned}$$

$$\mu_6 = -\lambda^4/64$$

$$\begin{aligned}\mu_7 = {} & (4)^{-1}(B_1 - C_1)[4B_3 + 10B_2 - 8B_1B_2 + 7B_1 + 4B_1^3 - 10B_1^2 \\ & + 2\lambda B_2 + 2\lambda B_1 - 2\lambda B_1^2 - \lambda^2 B_1/2 - \lambda^3/4 + \lambda^2/4 + \lambda/2 - 1/2] \\ & + (4)^{-1}(B_2 - C_2)[4B_2 - 4B_1^2 + 10B_1 + 2\lambda B_1 - 2\lambda + \lambda^2/2 - 7] \\ & + (2)^{-1}(2B_1 - \lambda - 5)(B_3 - C_3) + C_4 - B_4 + 3\lambda^2/32 - 1/16\end{aligned}$$

$$\mu_{10} = (64)^{-1} (B_1 - C_1) [C_1^3 + 5B_1C_1^2 + 35B_1^2C_1 - 105B_1^3 + 236B_1^2 + 160B_1B_2 - 24B_2C_1 - 8C_1C_2 - 40B_1C_1 - 4C_1^2 - 192B_2 - 64B_3 - 291B_1/2 - C_1/2 + 9] + (8)^{-1} (B_2 - C_2) [2C_2 - 10B_2 + 6C_1 - 30B_1 - 7B_1C_1 + 15B_1^2 + 73/4] + (2)^{-1} (B_3 - C_3) (C_1 - 3B_1 + 6) + B_4 - C_4 + 63/1024 \quad \text{[2.6] cont.}$$

$$2\mu_7 + \mu_8 - \mu_{10} = (64)^{-1} (B_1 - C_1) [2B_1 - 8\lambda^2 B_1 - 8\lambda^3 + 4\lambda^2 + 24\lambda - 1] + (8)^{-1} (B_1 - C_1)^2 [\lambda - 2\lambda B_1 - 3\lambda^2/4 + 23/16] + (64)^{-1} (B_1 - C_1)^3 (4 - 7B_1 - C_1) + (8)^{-1} (B_2 - C_2) (B_1 - C_1)^2 + (B_2 - C_2) (B_1 - C_1) (\lambda/4) + (32)^{-1} (4\lambda^2 - 1) (B_2 - C_2) + 25\lambda^2/128 - 47/1024$$

$$\mu_6 + \mu_7 + \mu_8 + \mu_9 + \mu_{10} = -(64)^{-1} (2\gamma + \lambda - 3) (2\gamma + \lambda - 1) \times (2\gamma + \lambda) (2\gamma + \lambda + 2) = (2)^{-1} \mu_4(3 - 8\mu_4).$$

Again $\Gamma(\alpha_p)$ and $\Gamma(1 + \rho_{p+1})$ are to be interpreted as $\Pi_{t=1}^{p} \Gamma(\alpha_t)$ and $\Pi_{t=1}^{p+1} \Gamma(1 + \rho_t)$, respectively. By induction, it can be shown that with the exception of $\pi\gamma$, only odd powers of N appear inside the cosine term of (2.5) with curly brackets, while only even powers of N appear in the exponential term.

Representations valid along the negative real axis and positive real axis to the right of unity can be similarly constructed. The connecting constants in this instance are determined from (2.5) by comparing the dominant terms in those overlapping regions where both representations are valid. Hence

$$ {}_{p+2}F_{p+1} \left(\begin{matrix} -n, n+\lambda, \alpha_p \\ 1 + \rho_{p+1} \end{matrix} \middle| -z \right) \sim \sum_{t=1}^{p} \frac{(n+\lambda)_{-\alpha_t}}{(n+1)_{\alpha_t}} \mathscr{L}^{(\alpha_t)}_{p+2,p+1}(e^{\epsilon\pi i} z)$$

$$+ \frac{\Gamma(1 + \rho_{p+1}) N^{2\gamma}}{\Gamma(\alpha_p) \Gamma(1/2)} [\sinh(\xi/2)]^{2\gamma} [\cosh(\xi/2)]^{-2\gamma-\lambda} \exp \{N^{-2}\varphi_2(\xi i) + 0(N^{-4})\}$$

$$\times \cosh \{N\xi - iN^{-1}\varphi_1(\xi i) - iN^{-3}\varphi_3(\xi i) + 0(N^{-5})\}, \qquad (2.7)$$

$$\cosh \xi = 1 + 2z, \qquad |\arg z| \leq \pi - \epsilon_3, \qquad \epsilon_3 > 0,$$

$$\epsilon = +(-) \quad \text{if} \quad \arg z < (>) 0.$$

$$ {}_{p+2}F_{p+1} \left(\begin{matrix} -n, n+\lambda, \alpha_p \\ 1 + \rho_{p+1} \end{matrix} \middle| z \right) \sim \sum_{t=1}^{p} \frac{(n+\lambda)_{-\alpha_t}}{(n+1)_{\alpha_t}} \mathscr{L}^{(\alpha_t)}_{p+2,p+1}(z)$$

$$+ (-)^n \frac{\Gamma(1 + \rho_{p+1}) N^{2\gamma}}{\Gamma(\alpha_p) \Gamma(1/2)} [\cosh(\eta/2)]^{2\gamma} [\sinh(\eta/2)]^{-2\gamma-\lambda}$$

$$\times \exp \{N^{-2}\varphi_2(\pi + \eta i) + 0(N^{-4})\}$$

$$\times \cosh \{N\eta - iN^{-1}[\varphi_1(\pi + \eta i) + \mu_1\pi/2] - iN^{-3}[\varphi_3(\pi + \eta i) - \mu_6\pi/2] + 0(N^{-5})\},$$

$$\cosh \eta = 2z - 1, \qquad |\arg(z - 1)| \lesseqgtr \pi - \epsilon_4, \qquad \epsilon_4 > 0. \qquad (2.8)$$

We remark that the generalized Jacobi functions

$$_{p+2}F_{p+1}\left(\begin{matrix}-\omega n+\mu,\ \omega n+\lambda,\ \alpha_p\\ 1+\rho_{p+1}\end{matrix}\middle| z\right); \qquad \omega\neq 0, \qquad |z|<1;$$

can be similarly treated, and their expansion agrees with (2.5) and (2.7) if in (2.6), except for N^2, λ is replaced by $\lambda+\mu$,

$$N^2=(\omega n-\mu)(\omega n+\lambda), \tag{2.9}$$

and where now

$$A_t\mathscr{L}^{(\alpha_t)}_{p+2,p+1}(z)\sim\frac{(\omega n+\lambda)_{-\alpha_t}(\alpha_p)_{-\alpha_t}}{(\omega n-\mu+1)_{\alpha_t}(1+\rho_{p+1})_{-\alpha_t}}$$

$$\times\ _{p+2}F_{p+1}\left(\begin{matrix}\alpha_t,\ \alpha_t-\rho_{p+1}\\ 1+\alpha_t+\omega n-\mu,\ 1+\alpha_t-\omega n-\lambda,\ 1+\alpha_t-\alpha_p\end{matrix}\middle|\frac{1}{z}\right). \tag{2.10}$$

Finally, note that the above representations for $p=0$ differ from the classical expression for the Jacobi polynomials in terms of $0(n^{-1})$. This difference comes from our choice of the large parameter. Classically, one puts

$$N=n+\frac{\lambda}{2}, \tag{2.11}$$

but from the differential equation (1.8), it is much more natural to set

$$N=n\left(1+\frac{\lambda}{n}\right)^{1/2}=n+\frac{\lambda}{2}+0(n^{-1}) \tag{2.12}$$

which agrees with (2.11) through terms of $0(n^{-1})$.

The choice $N^2=n(n+\lambda)$ is advantageous since N large leads to the dual interpretation that either n or λ or both n and λ are large. We have found that if in our asymptotic developments of (1.6), z is replaced by $z/(n+\lambda)$, $0\ll n\ll\lambda$, then by confluence on λ (that is, let $\lambda\to\infty$), we can get like developments for $_{p+1}F_p\left(\begin{smallmatrix}-n,\alpha_p\\ 1+\rho_p\end{smallmatrix}\middle| z\right)$. This includes results for the classical Laguerre polynomials. However, further discussion is deferred for a later paper.

Except for those values of z explicitly excluded, and the singular points zero, unity, and infinity, (2.5), (2.7), and (2.8) hold for all fixed values of z as $N\to\infty$. For N fixed and z varying, we require that the correction terms in the above representations be small, i.e., $|z|\geq 0(N^{-2})$, $|1-z|\leq 0(N^{-2})$ and $\log|z|\leq 0(N)$ for z near zero, unity, and infinity, respectively. A study of $F_n(z)$ for large n in the neighborhood of these singularities is deferred to a later paper. Also, Case II and Case III analyses (see remarks after (1.20)), as well as a study of the zeros and extrema for all cases, will be the subject of future papers.

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