Detecting infinitely many semisimple representations in a fixed finite dimension ✩

E.S. Letzter

Department of Mathematics, Temple University, Philadelphia, PA 19122-6094, USA

A B S T R A C T

Let n be a positive integer, and let k be a field (of arbitrary characteristic) accessible to symbolic computation. We describe an algorithmic test for determining whether or not a finitely presented k-algebra R has infinitely many equivalence classes of semisimple representations $R \to M_n(k')$, where k' is the algebraic closure of k. The test reduces the problem to computational commutative algebra over k, via famous results of Artin, Procesi, and Shirshov. The test is illustrated by explicit examples, with $n = 3$.

© 2008 Elsevier Inc. All rights reserved.

1. Introduction

Among the most fundamental tasks, when studying a given finitely presented algebra over a field k, is the parametrization of the irreducible finite dimensional representations. Typically, such parametrizations depend on whether or not the finite dimensional irreducible representations, partitioned according to their dimensions, occur in finite or infinite families. The focus in this paper is on general algorithmic approaches to this latter issue.

1.1. Before describing the work of this paper, we review some of the background and context. To start, assume that k is a field accessible to symbolic computation and that n is a positive integer. Set

$$R := k\{X_1, \ldots, X_s\}/\langle f_1, \ldots, f_t \rangle,$$

the free associative k-algebra in $X_1, \ldots, X_s$ modulo the ideal generated by $f_1, \ldots, f_t$. By an *n-dimensional representation of R* we will always mean a unital k-algebra homomorphism

$$R \to M_n(k'),$$

where k' is the algebraic closure of k. Irreducibility, semisimplicity, and equivalence of representations are defined over k'; see (2.2).

Since the presence of finite dimensional representations (without restriction on the dimension) over k is a Markov property [10,12] the question of existence of finite dimensional representations of R is algorithmically undecidable in general [4]. Consequently, the task of algorithmically studying the finite dimensional representation theory of R, without bounding the dimensions involved, appears to be hopeless. However, if we fix n and restrict our attention to dimensions $\leqslant n$, then the situation improves considerably (at least in principle). To start, the existence of representations in dimensions bounded by n is determined by finitely many (commutative) polynomial equations, and so can be approached (again in principle) using computational commutative algebra. In [8] tests determining the existence of irreducible n-dimensional representations were described, and in [6] tests determining the existence of nonsemisimple at-most-n-dimensional representations of R were described.

1.2. Our purpose in this paper is to describe a test, involving computational commutative algebra over k, for determining whether or not R has infinitely many equivalence classes of n-dimensional semisimple representations. The test is developed in Section 2 and presented in Section 3.

1.3. In Section 4, explicit illustrative examples are given, with $n = 3$ and $s = 2$. The calculations in these examples were performed with Macaulay 2 [9] on small computers ($\leqslant 8$ GB RAM).

1.4. Our approach can be sketched as follows. To start, by famous results of Artin [1] and Procesi [11], the equivalence classes of n-dimensional representations of R correspond exactly with the maximal ideals of the (commutative) trace ring of R, taken over k'; see (2.8). In particular, R has finitely many equivalence classes of n-dimensional semisimple representations if and only if this trace ring is finite dimensional over k'. Next, using Shirshov's Theorem [14], the trace ring's being finite or infinite dimensional depends only on a suitably truncated trace ring. (We use Belov's refinement of Shirshov's Theorem [3].) Finally, the finite-versus-infinite dimensionality of the truncated trace ring can be algorithmically determined using a variant of the subring membership test, working over k.

1.5. An algorithm, in characteristic zero, for determining whether or not there exist infinitely many equivalence classes of n-dimensional irreducible representations was outlined in [7]. In part, the present paper provides a generalization and simplification of this previous work. Moreover, the methods in [7] can be combined with the approach of the present paper to formulate a test for determining whether or not there exist infinitely many equivalence classes of irreducible n-dimensional representations of R, in arbitrary characteristic. However, the methods in [7] appear to be considerably more costly than the test described in the present paper. A detailed analysis, with examples, of tests for detecting infinitely many equivalence classes of irreducible n-dimensional representations is left for future work.

1.6. A warning: Since we are assuming that representations are unital maps, it is possible for R to have only finitely many (possibly zero) equivalence classes of n-dimensional semisimple representations but at the same time for R to have infinitely many equivalence classes of irreducible representations in some dimension less than n. Of course, if R has a 1-dimensional representation, then finiteness of the number of equivalence classes of n-dimensional semisimple representations ensures the finiteness of the number of equivalence classes of irreducible representations in dimensions $\leqslant n$.

2. Setup and proof of test

In this section we develop and prove our test to determine whether a finitely presented algebra over a field has infinitely many distinct equivalence classes of n-dimensional semisimple representations, for a fixed n. The algorithm is presented in Section 3.

2.1. Assume that n is a positive integer, that k is a field, and that k' is the algebraic closure of k. Let $k\{X_1, \ldots, X_s\}$ denote the free associative k-algebra in the noncommuting indeterminates $X_1, \ldots, X_s$.

Set

$$R := k\{X_1, \ldots, X_s\}/\langle f_1, \ldots, f_t \rangle,$$

for some fixed choice of $f_1, \ldots, f_t$ in $k\{X_1, \ldots, X_s\}$. We will use $\overline{X}_1, \ldots, \overline{X}_s$ to denote the respective images in R of $X_1, \ldots, X_s$.

2.2. Let L be any subfield of k', let m be a positive integer, and let Λ be an L-algebra.

(i) By an *m-dimensional representation* of Λ we will always mean a unital L-algebra homomorphism from Λ into the L-algebra $M_m(k')$ of $m \times m$ matrices over k'.

(ii) Representations $\rho, \rho' \colon \Lambda \to M_m(k')$ are *equivalent* if there exists a matrix $Q \in GL_m(k')$ such that

$$\rho'(a) = Q\,\rho(a)\,Q^{-1},$$

for all $a \in \Lambda$.

(iii) An m-dimensional representation $\rho \colon \Lambda \to M_m(k')$ is *irreducible* (cf. [1, §9]) provided $k'\rho(\Lambda) = M_m(k')$.

(iv) An n-dimensional representation $\rho \colon \Lambda \to M_n(k')$ is *semisimple* provided ρ is equivalent to a representation of the form

$$a \longmapsto \begin{bmatrix} \rho_1(a) & & & \\ & \rho_2(a) & & \\ & & \ddots & \\ & & & \rho_r(a) \end{bmatrix},$$

for suitable choices of positive integers $m_1, \ldots, m_r$, suitable choices of irreducible m_i-dimensional representations ρ_i of Λ, and all $a \in \Lambda$.

2.3. (i) Set

$$B' := k'\big[x_{ij}(\ell) \colon 1 \leqslant i, j \leqslant n, \ 1 \leqslant \ell \leqslant s\big],$$

and

$$B := k\big[x_{ij}(\ell) \colon 1 \leqslant i, j \leqslant n, \ 1 \leqslant \ell \leqslant s\big] \subseteq B',$$

where the $x_{ij}(\ell)$ are commuting indeterminates.

For $1 \leqslant \ell \leqslant s$, let $\mathbf{x}_\ell$ denote the $n \times n$ generic matrix $(x_{ij}(\ell))$, in $M_n(B)$. For $g = g(X_1, \ldots, X_s) \in k\{X_1, \ldots, X_s\}$, let $g(\mathbf{x}) = g(\mathbf{x}_1, \ldots, \mathbf{x}_s)$ denote the image of g, in $M_n(B) \subseteq M_n(B')$, under the canonical map

$$k\{X_1, \ldots, X_s\} \xrightarrow{\ X_\ell \longmapsto \mathbf{x}_\ell\ } M_n(B).$$

Identify B' with the center of $M_n(B')$, and identify B with the center of $M_n(B)$.

(ii) Set

$$RelMatrices := \big\{ f_1(\mathbf{x}), \ldots, f_t(\mathbf{x}) \big\}.$$

Let $RelIdeal(M_n(B'))$ be the ideal of $M_n(B')$ generated by $RelMatrices$, and let $RelIdeal(M_n(B))$ be the ideal of $M_n(B)$ generated by $RelMatrices$. Note that

$$RelIdeal\big(M_n(B)\big) = RelIdeal\big(M_n(B')\big) \cap M_n(B).$$

(iii) Let $RelEntries$ denote the set of entries of the matrices in $RelMatrices$. Let $RelIdeal(B)$ denote the ideal of B generated by $RelEntries$, and let $RelIdeal(B')$ denote the ideal of B' generated by $RelEntries$. Then

$$RelIdeal(B) = RelIdeal(B') \cap B.$$

Also,

$$RelIdeal(B') = RelIdeal\big(M_n(B')\big) \cap B', \quad \text{and} \quad RelIdeal(B) = RelIdeal\big(M_n(B)\big) \cap B.$$

(iv) Set

$$A' := k'\{\mathbf{x}_1, \dots, \mathbf{x}_s\},$$

the k'-subalgebra of $M_n(B')$ generated by the generic matrices $\mathbf{x}_1, \dots, \mathbf{x}_s$, and set

$$A := k\{\mathbf{x}_1, \dots, \mathbf{x}_s\} \subseteq A'.$$

Set

$$RelIdeal(A') := RelIdeal\big(M_n(B')\big) \cap A',$$

and

$$RelIdeal(A) := RelIdeal\big(M_n(B)\big) \cap A.$$

2.4. Let

$$\overline{A} := A/RelIdeal(A), \qquad \overline{B} := B/RelIdeal(B), \qquad \overline{M} := M_n(B)/RelIdeal\big(M_n(B)\big)$$

and

$$\overline{A}' := A'/RelIdeal(A'), \qquad \overline{B}' := B'/RelIdeal(B'), \qquad \overline{M}' := M_n(B')/RelIdeal\big(M_n(B')\big).$$

Identify $\overline{A}$, $\overline{B}$, and $\overline{M}$ with their respective natural images in $\overline{A}'$, $\overline{B}'$, and $\overline{M}'$. Since

$$RelIdeal\big(M_n(B')\big) = M_n\big(RelIdeal(B')\big),$$

we see that there is a natural isomorphism

$$\overline{M}' = M_n(B')/M_n\big(RelIdeal(B')\big) \cong M_n(\overline{B}').$$

We use this isomorphism to identify $\overline{M}'$ with $M_n(\overline{B}')$. Furthermore, $\overline{B}'$ is isomorphic to the image of B' in $\overline{M}'$; we identify $\overline{B}'$ with that image. In particular, $\overline{B}' = Z(\overline{M}')$, the center of $\overline{M}'$. We similarly identify $\overline{B}$ with $Z(\overline{M})$.

Let $\overline{\mathbf{x}}_1, \ldots, \overline{\mathbf{x}}_s$ denote, respectively, the images of $\mathbf{x}_1, \ldots, \mathbf{x}_s$ in $\overline{M} \subseteq \overline{M}'$. So $\overline{\mathbf{x}}_1, \ldots, \overline{\mathbf{x}}_s$ generate $\overline{A}'$ as a k'-algebra and generate $\overline{A}$ as a k-algebra.

2.5. We have a canonical k-algebra homomorphism

$$\pi' : R \xrightarrow{\overline{X}_\ell \longmapsto \overline{\mathbf{x}}_\ell} \overline{M} \xrightarrow{\text{inclusion}} \overline{M}'.$$

Note that $\pi'(R) = \overline{A}$ and that $\overline{A}'$ is generated as a k'-algebra by $\pi'(R)$.

2.6. We will say that a k'-algebra homomorphism $\alpha : \overline{M}' \to M_n(k')$ is *matrix-unital* provided α restricts to the identity on $M_n(k') \subseteq \overline{M}'$. More generally, if L is a subfield of k' and Λ is an L-subalgebra of $\overline{M}'$, we will say that an n-dimensional representation $\rho : \Lambda \to M_n(k')$ is *matrix-unital* (*with respect to* $\overline{M}'$) when ρ is the restriction of a matrix-unital map $\overline{M}' \to M_n(k')$. In other words, the matrix-unital n-dimensional representations of Λ all have the form

$$\Lambda \xrightarrow{\text{inclusion}} \overline{M}' \longrightarrow M_n(k'),$$

where the right-hand map is matrix-unital.

2.7. Every n-dimensional representation $\rho : R \to M_n(k')$ can be written in the form

$$R \xrightarrow{\pi'} \overline{M}' \xrightarrow{\rho^{\overline{M}'}} M_n(k'),$$

for a suitable, unique, matrix-unital representation $\rho^{\overline{M}'} : \overline{M}' \to M_n(k')$. (Conversely, of course, every matrix-unital n-dimensional representation of $\overline{M}'$ gives rise to a unique n-dimensional representation of R, written in the preceding form.) Since $\overline{A}'$ is the k'-subalgebra of $\overline{M}'$ generated by $\pi'(R)$, we see that the assignment

$$\rho \mapsto \rho^{\overline{A}'} := \rho^{\overline{M}'}|_{\overline{A}'}$$

induces a one-to-one correspondence between the equivalence classes of n-dimensional representations of R and the equivalence classes of n-dimensional matrix-unital representations of $\overline{A}'$. This correspondence depends only on the original choice of presentation of R given in (2.1).

Furthermore, an n-dimensional representation $\rho : R \to M_n(k')$ will be semisimple if and only if the corresponding matrix-unital representation $\rho^{\overline{A}'}$ is semisimple, and ρ will be irreducible if and only if $\rho^{\overline{A}'}$ is irreducible.

2.8. (i) Let $\overline{Monomials} \subseteq \overline{M} \subseteq \overline{M}'$ denote the set of monomials (i.e., matrix products) of length greater than or equal to 1, in the $\overline{\mathbf{x}}_1, \ldots, \overline{\mathbf{x}}_s$. Let $\overline{CharCoefs} \subseteq \overline{B} \subseteq \overline{B}'$ denote the set of nonscalar coefficients of characteristic polynomials of monomials in $\overline{Monomials}$. Let $\overline{T}$ denote the k-subalgebra of $\overline{B}$ generated by $\overline{CharCoefs}$, and let $\overline{T}'$ denote the k'-subalgebra of $\overline{T}'$ generated by $\overline{CharCoefs}$. It follows from Shirshov's Theorem [14] (see [11, Proposition 3.1]) that $\overline{T}'$ is a finitely generated k'-algebra.

In the literature, $\overline{T}'$ is referred to as a *trace ring*, since in characteristic zero it is generated by traces. More precisely, in characteristic zero, Razmyslov proved that $\overline{T}'$ is generated by the traces of the monomials in $\overline{Monomials}$ of length $\leqslant n^2$; this upper bound is the best known [13].

(ii) Observe that any matrix-unital representation of $\overline{M}'$ will map coefficients of characteristic polynomials of matrices in $\overline{M}'$ to coefficients of characteristic polynomials of matrices in $M_n(k')$.

(iii) Now let $\rho : R \to M_n(k')$ be an arbitrary n-dimensional representation, with corresponding matrix-unital map $\rho^{\overline{M}'} : \overline{M}' \to M_n(k')$. By (ii), the restriction of $\rho^{\overline{M}'}$ to $\overline{T}'$ produces a k'-algebra homomorphism $\rho^{\overline{T}'} : \overline{T}' \to k'$. Since coefficients of characteristic polynomials are invariant under conjugation, we further see that $\rho^{\overline{T}'}$ depends only on the equivalence class of ρ. Consequently, the assignment $\rho \mapsto \rho^{\overline{T}'}$ provides a well defined function from the set of equivalence classes of n-dimensional representations of R to the set of k'-algebra homomorphisms from $\overline{T}'$ onto k'. Moreover, the assignment $\rho \mapsto \ker \rho^{\overline{T}'}$ then provides a well defined function from the set of equivalence classes of n-dimensional representations of R to max $\overline{T}'$.

(iv) Key to Artin's [1] and Procesi's [11] study of finite dimensional representations is their proof that the function $\rho \mapsto \ker \rho^{\overline{T}'}$ induces a bijection from the set of equivalence classes of semisimple n-dimensional representations of R onto the maximal spectrum of $\overline{T}'$.

In particular, there are only finitely many equivalence classes of n-dimensional semisimple representations of R if and only if the finitely generated commutative k'-algebra $\overline{T}'$ is finite dimensional over k'.

2.9. Let $\overline{Monomials_n}$ denote the set of monomials in $\overline{Monomials}$ of length less than or equal to n. (The use of the number n will be explained below.) Let $\overline{CharCoefs_n}$ denote the set of nonscalar coefficients of characteristic polynomials of monomials in $\overline{Monomials_n}$. Let $\overline{T}_n$ denote the k-subalgebra of $\overline{B}$ generated by $\overline{CharCoefs_n}$, and let $\overline{T}'_n$ denote the k'-subalgebra of $\overline{B}'$ generated by $\overline{CharCoefs_n}$.

Set

$$\overline{S}'_n = \overline{T}'_n\{\overline{\mathbf{x}}_1, \ldots, \overline{\mathbf{x}}_s\}.$$

We can conclude as follows that $\overline{S}'_n$ is a finitely generated module over the central subalgebra $\overline{T}'_n$: To start, by Belov's refinement [3] of Shirshov's Theorem (see [5, §9.2]), it follows that there exists an integer h, depending only on n and s, such that $\overline{A}'$ is spanned as a k'-vector space by the products $w_1^{a_1} \cdots w_m^{a_m}$, where $m \leqslant h$ and where $w_1, \ldots, w_m$ are monomials of degree $\leqslant n$. (In Shirshov's original theorem the degree bound on $w_1, \ldots, w_m$ is $2n - 1$.) By the Cayley–Hamilton Theorem, each $w_i^{a_i}$, for $a_i \geqslant n$, can be expressed as a $\overline{T}'_n$-linear combination of powers of w_i of degree less than n. Hence $\overline{S}'_n$ is a finitely generated $\overline{T}'_n$-module. The length n used in defining $\overline{CharCoefs_n}$ and $\overline{T}'_n$ is the minimum length required to apply Belov's result.

The following is now a corollary to 2.8 and 2.9

2.10. Lemma. *R has only finitely many equivalence classes of semisimple n-dimensional representations if and only if $\overline{T}'_n$ is a finite dimensional k'-algebra.*

Proof. Suppose first that R has only finitely many equivalence classes of semisimple n-dimensional representations. By 2.8(iv), $\overline{T}'$ is finite dimensional over k', and so $\overline{T}'_n$ is finite dimensional over k'.

Next, suppose that $\overline{T}'_n$ is finite dimensional over k'. By 2.9, $\overline{S}'_n$ is finite dimensional over k', and so $\overline{A}' \subseteq \overline{S}'_n$ is finite dimensional over k'. Consequently, it follows from 2.7 that R has only finitely many equivalence classes of n-dimensional semisimple representations. $\quad\square$

2.11. By 2.10, to algorithmically decide whether or not R has infinitely many distinct equivalence classes of n-dimensional semisimple representations it remains to find an effective means of determining when $\overline{T}'_n$ is finite dimensional over k'. To start, observe that $\overline{T}'_n$ is finite dimensional over k' if and only if each $d \in \overline{CharCoefs_n}$ is algebraic over k', if and only if each $d \in \overline{CharCoefs_n}$ is algebraic over k (since k' is algebraic over k).

Now let $Monomials_n \subseteq M \subseteq M'$ denote the set of monomials (i.e., matrix products) of length greater than or equal to 1, but less than or equal to n, in the $\mathbf{x}_1, \ldots, \mathbf{x}_s$. Let $CharCoefs_n \subseteq B \subseteq B'$ denote the set of nonscalar coefficients of characteristic polynomials of monomials in $Monomials_n$. It follows from the preceding paragraph that R has at most finitely many equivalence classes of n-dimensional

semisimple representations if and only if each $c \in CharCoefs_n$ is algebraic, modulo $RelIdeal(B)$, over k. We can use the following variant of the subring membership test to determine whether a given $c \in CharCoefs_n$ is algebraic, modulo $RelIdeal(B)$, over k.

2.12. (Cf. [2, pp. 269–270].) Let U denote the commutative polynomial ring $k[y_1, \ldots, y_m]$, let $a_1, \ldots, a_u \in U$, and let I be the ideal of U generated by $a_1, \ldots, a_u$. Choose $g \in U$. Then g is algebraic, modulo I, over k if and only if $I \cap k[g] \neq 0$. Now view U as a subring of $\widetilde{U} = k(v)[y_1, \ldots, y_m]$, where v is an indeterminate. Then $I \cap k[g] \neq 0$ if and only if 1 is contained in the ideal

$$J := (v - g).\widetilde{U} + I.\widetilde{U} = \langle v - g, a_1, \ldots, a_u \rangle$$

of $\widetilde{U}$.

   Assuming that the ideal membership test can be applied to (commutative) polynomial rings over $k(v)$, we can effectively determine whether or not 1 is contained in J, and we can thus determine whether or not $g \in U$ is algebraic, modulo I, over k.

2.13. In characteristic 0, we may replace $CharCoefs_n$ in 2.11 with the set of traces of elements of $Monomials$ of length $\leqslant n^2$. This replacement is possible because, in characteristic zero, $\overline{T}'$ is generated by the traces of those elements of $\overline{Monomials}$ of length $\leqslant n^2$, following 2.8(i). Details are left to the interested reader.

3. Test for detecting infinitely many semisimple representations in a fixed finite dimension

   Retain the notation of the preceding section (although we have attempted to make the discussion below reasonably self contained). It immediately follows from 2.11 and 2.12 that the following procedure will determine whether or not the finitely presented algebra R has infinitely many distinct equivalence classes of semisimple n-dimensional representations.

Inputs: A positive integer n, a field k (suitably accessible to symbolic computations) with algebraic closure k', a finitely presented k-algebra

$$R = k\{X_1, \ldots, X_s\}/\langle f_1(X_1, \ldots, X_s), \ldots, f_t(X_1, \ldots, X_s)\rangle.$$

Output: YES if there are infinitely many distinct equivalence classes of semisimple representations $R \to M_n(k')$; NO otherwise

begin

   $C := k(v)[x_{ij}(\ell): \ 1 \leqslant i, j \leqslant n, 1 \leqslant \ell \leqslant s]$

   RelEntries := set of entries of the matrices $f_1(\mathbf{x}_1, \ldots, \mathbf{x}_s), \ldots, f_t(\mathbf{x}_1, \ldots, \mathbf{x}_s)$,
   where $\mathbf{x}_\ell$ denotes the $n \times n$ generic matrix $(x_{ij}(\ell))$, for $1 \leqslant \ell \leqslant s$

   Monomials$_n$:= the set of matrix products of length greater than or equal to 1,
   but less than or equal to n, in the $\mathbf{x}_1, \ldots, \mathbf{x}_s$

   CharCoefs$_n$:= the set of nonscalar coefficients of characteristic polynomials
   of matrices in *Monomials*$_n$

   reply := "NO"

   set := *CharCoefs*$_n$

      while *set* $\neq \emptyset$ **do**

      Choose $c \in set$

      $J :=$ ideal of C generated by $c - v$ and *RelEntries*

         If $1 \notin J$ (applying Ideal Membership Test)

> **then** *reply* := "YES" **and** *set* := ∅
>
> **else** *set* := *set* \ {*c*}
>
> **end**
>
> **end**
>
> **return** *reply*

end

3.1. To give some relative measure of the complexity of the above algorithm, note that each generator of the ideal *J* will be a (commutative) polynomial, over $k(v)$, of degree no greater than $\max(n^2, e)$, where *e* is the maximum total degree of the $f_1, \ldots, f_t$. In particular, the members of *CharCoefs$_n$* used in the algorithm are polynomials of degree $\leqslant n^2$.

4. Examples: 3-dimensional representations of 2-generator algebras

Retain the notation of the previous sections. To illustrate the procedure in Section 3, we examine the 3-dimensional representations of algebras of the form

$$R = k\{X, Y\}/\langle f_1(X, Y), \ldots, f_t(X, Y)\rangle.$$

To start, set $\mathbf{x} = (x_{ij})$ and $\mathbf{y} = (y_{ij})$, in $M_3(C)$, where

$$C = k(v)[x_{ij}, y_{ij}\colon 1 \leqslant i, j \leqslant 3].$$

Then *RelEntries*, in this situation, is a set of 9 (commutative) polynomials, in $9t$ variables, with coefficients in *k*. The maximum degree appearing is the maximum of the total degrees of the $f_1, \ldots, f_t$. Also, *Monomials$_3$* consists of the 14 monomials in $\mathbf{x}$ and $\mathbf{y}$ of length 1, 2, or 3.

The characteristic polynomial of a 3×3 matrix $\mathbf{w} = (w_{ij})$ is

$$\lambda^3 - \mathrm{trace}(\mathbf{w})\lambda + (w_{11}w_{22} + w_{11}w_{33} + w_{22}w_{33} - w_{12}w_{21} - w_{13}w_{31} - w_{23}w_{32})\lambda - \det(\mathbf{w}),$$

and so *CharCoefs$_3$* is a set of 42 distinct polynomials, in 18 variables, with coefficients in *k*. The maximum degree appearing is 9.

To perform the algorithm in Section 3, we need to check, for $c \in$ *CharCoefs$_3$*, whether 1 is contained in the ideal of *C* generated by $(c - v)$ and *RelEntries*. If for all $c \in$ *CharCoefs$_3$* these ideals contain 1, then there are only finitely many equivalence classes of semisimple 3-dimensional representations $R \to M_3(k')$. If there is at least one of these ideals that does not contain 1, then there are infinitely many equivalence classes.

4.1. For a concrete example, let

$$R = k\{X, Y\}/\langle X^2 - 1, Y^3 - 1\rangle,$$

the group algebra, over *k*, of $PSL_2(\mathbb{Z})$. Using Macaulay 2 [9], for $k = \mathbb{Q}$, $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_5$, and $\mathbb{F}_7$, we found that 1 is not contained in the ideal of *C* generated by *RelEntries* and $\mathrm{trace}(\mathbf{xy}) - v$. Therefore, for these choices of *k*, *R* has infinitely many equivalence classes of semisimple 3-dimensional representations. (All of the calculations discussed in this section were performed on computers with $\leqslant 8$ GB RAM.)

4.2. For a second example, consider the case when

$$R = k\{X, Y\}/\langle f_1, f_2 \rangle,$$

for

$$f_1 = (XY - YX)X - X(XY - YX) - 2X, \qquad f_2 = (XY - YX)Y - Y(XY - YX) + 2Y.$$

It is well known that R, now, is isomorphic to the enveloping algebra of $\mathfrak{sl}_2(k)$. It is also well known, when k has characteristic zero, that $\mathfrak{sl}_2(k')$ has exactly one irreducible representation (up to equivalence) in each finite dimension.

We used Macaulay 2 to implement the algorithm for $k = \mathbb{Q}$, $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_5$, and $\mathbb{F}_7$. The procedure showed that there are only finitely many equivalence classes of 3-dimensional representations when $k = \mathbb{Q}$, $\mathbb{F}_5$, and $\mathbb{F}_7$. For $k = \mathbb{F}_2$, the procedure found that 1 is not in the ideal of C generated by *RelEntries* and trace($\mathbf{x}$) $- v$, thus showing the existence of infinitely many distinct equivalence classes of semisimple 3-dimensional representations. For $k = \mathbb{F}_3$, the procedure found that 1 is not contained in the ideal of C generated by *RelEntries* and

(the degree-2 coefficient of the characteristic polynomial of $\mathbf{xy}$) $- v$,

demonstrating the existence of infinitely many distinct equivalence classes of semisimple 3-dimensional representations.

Acknowledgment

References

[1] M. Artin, On Azumaya algebras and finite dimensional representations of rings, J. Algebra 11 (1969) 532–563.
[2] T. Becker, V. Weispfenning, Gröbner Bases: A Computational Approach to Commutative Algebra, Grad. Texts in Math., vol. 141, Springer-Verlag, New York, 1993.
[3] A.Ya. Belov, On a Shirshov basis of relatively free algebras of complexity n, Mat. Sb. 135 (1988) 373–384.
[4] L.A. Bokut', Unsolvability of certain algorithmic problems in a class of associative rings, Algebra Logika 9 (1970) 137–144 (in Russian).
[5] V.S. Drensky, Free Algebras and PI-Algebras: Graduate Course in Algebra, Springer-Verlag, Singapore, 1999.
[6] E.S. Letzter, Effective detection of nonsplit module extensions, J. Pure Appl. Algebra 198 (2005) 225–236.
[7] E.S. Letzter, Counting equivalence classes of irreducible representations, AMA Algebra Montp. Announc. (2001).
[8] E.S. Letzter, Constructing irreducible representations of finitely presented algebras, J. Symbolic Comput. 32 (2001) 255–262.
[9] D.R. Grayson, M.E. Stillman, Macaulay 2, A Software System for Research in Algebraic Geometry, available at http://www.math.uiuc.edu/Macaulay2/.
[10] A. Markov, The impossibility of certain algorithms in the theory of associative systems, Dokl. Akad. Nauk SSSR (N.S.) 77 (1951) 19–20 (in Russian).
[11] C. Procesi, Finite dimensional representations of algebras, Israel J. Math. 19 (1974) 169–182.
[12] M.O. Rabin, Recursive unsolvability of group theoretic problems, Ann. of Math. (2) 67 (1958) 172–194.
[13] Yu.P. Razmyslov, Trace identities of full matrix algebras over a field of characteristic zero, Izv. Akad. Nauk SSSR 38 (1974) 723–756.
[14] A.I. Širšov, On rings with identity relations, Mat. Sb. (N.S.) 43 (85) (1957) 277–283.