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Blow-up of the energy at infinity for 2 by 2 systems

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ABSTRACT

The goal of this paper is to study the long time behavior of the energy of solutions to 2 by 2 linear hyperbolic systems. Some blow-up results at $t = \infty$ are given for a large class of initial data of the Cauchy problem. We shall prove the optimality of these results for a special class of systems. We present also an example of generalized energy conservation under a $\mathcal{C}^m$ stabilization condition.

1. Introduction

In the recent paper [1] the authors have considered the Cauchy problem for the strictly hyperbolic system

$$\partial_t U - \lambda(t)A(t)\partial_x U + B(t)U = 0, \qquad U(0,x) = U_0(x). \tag{1.1}$$

They developed an approach which gives information about

1. upper and lower bounds for the energy $\|U(t,\cdot)\|_{L^2}$,
2. results about *generalized energy conservation*, that is, the following *a priori* estimate holds

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$$C_1\|U_0\|_{L^2} \leqslant \left\|U(t,\cdot)\right\|_{L^2} \leqslant C_2\|U_0\|_{L^2}, \quad t \geqslant 0, \tag{1.2}$$

with positive constants C_1 and C_2 which are independent of U_0,

3. scattering results.

What remained open in [1] was an answer to the question if the *upper or lower bounds for a possible energy growth are sharp* (we refer the interested reader to [6] concerning the optimality for wave models). The main goal of this paper is to give an answer to this question.

We consider in $[0,\infty)\times\mathbb{R}$ the Cauchy problem for the 2×2 homogenous system

$$\partial_t U - \lambda(t)A(t)\partial_x U = 0, \qquad U(0,x) = U_0(x) \tag{1.3}$$

under the following basic assumptions:

Hypothesis 1. The matrix

$$A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$$

is real-valued, continuous, bounded, and uniformly strictly hyperbolic, that is, there exists a positive constant m_0 such that

$$\Delta(t) := \big(a(t)-d(t)\big)^2 + 4b(t)c(t) \geqslant m_0 > 0, \quad t \geqslant 0. \tag{1.4}$$

We denote $\|A\|_{L^\infty} := \sup_{t\geqslant 0}\|A(t)\|$.

Hypothesis 2. We assume that $\lambda \in \mathcal{C}^1([0,\infty))$ is real-valued, strictly positive and monotonic. Let

$$\Lambda(t) := 1 + \int_0^t \lambda(\tau)\,d\tau, \quad t \geqslant 0,$$

be a strictly positive primitive of λ. We assume that $\lim_{t\to\infty}\Lambda(t) = +\infty$ and that

$$\left|\lambda'(t)\right| \leqslant M_0\frac{\lambda^2(t)}{\Lambda(t)} \tag{1.5}$$

for some $M_0 \geqslant 0$.

Hypothesis 3. Let

$$\widetilde{A}(t) := A(t) - \frac{1}{2}\operatorname{tr} A(t) = \begin{pmatrix} \frac{a(t)-d(t)}{2} & b(t) \\ c(t) & \frac{d(t)-a(t)}{2} \end{pmatrix}. \tag{1.6}$$

We assume that $\widetilde{A} \in \mathcal{C}^2([0,\infty))$ and that

$$\left\|\widetilde{A}^{(k)}(t)\right\| \leqslant M_k\left(\frac{\lambda(t)}{\Lambda(t)}\right)^k, \quad t \geqslant 0,\ k = 1,2, \tag{1.7}$$

for some $M_1, M_2 \geqslant 0$.

We say that the oscillations in the entries of $\widetilde{A}(t)$ described in Hypothesis 3 are *very slow*. Nevertheless, Hypothesis 3 is not sufficient by itself to describe all the effects coming from the oscillations; indeed, the interaction of oscillations may lead to a blow-up result of the energy. In [1] it is proved that such a blow-up result can be excluded if one assumes together with Hypotheses 1 to 3, that there exists a constant $C \geqslant 0$ such that

$$\left| \int_0^t \Re\psi(\tau)\, d\tau \right| \leqslant C, \quad t \geqslant 0, \tag{1.8}$$

where the function $\psi(t)$, coming out from the diagonalization procedure of $A(t)$, is defined by

$$\psi(t) = \frac{(c - b - i\sqrt{\Delta})((a-d)(b+c)' - (a-d)'(b+c))}{2\sqrt{\Delta}((b+c)^2 + (a-d)^2)}. \tag{1.9}$$

In particular, it is proved that the *generalized energy conservation* property holds. In this paper we show how to obtain a blow-up result for the energy for a large class of initial data by assuming an integral condition for the function ψ. First, we present a special system for which we are able to give a very precise description of the energy behavior. Following Example 1.6 and Theorem 2.5 from [5] we consider the Cauchy problem (1.3) with $\lambda(t)$ satisfying Hypothesis 2 and

$$A(t) = \begin{pmatrix} -\cos\omega(t) & \sin\omega(t) + 1/\sqrt{2} \\ \sin\omega(t) - 1/\sqrt{2} & \cos\omega(t) \end{pmatrix}; \tag{1.10}$$

it follows

$$a(t) - d(t) = -2\cos\omega(t), \qquad b(t) - c(t) = \sqrt{2}, \qquad b(t) + c(t) = 2\sin\omega(t),$$
$$\Delta(t) = 2, \qquad \Re\psi(t) = \omega'(t)/2.$$

It is clear that Hypothesis 1 is verified. Now let

$$\omega(t) = \ell^r(t)\bigl(2 - \cos\ell^{1-r}(t)\bigr) \quad \text{for some } r \in (0,1), \tag{1.11}$$

where $\ell(t) := \log(\Lambda(t))$. We write $\Re\psi := \varphi_1 + \varphi_2$, where

$$\varphi_1(t) = -\frac{r}{2}\ell'(t)\ell^{-(1-r)}(t)\bigl(2 - \cos\ell^{1-r}(t)\bigr) \text{ is negative}$$

and

$$\varphi_2(t) = -\frac{1-r}{2}\ell'(t)\sin\ell^{1-r}(t) \text{ has an oscillating sign.}$$

Hypothesis 3 is satisfied since $\ell'(t) = \lambda(t)/\Lambda(t)$. Obviously,

$$\frac{1}{2}\ell^r(t) \leqslant \int_0^t \Re\psi(\tau)\, d\tau = \frac{1}{2}\omega(t) \leqslant \frac{3}{2}\ell^r(t). \tag{1.12}$$

Moreover, we can prove (see Section 3) that

$$\int_0^t \left|\Re\psi(\tau)\right| d\tau \approx \ell(t). \tag{1.13}$$

Theorem 1. *If we choose the matrix $A(t)$ as in* (1.10), (1.11) *together with Hypothesis* 2, *then the solution of the Cauchy problem* (1.3) *satisfies the following two-sided estimate for any initial datum $U_0 \in M_+(N)$, where $M_+(N)$ is a set equipotent to $L^2(\mathbb{C})$ (see later, Definition* 1*), and for any $t \geqslant T(U_0)$, sufficiently large*:

$$C' \exp\bigl(\ell^r(t)/2\bigr)\|U_0\|_{L^2} \leqslant \bigl\|U(t,\cdot)\bigr\|_{L^2} \leqslant C_0 \exp\bigl(C_1 \ell^r(t)\bigr)\|U_0\|_{L^2}.$$

The proof of this result is divided into two parts: the estimate from above will be carried out thanks to the special structure of system (1.10) and it is proved in Theorem 2, whereas the estimate from below is obtained from a blow-up result for more general systems in Theorem 3 for initial data in the set $M_\pm(N)$ which will be introduced in Definition 1.

For the oscillating behavior from Hypothesis 3 we can allow faster oscillations for the entries of $\widetilde{A}(t)$. In such a case we will replace Hypothesis 3 with the following one:

Hypothesis 4. We assume that $\widetilde{A}(t) \in \mathcal{C}^2$ in (1.6) satisfies

$$\bigl\|\widetilde{A}^{(k)}(t)\bigr\| \leqslant M_k \left(\frac{\lambda(t)\nu(t)}{\Lambda(t)}\right)^k, \quad t \geqslant 0,\ k = 1, 2, \tag{1.14}$$

for some $M_1, M_2 \geqslant 0$, where $\nu(t) \in \mathcal{C}^1$ is a real-valued strictly positive function such that

$$\lim_{t\to\infty} \nu(t) = +\infty, \qquad \nu(t) = o\bigl(\Lambda(t)\bigr) \quad \text{as } t \to \infty, \tag{1.15}$$

$$0 \leqslant \nu'(t) \leqslant \delta \frac{\lambda(t)\nu(t)}{\Lambda(t)}, \quad t \geqslant 0,\ \text{for some } \delta \in (0, 1/2). \tag{1.16}$$

We say that the oscillations in the entries of $\widetilde{A}(t)$ which are described in Hypothesis 4 are *not very slow*.

Remark 1.1. If (1.16) holds for $\nu(t)$, then $0 \leqslant \nu'/\nu \leqslant \delta\lambda/\Lambda$, therefore $\log(\nu(t)/\nu(0)) \leqslant \delta \log \Lambda(t)$, that is,

$$\nu(t) \leqslant \bigl(\Lambda(t)\bigr)^\delta \nu(0). \tag{1.17}$$

Example 1.2. The function $\nu(t) = (\log(\Lambda(t) + c_\gamma))^\gamma$ satisfies (1.15), (1.16) for any $\gamma > 0$ and for a suitable constant $c_\gamma > 0$ depending on γ. Indeed,

$$\nu'(t) = \gamma\bigl(\log\bigl(\Lambda(t) + c_\gamma\bigr)\bigr)^{\gamma-1} \frac{\lambda(t)}{\Lambda(t) + c_\gamma},$$

hence (1.16) holds provided that $c_\gamma > e^{2\gamma} - 1$, that is,

$$\frac{\gamma}{\log(1 + c_\gamma)} < \frac{1 + c_\gamma}{2}.$$

Example 1.3. The function $\nu(t) = (\Lambda(t))^\delta$ with $\delta < 1/2$ satisfies (1.15), (1.16).

2. Main results

2.1. Energy estimates for the special system (1.3), (1.10)

We consider the system (1.3), (1.10) and we allow *oscillations which are not very slow*, that is, we replace (1.11) by

$$\omega(t) = \ell^r(t)\big(2 - \cos \ell^p(t)\big), \quad r \in (0, 1),\ p \geqslant 1 - r. \tag{2.1}$$

We define $\gamma := r + p - 1$ and we remark that $\gamma \geqslant 0$. Let $\ell(t) := \log(\Lambda(t) + c_\gamma)$ with $c_\gamma > e^{2\gamma} - 1$ as in Example 1.2. We write $\Re\psi := \varphi_1 + \varphi_2$, where

$$\varphi_1(t) = -\frac{r}{2}\ell'(t)\ell^{r-1}(t)\big(2 - \cos \ell^p(t)\big) \text{ is negative}$$

and

$$\varphi_2(t) = -\frac{p}{2}\ell'(t)\ell^\gamma(t)\sin \ell^p(t) \text{ has an oscillating sign.}$$

Hypothesis 3, that is, (1.11) corresponds to $\gamma = 0$. Hypothesis 4 corresponds to $\nu(t) = \ell^\gamma(t)$ for $\gamma > 0$ as in Example 1.2. It is clear that (1.12) still holds. Moreover, we can prove (see Section 3) that

$$\int_0^t \big|\Re\psi(\tau)\big|\, d\tau \approx \ell^{p+r}(t) = \ell^{\gamma+1}(t). \tag{2.2}$$

We are able to derive the following *a priori* estimate for the solution U.

Theorem 2. *We assume* (1.10) *and* (2.1)*. Let $p \leqslant 2(1 - r)$, that is, $\gamma \leqslant 1 - r$. Then there exist two constants $C_0, C_1 \geqslant 0$ such that the solution of the Cauchy problem* (1.3) *satisfies the following estimate*:

$$\big\|U(t,\cdot)\big\|_{L^2} \leqslant C_0 \exp\Big(C_1\big(\log\big(\Lambda(t) + c_\gamma\big)\big)^{\gamma+r}\Big)\|U_0\|_{L^2}, \quad t \geqslant 0. \tag{2.3}$$

In particular,

- *if $\gamma + r < 1$, then for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$\big\|U(t,\cdot)\big\|_{L^2} \leqslant C_\varepsilon\big(\Lambda(t) + c_\gamma\big)^\varepsilon\|U_0\|_{L^2}, \quad t \geqslant 0;$$

- *if $\gamma + r = 1$, then we have*

$$\big\|U(t,\cdot)\big\|_{L^2} \leqslant C_0\big(\Lambda(t) + c_\gamma\big)^{C_1}\|U_0\|_{L^2}, \quad t \geqslant 0.$$

We remark that this result for the special system (1.3), (1.10) is more precise than the statements of Theorem 4 in [1] for more general systems. Applying Theorem 4 from [1] implies

$$\big\|U(t,\cdot)\big\|_{L^2} \leqslant C_0 \exp\Big(C_1\big(\log\big(\Lambda(t) + c_\gamma\big)\big)^{\gamma+1}\Big)\|U_0\|_{L^2}, \quad t \geqslant 0.$$

2.2. Blow-up for systems with very slow oscillations

The basic strategy in this paper relies into considering the Fourier transform of (1.3) with respect to x, that is,

$$\partial_t \widehat{U}(t,\xi) = i\xi\lambda(t)A(t)\widehat{U}(t,\xi), \qquad \widehat{U}(0,\xi) = \widehat{U_0}(\xi) \tag{2.4}$$

estimating its fundamental solution $E(t,s,\xi)$ with a different approach in the *pseudo-differential zone* and the *hyperbolic zone*, a suitable division of the extended phase space in (t,ξ). In the first zone we will estimate $E(t,s,\xi)$ by a direct way, whereas in the hyperbolic zone we are going to use a diagonalization procedure. Indeed, thanks to Hypothesis 1 we are able to find a diagonalizer $H(t)$ for the matrix $A(t)$, namely

$$H^{-1}(t)A(t)H(t) = \begin{pmatrix} \mu_+(t)+d(t) & 0 \\ 0 & \mu_-(t)+d(t) \end{pmatrix}, \quad t \geqslant 0,$$

where

$$\mu_\pm(t) := \frac{a(t)-d(t) \pm \sqrt{\Delta(t)}}{2},$$

such that $H(t)$ is bounded and uniformly regular. Following [5] we define

$$H(t) := (1+i)\begin{pmatrix} b(t) & \mu_-(t) \\ -\mu_-(t) & c(t) \end{pmatrix} + (1-i)\begin{pmatrix} \mu_+(t) & b(t) \\ c(t) & -\mu_+(t) \end{pmatrix}, \tag{2.5}$$

and we remark that $|\det H(t)| \geqslant 2m_0 > 0$ with m_0 as in (1.4). Since

$$\det H(t) = 2\sqrt{\Delta(t)}\big(c(t) - b(t) + i\sqrt{\Delta(t)}\big);$$

after replacing $U(t,x) = (\det H(t))^{-\frac{1}{2}} H(t) U^\#(t,x)$ the system in (1.3) is equivalent to

$$\partial_t U^\# - \lambda(t)\begin{pmatrix} \mu_+ + d & 0 \\ 0 & \mu_- + d \end{pmatrix}\partial_x U^\# + \psi(t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}U^\# + \begin{pmatrix} 0 & h_+(t) \\ h_-(t) & 0 \end{pmatrix}U^\# = 0, \tag{2.6}$$

where $\psi(t)$ is as in (1.9), and

$$h_\pm(t) = \frac{\det H(t)}{2\Delta(t)}\left(\frac{\sqrt{\Delta(t)}(i(d(t)-a(t)) \pm (b(t)+c(t)))}{\det H(t)}\right)'. \tag{2.7}$$

Thanks to Hypothesis 3 we have

$$\frac{|h_\pm(t)|}{|\xi|\lambda(t)\sqrt{\Delta(t)}} \leqslant \frac{M_3}{|\xi|\Lambda(t)} \tag{2.8}$$

for some $M_3 \geqslant 0$ depending only on the constants m_0 in (1.4) and M_1 in (1.7). In correspondence to some positive N, with $N > 2M_3$, we define

$$t_{|\xi|} = \begin{cases} \Lambda^{-1}(N/|\xi|) & \text{if } |\xi| \leqslant N, \\ 0 & \text{otherwise,} \end{cases} \tag{2.9}$$

and

$$Z_{\mathrm{pd}}(N) = \{t \leqslant t_{|\xi|}\}, \qquad Z_{\mathrm{hyp}}(N) = \{t \geqslant t_{|\xi|}\}. \tag{2.10}$$

In $Z_{\mathrm{hyp}}(N)$ we introduce the *refined diagonalizer*

$$K(t,\xi) := \begin{pmatrix} 1 & \frac{h_+(t)}{i\xi\lambda(t)\sqrt{\Delta(t)}} \\ -\frac{h_-(t)}{i\xi\lambda(t)\sqrt{\Delta(t)}} & 1 \end{pmatrix}, \tag{2.11}$$

and, from (2.8), we derive $|\det K(t,\xi)| \geqslant 3/4$ and $\|K(t,\xi)\| = 1$, that is, $K(t,\xi)$ is uniformly regular and bounded. Via the change of variables

$$\widehat{U}(t,\xi) = K(t,\xi)\widehat{U^{\#}}(t,\xi) = K(t,\xi)\bigl(\det H(t)\bigr)^{-1/2} H(t)W(t,\xi) \tag{2.12}$$

the system in (2.4) is in $Z_{\mathrm{hyp}}(N)$ equivalent to

$$\partial_t W - \begin{pmatrix} \varphi_+(t,\xi) & 0 \\ 0 & \varphi_-(t,\xi) \end{pmatrix} iW + \Re\psi(t) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} W + J(t,\xi)W = 0, \tag{2.13}$$

where

$$\varphi_\pm(t,\xi) = \bigl(\mu_\pm(t) + d(t)\bigr)\lambda(t)\xi \pm \Im\psi(t)$$

are real-valued and the matrix $J(t,\xi)$ satisfies the following estimate:

$$\bigl\| J(t,\xi) \bigr\| \leqslant \frac{M_4\lambda(t)}{|\xi|\Lambda^2(t)} \tag{2.14}$$

for some $M_4 \geqslant 0$ that depends only on the constants m_0 in (1.4) and M_k for $k = 0,1,2$ in (1.5) and (1.7).

Definition 1. Let $N > 2M_3$. We define by $M_+(N)$ (resp. $M_-(N)$) the set of initial data $U_0 \in L^2(\mathbb{R},\mathbb{C}^2)$ such that the solution $U(t,x)$ of (1.3) with initial datum U_0 verifies

$$\widehat{U}(t_{|\xi|},\xi) = K(t_{|\xi|},\xi)\bigl(\det H(t_{|\xi|})\bigr)^{-\frac{1}{2}} H(t_{|\xi|})Y(\xi), \tag{2.15}$$

where $Y(\xi) = (y(\xi),0)$ (resp. $Y(\xi) = (0,y(\xi))$) for some $y \in L^2(\mathbb{R},\mathbb{C})$.

It is clear that $M_+(N)$ and $M_-(N)$ are equipotent to $L^2(\mathbb{R},\mathbb{C})$ thanks to the well-posedness of the Cauchy problem (1.3). We are now in a position to estimate from below the blow-up rate of the energy by using a time-dependent increasing function.

Theorem 3. *We assume Hypotheses* 1 *to* 3*. We assume that the function* $t \mapsto \int_0^t \Re\psi(\tau)\,d\tau$ *has a constant sign and it satisfies*

$$m_1\nu_1(t) \leqslant \int_0^t \Re\psi(\tau)\,d\tau \leqslant m_2\nu_1(t) \quad \left(\textit{resp.}\ -m_2\nu_1(t) \leqslant \int_0^t \Re\psi(\tau)\,d\tau \leqslant -m_1\nu_1(t)\right) \tag{2.16}$$

for some positive constants m_1, m_2, *where* $\nu_1(t)$ *is a strictly positive, increasing function with* $\lim_{t\to\infty}\nu_1(t) = +\infty$. *Moreover, we assume that there exists a function* $\theta(t)$ *such that*

$$\left|\int_s^t \theta(\tau)\,d\tau\right| \leqslant M_5, \qquad \Re\psi(t)+\theta(t)\geqslant 0 \quad \bigl(\textit{resp.}\ \Re\psi(t)+\theta(t)\leqslant 0\bigr) \tag{2.17}$$

for some constant $M_5 \geqslant 0$. *Let* $N > \overline{N}$, *where*

$$\overline{N} = \max\bigl\{2M_3, 4M_4 e^{2M_5+1}\bigr\}, \tag{2.18}$$

and M_3, M_4 *are from* (2.8) *and* (2.14). *Then there exists a positive constant* C *such that for any initial datum* $U_0 \in M_+(N)$ (*resp.* $U_0 \in M_-(N)$) *we can find a constant* $T(U_0)\geqslant 0$ *such that the solution* $U(t,x)$ *of* (1.3) *with initial datum* U_0 *satisfies*

$$\bigl\|U(t,\cdot)\bigr\|_{L^2} \geqslant C\exp\bigl(m_1\nu_1(t) - m_2\nu_1\bigl(T(U_0)\bigr)\bigr)\|U_0\|_{L^2} \quad \textit{for } t\geqslant T(U_0). \tag{2.19}$$

We remark that C *is independent of* U_0.

By using Hypothesis 3 to estimate $\Re\psi(t)$ we can directly check that the function $\nu_1(t)$ is bounded from above by $c\log\Lambda(t)$ for some $c>0$.

Remark 2.1. If we fix $\kappa>0$, and for any $U_0\in M_\pm(N)$ we choose $\epsilon=\epsilon(\kappa)>0$ such that the corresponding function y (see Definition 1) verifies

$$\int_{|\xi|\geqslant\epsilon} \bigl|y(\xi)\bigr|^2\,d\xi = \kappa \int_{|\xi|\leqslant\epsilon} \bigl|y(\xi)\bigr|^2\,d\xi, \tag{2.20}$$

then we can take $T(U_0)=t_\epsilon$ in (2.19) in Theorem 3 with t_ϵ as in (2.9).

Remark 2.2. If (2.17) holds true, then $\Re\psi$ satisfies

$$\int_s^t \Re\psi(\tau)\,d\tau \geqslant -M_5 \quad \left(\text{resp.} \int_s^t \Re\psi(\tau)\,d\tau\leqslant M_5\right), \quad t\geqslant s\geqslant 0. \tag{2.21}$$

Remark 2.3. In order to construct the function θ we remark that (2.17) is satisfied if there exist a function $\theta(t)$ such that $\Re\psi+\theta\geqslant 0$ (resp. $\Re\psi+\theta\leqslant 0$) and a strictly increasing sequence $\{t_j\}_{j\geqslant 0}$ with $t_1=0$ and $t_j\to\infty$ such that

$$\int_{t_{2k-1}}^{t_{2k+1}} \theta(\tau)\,d\tau = 0, \qquad \left|\int_{t_{2k-1}}^{t} \theta(\tau)\,d\tau\right| \leqslant M_5, \quad t\in(t_{2k-1},t_{2k+1}).$$

2.3. Blow-up with oscillations which are not very slow

In the context of oscillations which are not very slow we divide the extended phase space by using a function $t_{|\xi|}$ that is different from the one in (2.9).

Definition 2. If Hypothesis 4 holds, then the $\mathcal{C}^1$ function

$$\Theta : [0,\infty) \to \big[1/\nu(0), +\infty\big), \quad \Theta(t) = \frac{\Lambda(t)}{\nu(t)},$$

is strictly increasing since, by virtue of (1.16), we have

$$\Theta'(t) = \frac{\lambda(t)\nu(t) - \Lambda(t)\nu'(t)}{\nu^2(t)} \geqslant (1-\delta)\frac{\lambda(t)}{\nu(t)} > 0.$$

We remark that $\lim_{t\to\infty} \Theta(t) = +\infty$ thanks to (1.15), that is, Θ is invertible and $\Theta(t) = o(\Lambda(t))$ as $t \to \infty$.

Analogously to (2.8) from Hypothesis 4 it follows

$$\frac{|h_\pm(t)|}{|\xi|\lambda(t)\sqrt{\Delta(t)}} \leqslant \frac{M_3\nu(t)}{|\xi|\Lambda(t)} \tag{2.22}$$

for some $M_3 \geqslant 0$ depending only on the constants m_0 in (1.4) and M_1 in (1.14). In correspondence to any $N > 0$ with $N > 2M_3$ we define

$$t_{|\xi|} = \begin{cases} \Theta^{-1}(N/|\xi|) & \text{if } |\xi| \leqslant N\nu(0), \\ 0 & \text{otherwise}, \end{cases} \tag{2.23}$$

with $\Theta(t)$ as in Definition 2 and $Z_{\mathrm{pd}}(N), Z_{\mathrm{hyp}}(N)$ as in (2.10). As in the case for *very slow oscillations* the *refined diagonalizer* $K(t,\xi)$ in (2.11) is in $Z_{\mathrm{hyp}}(N)$ uniformly regular with $|\det K(t,\xi)| \geqslant 3/4$ and bounded with $\|K(t,\xi)\| = 1$. After the change of variables (2.12) the system in (2.4) is equivalent to (2.13) in $Z_{\mathrm{hyp}}(N)$, where the matrix $J(t,\xi)$ satisfies the following estimate:

$$\big\|J(t,\xi)\big\| \leqslant \frac{M_4\lambda(t)\nu^2(t)}{|\xi|\Lambda^2(t)} \tag{2.24}$$

for some $M_4 \geqslant 0$ that depends only on the constants m_0 in (1.4) and M_k for $k = 0,1,2$ in (1.5) and (1.14).

Lemma 2.1. *Let $\nu(t)$ be as in Hypothesis 4 and let $\epsilon > 0$. For any constant $M > 0$ there exists a constant $N(\epsilon, M)$ such that*

$$N \geqslant M\nu(t_\epsilon) \quad \textit{for any } N \geqslant N(\epsilon, M). \tag{2.25}$$

In fact, we remark that t_ϵ depends both on ϵ and N. With the notation from Lemma 2.1 we put

$$N_\epsilon = \max\big\{2M_3, N\big(\epsilon, 4M_4\exp(2M_5+1)\big)\big\}, \tag{2.26}$$

where M_3, M_4 and M_5 are as in (2.22), (2.24), and (2.17).

We are now in a position to derive some blow-up results for the energy in the case of oscillations which are *not very slow*. Our philosophy is that we are going to replace the classical estimate that involves the L^2 norm of the data and of the solution by some inequalities that compare the L^2 norm of the solution with some suitable behavior in the phase space of the initial data. In Theorem 4 we estimate from below the L^2 norm of the solution by the weighted norm of the initial data which is introduced in the following definition.

Definition 3. Let $g : (0, +\infty) \to (-\infty, 0)$ be a continuous increasing function with $g(\rho) \to -\infty$ as $\rho \to 0$. We define the following weighted norm on L^2:

$$\|U\|_g^2 := \int_{\mathbb{R}} \exp\bigl(2g\bigl(|\xi|\bigr)\bigr)\bigl|\widehat{U}(\xi)\bigr|^2\, d\xi.$$

By Plancherel's theorem, the norm $\|\cdot\|_g$ is weaker than the usual norm $\|\cdot\|_{L^2}$, since $\|U\|_g \leqslant \|\widehat{U}\|_{L^2}$ for any $U \in L^2$. On the contrary, we can easily prove that $\|\cdot\|_g$ is not equivalent to $\|\cdot\|_{L^2}$, that is, for any $C > 0$ there exists $U \in L^2$ such that $\|U\|_{L^2} \geqslant C\|U\|_g$. In particular, this implies that $(L^2, \|\cdot\|_g)$ is not complete.

As an example we propose $g(|\xi|) = s\log(|\xi|/\langle\xi\rangle)$. Then $\|U\|_g^2$ is equivalent to

$$\int_{|\xi|\leqslant 1} |\xi|^{2s}\bigl|\widehat{U}(\xi)\bigr|^2\, d\xi + \int_{|\xi|\geqslant 1} \bigl|\widehat{U}(\xi)\bigr|^2\, d\xi,$$

that is, the natural norm on the space

$$\chi(D_x)\dot{H}^s + \bigl(1 - \chi(D_x)\bigr)L^2,$$

where χ is chosen as usually as a smooth cut-off function localizing near small frequencies.

Theorem 4. *We assume Hypotheses* 1, 2 *and* 4. *Moreover, we assume* (2.16), (2.17) *and that* $\nu(t) = o(\nu_1(t))$ *as* $t \to \infty$. *We fix* $\kappa > 0$. *Let* $y \in L^2(\mathbb{R}, \mathbb{C})$ *and let* $\epsilon > 0$ *be such that* (2.20) *is satisfied. Let* N_ϵ *be as in* (2.26) *and let* $N \geqslant N_\epsilon$. *Let* $t_{|\xi|}$ *be as in* (2.23) *and let* $U_0 \in L^2(\mathbb{R}, \mathbb{C}^2)$ *be such that the solution* $U(t, x)$ *of* (1.3) *with initial datum* U_0 *satisfies* (2.15). *Finally, we define*

$$g\bigl(|\xi|\bigr) := -N\nu(t_{|\xi|})\|A\|_{L^\infty}.$$

Then the solution of (1.3) *with initial datum* U_0 *fulfills*

$$\bigl\|U(t, \cdot)\bigr\|_{L^2} \geqslant C\exp\bigl(m_1\nu_1(t) - m_2\nu_1(t_\epsilon)\bigr)\|U_0\|_g, \quad t \geqslant t_\epsilon, \tag{2.27}$$

where m_1 *and* m_2 *are as in* (2.16) *and* C *is a constant that is independent of* y.

In the next Theorem 5 we shall use for the initial data a weighted L^2 norm, but now the weight depends on t itself. Therefore, we define for a fixed $N > N_\epsilon$ the function ρ_t to be the inverse function of t_ρ for $\rho \in (0, N\nu(0)]$, namely,

$$\rho_t = \frac{N}{\Theta(t)} = \frac{N\nu(t)}{\Lambda(t)} \in \bigl(0, N\nu(0)\bigr] \quad \text{with } \rho_t \to 0 \text{ for } t \to \infty. \tag{2.28}$$

Definition 4. Let $h:[0,+\infty)\to[0,+\infty)$ be a continuous increasing function with $h(t)\to+\infty$ as $t\to\infty$. For any $t\in[0,\infty)$ we define the weighted L^2-norm

$$\|U\|_{t,h}^2 := \int_{|\xi|\geqslant\rho_t} \big|\widehat{U}(\xi)\big|^2\,d\xi + \exp\big(-2h(t)\big)\int_{|\xi|\leqslant\rho_t} \big|\widehat{U}(\xi)\big|^2\,d\xi, \quad U\in L^2. \tag{2.29}$$

It is clear that for any $U\in L^2$ it holds $\|U\|_{t,h}\to\|\widehat{U}\|_{L^2}$ as $t\to\infty$ as a pointwise-limit. For this reason $\|U_0\|_{t,h}$ is not equivalent to $\|U_0\|_g$. In particular, for a fixed t the norm $\|\cdot\|_{t,h}$ is equivalent to the usual norm $\|\cdot\|_{L^2}$, since for any $U\in L^2$ it holds

$$e^{-h(t)}\|\widehat{U}\|_{L^2}\leqslant\|U\|_{t,h}\leqslant\|\widehat{U}\|_{L^2},$$

but the lower bound is not uniform with respect to t.

Theorem 5. *We assume Hypotheses* 1, 2 *and* 4. *We fix $\epsilon>0$. Moreover, we assume* (2.16) *and* (2.17) *and that $\nu(t)=o(\nu_1(t))$ as $t\to\infty$. Let $N\geqslant N_\epsilon$ with N_ϵ as in* (2.26). *Let $t_{|\xi|}$ be as in* (2.23) *and ρ_t be as in* (2.28). *Then there exists a constant $C>0$ such that for any initial data $U_0\in M_+(N)$ (resp. $U_0\in M_-(N)$), the solution $U(t,x)$ of* (1.3) *with initial data U_0 satisfies*

$$\big\|U(t,\cdot)\big\|_{L^2}\geqslant C\exp\big(m_1\nu_1(t)-N\nu(t)\|A\|_{L^\infty}-m_2\nu_1(t_\epsilon)\big)\|U_0\|_{t,h}, \quad t\geqslant t_\epsilon, \tag{2.30}$$

where m_1 and m_2 are as in (2.16), *the constant $C>0$ is independent of ϵ, and $h(t)=m_1\nu_1(t)$.*

Remark 2.4. Theorem 5 is written in a non-standard form in comparison with Theorem 4. In both cases, the difficulty arising with oscillations which are *not very slow* is managed by using a weighted norm for the initial datum U_0. The weight itself depends on the speed of the oscillations. It is clear that if $\widehat{U_0}(\xi)$ is more concentrated in a small neighborhood of $\xi=0$, namely in the ball $B_{2\epsilon}(0)$ for small $\epsilon>0$, then the estimate of $\|U(t,\cdot)\|_{L^2}$ is worst and it holds only for large time.

Let $\epsilon>0$. We remark that in Theorem 4, for any $y\in L^2(\mathbb{R},\mathbb{C})$ such that

$$\operatorname{supp} y\subset B_{2\epsilon}(0), \qquad \int_{\epsilon\leqslant|\xi|\leqslant2\epsilon}\big|y(\xi)\big|^2\,d\xi\neq0,$$

we can choose an initial datum U_0 with $\operatorname{supp}\widehat{U_0}\subset B_{2\epsilon}(0)$, such that (2.27) holds for $t\geqslant t_\epsilon$. On the other hand, in Theorem 5 we can state (2.30) for any $U_0\in M_\pm(N)$, that is, $\operatorname{supp}\widehat{U_0}$ can be arbitrarily small. Nevertheless, the estimate (2.30) is non-trivial only for large t with respect to the radius of $\operatorname{supp}\widehat{U_0}$, due to the definition of $\|\cdot\|_{t,h}$ given in (2.29). Indeed, if $\operatorname{supp}\widehat{U_0}\subset B_{\epsilon_1}(0)$ for some $\epsilon_1\in(0,\epsilon)$, then for any $t\in[t_\epsilon,t_{\epsilon_1}]$, it holds

$$\exp\big(m_1\nu_1(t)\big)\|U_0\|_{t,h}=\|\widehat{U_0}\|_{L^2}.$$

In facts, the estimate (2.30) allows a more general statement.

3. Proof of the energy estimate in Theorem 2

First we prove (2.2) (we recall that (1.13) is the special case with $\gamma=0$). We can directly check that

$$\int_0^t \left|\Re\psi(\tau)\right| d\tau \leqslant C \int_0^t \ell'(\tau)\ell^\gamma(\tau)\, d\tau \leqslant C'\ell^{\gamma+1}(t).$$

To prove the estimate from below we define the sequence $\{t_j\}_{j\geqslant j_0}$ (with $j_0 := 2k_0 - 1$ large enough to make the sequence well defined) by

$$\ell^p(t_j) = j\pi/2 + 3\pi/4.$$

After using the change of variables

$$\sigma = \ell^p(\tau), \qquad \frac{d\sigma}{d\tau} = p\ell^{p-1}(\tau)\ell'(\tau) \tag{3.1}$$

we get for any $k \geqslant k_0$

$$p \int_{t_{2k-1}}^{t_{2k}} \ell'(\tau)\ell^{r+p-1}(\tau)\sin \ell^p(\tau)\, d\tau = \int_{k\pi+\pi/4}^{k\pi+3\pi/4} \sigma^{r/p}\sin\sigma\, d\sigma \approx (-1)^k (k\pi)^{r/p}.$$

Therefore, by using $-\varphi_1 \leqslant c|\varphi_2|$ in $[t_{2k-1}, t_{2k}]$ for any k we conclude for any $t \geqslant t_{j_0}$

$$\begin{aligned}\int_0^t \left|\Re\psi(\tau)\right| d\tau &\geqslant c\sum_{k=k_0}^{l} \int_{t_{2k-1}}^{t_{2k}} (-1)^k\varphi_2(\tau)\, d\tau \geqslant c_1 \sum_{k=k_0}^{l} (k\pi)^{r/p} \\ &\geqslant c_2 (l\pi)^{1+r/p} \approx \ell^{p+r}(t_{2l}) \approx \ell^{p+r}(t),\end{aligned}$$

where $t \in [t_{2l}, t_{2(l+1)})$. This concludes the proof of (2.2).

Now we shall prove Theorem 2. First we prepare some integral estimates. Let $\ell^p(t_j) = j\pi$, that is,

$$t_j := \Lambda^{-1}\big(\exp\big((j\pi)^{1/p}\big) - c_\gamma\big) \nearrow \infty, \quad j \geqslant j_0,$$

with $j_0 = j_0(p, c_\gamma) = 2k_0 - 1$ sufficiently large to be well defined. It is clear that $\varphi_2(t_j) = 0$ and that φ_2 is strictly positive (resp. negative) for $t \in (t_{2k-1}, t_{2k})$ (resp. $t \in (t_{2k}, t_{2k+1})$) with $k \geqslant k_0$. Moreover, by using (3.1) we derive

$$\int_{t_{2k-1}}^{t_{2k+1}} \varphi_2(\tau)\, d\tau = -\frac{1}{2} \int_{(2k-1)\pi}^{(2k+1)\pi} \sigma^{r/p}\sin\sigma\, d\sigma < 0. \tag{3.2}$$

We define

$$I_j = \|\varphi_2\|_{L^1(t_{j-1},t_j)} = \int_{t_{j-1}}^{t_j} (-1)^j \varphi_2(\tau)\, d\tau,$$

$$\theta^*(t) = -\varphi_2(t)\frac{I_{2k} + I_{2k+1}}{2} \times \begin{cases} 1/I_{2k}, & t \in [t_{2k-1}, t_{2k}], \\ 1/I_{2k+1}, & t \in [t_{2k}, t_{2k+1}]. \end{cases}$$

By (3.2) it holds $I_{2k} < I_{2k+1}$. From

$$\int_{t_{2k-1}}^{t_{2k+1}} \varphi_2(\tau)\,d\tau = I_{2k} - I_{2k-1} < 0$$

it can be proved that

$$0 > \varphi_2(t) + \theta^*(t) = \left|\varphi_2(t)\right| \frac{1}{2} \int_{t_{2k-1}}^{t_{2k+1}} \varphi_2(\tau)\,d\tau \times \begin{cases} 1/I_{2k}, & t \in [t_{2k-1}, t_{2k}], \\ 1/I_{2k+1}, & t \in [t_{2k}, t_{2k+1}]. \end{cases} \tag{3.3}$$

Indeed, let $t \in [t_{2k-1}, t_{2k}]$. Then we have $\varphi_2(t) = |\varphi_2(t)|$ and

$$\varphi_2(t) + \theta^*(t) = \varphi_2(t)\left(1 - \frac{I_{2k} + I_{2k+1}}{2I_{2k}}\right) = \varphi_2(t) \frac{1}{2I_{2k}} \int_{t_{2k-1}}^{t_{2k+1}} \varphi_2(\tau)\,d\tau.$$

Analogously we prove (3.3) for $t \in [t_{2k}, t_{2k+1}]$, where $\varphi_2(t) = -|\varphi_2(t)|$. By using (3.2) we conclude from (3.3) that

$$\int_{t_{2k-1}}^{t_{2k+1}} \left|\varphi_2(\tau) + \theta^*(\tau)\right| = -\int_{t_{2k-1}}^{t_{2k+1}} \varphi_2(\tau)\,d\tau = \frac{1}{2} \int_{(2k-1)\pi}^{(2k+1)\pi} \sigma^{r/p} \sin\sigma\,d\sigma. \tag{3.4}$$

Remark 3.1. The role of the function $\theta^*(t)$ is quite similar to the one of the function $\theta(t)$ that appears in Theorems 3, 4, 5 in the setting of blow-up results. Indeed, here we also have

$$\Re\psi + \theta^* = \varphi_1 + \varphi_2 + \theta^* < \varphi_1 \leqslant 0.$$

Nevertheless, we do not have an integral estimate on $\Re\psi$ as the one in (2.16) (which we need to prove blow-up). In fact, estimate (1.12) describes exactly such an integral behavior.

Lemma 3.1. *For any $k \geqslant k_0$ we have*

$$\int_{t_{2k-1}}^{t_{2k+1}} \theta^*(\tau)\,d\tau = 0, \tag{3.5}$$

$$\int_{t_{2k-1}}^{t_{2k+1}} \left|\varphi_2(\tau)\right|d\tau \leqslant C\ell^r(t_{2k-1}), \tag{3.6}$$

and for any $s \leqslant t$ it holds

$$\int_s^t \left|\varphi_2(\tau)+\theta^*(\tau)\right| d\tau \leqslant C\ell^r(t), \tag{3.7}$$

$$\left|\int_s^t \theta^*(\tau)\, d\tau\right| \leqslant 2M_6\ell^r(t). \tag{3.8}$$

Proof. The proof of (3.5) is straight-forward. Indeed,

$$\int_{t_{2k-1}}^{t_{2k+1}} \theta^*(\tau)\, d\tau = -\int_{t_{2k-1}}^{t_{2k}} \varphi_2(\tau)\frac{I_{2k}+I_{2k+1}}{2I_{2k}}\, d\tau - \int_{t_{2k}}^{t_{2k+1}} \varphi_2(\tau)\frac{I_{2k}+I_{2k+1}}{2I_{2k+1}}\, d\tau = 0.$$

By using (3.1) we can prove (3.6) since

$$\begin{aligned}\int_{t_{2k-1}}^{t_{2k+1}} \left|\varphi_2(\tau)\right| d\tau &\leqslant \int_{t_{2k-1}}^{t_{2k+1}} \frac{p}{2}\ell'(\tau)\ell^{r+p-1}(\tau)\, d\tau = \frac{1}{2}\int_{(2k-1)\pi}^{(2k+1)\pi} \sigma^{r/p}\, d\sigma \\ &= \frac{1+r/p}{2}\Big(\big((2k+1)\pi\big)^{1+r/p} - \big((2k-1)\pi\big)^{1+r/p}\Big) \approx (1+r/p)\pi\big((2k+1)\pi\big)^{r/p}.\end{aligned}$$

To derive (3.7) it is sufficient to prove that

$$\int_{t_{2k_0-1}}^{t_{2k+1}} \left|\varphi_2(\tau)+\theta^*(\tau)\right| d\tau \leqslant \ell^r(t_{2k+1})$$

since $\ell^r(t) \approx \ell^r(t_{2k+1})$ for $t \in [t_{2k-1}, t_{2k+1}]$. By using (3.4) we obtain

$$\int_s^t \left|\varphi_2(\tau)+\theta^*(\tau)\right| d\tau \leqslant \int_{t_{2k_0-1}}^{t_{2k+1}} \left|\varphi_2(\tau)+\theta^*(\tau)\right| d\tau = \frac{1}{2}\int_{(2k_0-1)\pi}^{(2k+1)\pi} \sigma^{r/p}\sin\sigma\, d\sigma.$$

After integrating by parts twice we get

$$\begin{aligned}\frac{1}{2}\int_{(2k_0-1)\pi}^{(2k+1)\pi} \sigma^{r/p}\sin\sigma\, d\sigma &= \frac{1}{2}\big[\sigma^{r/p}\big]_{(2k_0-1)\pi}^{(2k+1)\pi} + \frac{r}{2p}\int_{(2k_0-1)\pi}^{(2k+1)\pi} \sigma^{-(1-r/p)}\cos\sigma\, d\sigma \\ &= \frac{1}{2}\big[\sigma^{r/p}\big]_{(2k_0-1)\pi}^{(2k+1)\pi} + 0 + \frac{r(1-r/p)}{2p}\int_{(2k_0-1)\pi}^{(2k+1)\pi} \sigma^{-(2-r/p)}\sin\sigma\, d\sigma.\end{aligned}$$

The conclusion of the proof follows by putting the last integral on the left-hand side by taking account of

$$\frac{r(1-r/p)}{2p\sigma^2} \leqslant \frac{r(1-r/p)}{2p\pi^2} \leqslant \frac{1}{4\pi^2}.$$

Now, let m and k be such that $s \in [t_{2m-1}, t_{2m+1})$ and $t \in [t_{2k-1}, t_{2k+1})$. By using (3.5) in $[t_{2m+1}, t_{2k-1}]$ and (3.6) in $[t_{2m-1}, t_{2m+1}]$ and in $[t_{2k-1}, t_{2k+1}]$ we are able to prove (3.8). □

3.1. Subzones of the hyperbolic zone

To prove the energy estimates in Theorem 2 we divide $Z_{\mathrm{hyp}}(N)$ in two subzones and we follow the ideas of the proof of Theorem 3 (see later, Section 4). Let $t_{|\xi|}$ be defined as in (2.23) by

$$\Lambda(t_{|\xi|})|\xi| = N\ell^{\gamma}(t_{|\xi|}),$$

whereas the function $\tilde{t}_{|\xi|}$ is defined by

$$\Lambda(\tilde{t}_{|\xi|})|\xi| = N\ell^{\gamma}(\tilde{t}_{|\xi|})\exp\bigl(L\ell^{r}(\tilde{t}_{|\xi|})\bigr)$$

for some $L \geqslant 4M_6$, where M_6 is as in (3.8). We recall from Definition 2 that $\Theta(t) = \Lambda(t)/\ell^{\gamma}(t)$ is increasing, therefore from

$$\frac{\Theta(\tilde{t}_{|\xi|})}{\Theta(t_{|\xi|})} = \exp\bigl(L\ell^{r}(\tilde{t}_{|\xi|})\bigr) > 1$$

it follows that $t_{|\xi|} \leqslant \tilde{t}_{|\xi|}$ (it is easy to prove that $\tilde{t}_{|\xi|}$ is well defined, too). Let $q : (0,\infty) \to \mathbb{N}^*$ be such that $q = q(|\xi|)$ satisfies $t_{2q-1} < \tilde{t}_{|\xi|} \leqslant t_{2q+1}$. We divide $Z_{\mathrm{hyp}}(N)$ into the two subzones, the *oscillation's subzone* $Z_{\mathrm{osc}}(N)$ and the *interaction's subzone* $Z_{\mathrm{intac}}(N)$ which are defined as follows

$$Z_{\mathrm{osc}}(N) = \{t_{|\xi|} \leqslant t \leqslant t_{2q+1}\} \quad \text{and} \quad Z_{\mathrm{intac}}(N) = \{t \geqslant t_{2q+1}\}.$$

Lemma 3.2. *It holds*

$$\int_{t_{|\xi|}}^{t_{2q+1}} \bigl|\varphi_2(\tau)\bigr|\,d\tau \leqslant C'L\ell^{r+\gamma}(\tilde{t}_{|\xi|}). \tag{3.9}$$

Proof. Let m be such that $t_{2m-1} \leqslant t_{|\xi|} < t_{2m+1}$. Analogously to the proof of (3.6) we get

$$\begin{aligned}
\int_{t_{|\xi|}}^{t_{2q+1}} \bigl|\varphi_2(\tau)\bigr|\,d\tau &\leqslant \frac{1+r/p}{2}\Bigl(\bigl((2q+1)\pi\bigr)^{1+r/p} - \bigl((2m-1)\pi\bigr)^{1+r/p}\Bigr) \\
&\approx \bigl((q+1-m)\pi\bigr)\bigl((2q+1)\pi\bigr)^{r/p} \\
&\approx \bigl(\ell^{p}(\tilde{t}_{|\xi|}) - \ell^{p}(t_{|\xi|})\bigr)\ell^{r}(\tilde{t}_{|\xi|}) \approx \bigl(\ell(\tilde{t}_{|\xi|}) - \ell(t_{|\xi|})\bigr)\ell^{r+p-1}(\tilde{t}_{|\xi|}) \\
&\approx \log\biggl(\frac{\Lambda(\tilde{t}_{|\xi|})}{\Lambda(t_{|\xi|})}\biggr)\ell^{\gamma}(\tilde{t}_{|\xi|}) = \biggl[L\ell^{r}(\tilde{t}_{|\xi|}) + \log\biggl(\frac{\ell^{\gamma}(\tilde{t}_{|\xi|})}{\ell^{\gamma}(t_{|\xi|})}\biggr)\biggr]\ell^{\gamma}(\tilde{t}_{|\xi|}) \approx L\ell^{r+\gamma}(\tilde{t}_{|\xi|})
\end{aligned}$$

what we wanted to show. □

First we estimate the fundamental solution to (2.4) in $Z_{\mathrm{intac}}(N)$. If we put $W = TZ$, where W is as in (2.13), and choose

$$T(t,\xi) := \begin{pmatrix} \exp(-\int_{t_{2q+1}}^t \theta^*(\tau)\,d\tau) & 0 \\ 0 & \exp(\int_{t_{2q+1}}^t \theta^*(\tau)\,d\tau) \end{pmatrix},$$

then we get

$$\partial_t Z - \begin{pmatrix} \varphi_+(t,\xi) & 0 \\ 0 & \varphi_-(t,\xi) \end{pmatrix} iZ + \big(\Re\psi(t) + \theta^*(t)\big) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} Z + \widetilde{J}(t,\xi)Z = 0.$$

Thanks to (3.8), from $L \geqslant 4M_6$ it follows

$$\big\|\widetilde{J}(t,\xi)\big\| \leqslant C\frac{\ell^{2\gamma}(t)\exp(L\ell^r(t))\lambda(t)}{|\xi|\Lambda^2(t)}.$$

Hence,

$$\partial_t|Z|^2 \leqslant 2\big(\big|\Re\psi(t) + \theta^*(t)\big| + \|\widetilde{J}\|\big)|Z|^2 \leqslant 2\bigg(\big|\varphi_2(t) + \theta^*(t)\big| + \frac{\ell^{r-1}(t)\lambda(t)}{\Lambda(t)} + \big\|\widetilde{J}(t,\xi)\big\|\bigg)|Z|^2.$$

Integrating by parts yields

$$\int_{t_{2q+1}}^t \big\|\widetilde{J}(\tau,\xi)\big\|\,d\tau \leqslant C\frac{\ell^{2\gamma}(t_{2q+1})\exp(L\ell^r(t_{2q+1}))}{|\xi|\Lambda(t_{2q+1})} \approx \frac{C}{N}\ell^\gamma(\tilde{t}_{|\xi|}), \quad t \geqslant t_{2q+1}.$$

By Gronwall's lemma and by using (3.7) we conclude

$$\big|Z(t,\xi)\big| \leqslant \exp\big(C\ell(t)\big)\big|Z(t_{2q+1},\xi)\big| \leqslant C\exp\big(C_1\big(\ell(t)\big)^{\max\{\gamma,r\}}\big)\big|Z(t_{2q+1},\xi)\big|.$$

In $Z_{\mathrm{osc}}(N)$ it is sufficient to use (3.9) (we recall that $r + \gamma = 2r + p - 1$) together with the estimate

$$\int_{t_{|\xi|}}^t \big\|J(\tau,\xi)\big\|\,d\tau \leqslant \frac{C\ell^{2\gamma}(t_{|\xi|})}{|\xi|\Lambda(t_{|\xi|})} = \frac{C}{N}\ell^\gamma(t_{|\xi|}),$$

whereas in $Z_{\mathrm{pd}}(N)$ we use a straight-forward estimate.

4. Proof of blow-up results

4.1. Proof of Theorem 3

We look for the fundamental solution $E = E(t,s,\xi)$ to (2.4). It solves for any $s,t \geqslant 0$ and $\xi \in \mathbb{R}$ the Cauchy problem

$$\partial_t E(t,s,\xi) = i\xi\lambda(t)A(t)E(t,s,\xi), \qquad E(s,s,\xi) = \mathrm{I}_2.$$

We can directly estimate $E(t,s,\xi)$ in $Z_{\mathrm{pd}}(N)$. Indeed, from the boundedness of $A(t)$ and the positivity of $\lambda(t)$ it follows

$$\left\|E_{\mathrm{pd}}(t,s,\xi)\right\| \leqslant \exp\left(|\xi| \int_0^{t_{|\xi|}} \lambda(\tau)\left\|A(\tau)\right\| d\tau\right) \leqslant \exp\bigl(|\xi| \Lambda(t_{|\xi|})\|A\|_{L^\infty}\bigr) = \exp\bigl(N\|A\|_{L^\infty}\bigr).$$

We can apply Liouville's formula to estimate $\|E_{\mathrm{pd}}^{-1}(t,s,\xi)\|$, so that

$$\exp\bigl(-N\|A\|_{L^\infty}\bigr) \leqslant \left\|E_{\mathrm{pd}}(t,s,\xi)\right\| \leqslant \exp\bigl(N\|A\|_{L^\infty}\bigr), \quad s,t \leqslant t_{|\xi|}. \tag{4.1}$$

Now let $y \in L^2$ and let the initial data U_0 be defined as in Definition 1. We claim that the solution $V = V(t,\xi)$ of the Cauchy problem

$$\begin{cases} \partial_t V - i\xi\lambda(t)A(t)V = 0, & t \geqslant t_{|\xi|}, \\ V(t_{|\xi|},\xi) = \widehat{U}(t_{|\xi|},\xi), \end{cases} \tag{4.2}$$

verifies in $Z_{\mathrm{hyp}}(N)$ the estimate

$$\left|V(t,\xi)\right| \geqslant C_1 \exp\bigl(m_1\nu_1(t) - m_2\nu_1(t_{|\xi|})\bigr)\left|y(\xi)\right|, \quad t \geqslant t_{|\xi|}, \tag{4.3}$$

where the constant C_1 is independent of ξ. Via the change of variables (2.12) the Cauchy problem (4.2) becomes (2.13) with the initial data $W(t_{|\xi|},\xi) = Y(\xi)$. For some positive $\rho = \rho(N)$ that we will fix later we define

$$T(t,\xi) := \begin{pmatrix} \exp(-\int_{t_{|\xi|}}^t (\theta(\tau) + \frac{\rho\lambda(\tau)}{|\xi|\Lambda^2(\tau)})\,d\tau) & 0 \\ 0 & \exp(\int_{t_{|\xi|}}^t (\theta(\tau) + \frac{\rho\lambda(\tau)}{|\xi|\Lambda^2(\tau)})\,d\tau) \end{pmatrix}.$$

It follows

$$\left\|T(t,\xi)\right\|, \left\|T^{-1}(t,\xi)\right\| \leqslant \exp(M_5 + \rho/N), \quad t \geqslant t_{|\xi|}.$$

If we put $W = T(t,\xi)Z$, then we get

$$\begin{cases} \partial_t Z - \begin{pmatrix} \varphi_+(t,\xi) & 0 \\ 0 & \varphi_-(t,\xi) \end{pmatrix} iZ + \left(\Re\psi(t) + \theta + \dfrac{\rho\lambda(t)}{|\xi|\Lambda^2(t)}\right)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} Z + \widetilde{J}(t,\xi)Z = 0, \\ Z(t_{|\xi|},\xi) = Y(\xi), \end{cases} \tag{4.4}$$

where, thanks to (2.14), the matrix $\widetilde{J} = T^{-1}JT$ verifies

$$\left\|\widetilde{J}(t,\xi)\right\| \leqslant \frac{M_4 \exp(2M_5 + 2\rho/N)\lambda(t)}{|\xi|\Lambda^2(t)}. \tag{4.5}$$

We consider the case $\Re\psi + \theta \geqslant 0$. Following [5] we define in $Z_{\mathrm{hyp}}(N)$ the *Lyapunov functional*

$$S(t,\xi) := \left|z_1(t,\xi)\right|^2 - \left|z_2(t,\xi)\right|^2,$$

where $Z(t,\xi) = (z_1(t,\xi), z_2(t,\xi))$ solves (4.4). Then we derive

$$\partial_t S(t,\xi) \geqslant 2\left(\Re\psi(t) + \theta(t) + \frac{\rho\lambda(t)}{|\xi|\Lambda^2(t)} - 2\big\|\widetilde{J}(t,\xi)\big\|\right)\big|Z(t,\xi)\big|^2.$$

We fix $\rho = N/2$, $N \geqslant \overline{N}$, that attains the maximum of the function $f(\rho) = \rho\exp(-2\rho/N)$. For such a choice, by virtue of (2.18), it holds

$$\rho \geqslant 2M_4 \exp(2M_5 + 2\rho/N). \tag{4.6}$$

But this allows to conclude

$$\partial_t S(t,\xi) \geqslant 2\big(\Re\psi(t) + \theta(t)\big)\big|Z(t,\xi)\big|^2 \geqslant 2\big(\Re\psi(t) + \theta(t)\big)S(t,\xi).$$

Thanks to Gronwall's inequality, to Remark 2.2 and to the choice of initial data $Y(\xi) = (y(\xi), 0)$ it follows

$$\begin{aligned} S(t,\xi) &\geqslant \exp\left(2\int_{t_{|\xi|}}^{t} \big(\Re\psi(\tau) + \theta(\tau)\big)\,d\tau\right) S(t_{|\xi|},\xi) \\ &= \left(2\int_0^t \left(\Re\psi(\tau)\,d\tau - 2\int_0^{t_{|\xi|}} \Re\psi(\tau)\,d\tau + 2\int_{t_{|\xi|}}^{t} \theta(\tau)\right)d\tau\right)\big|y(\xi)\big|^2 \\ &\geqslant \exp\big(2m_1\nu_1(t) - 2m_2\nu_1(t_{|\xi|}) - 2M_5\big)\big|y(\xi)\big|^2, \quad t \geqslant t_{|\xi|}. \end{aligned}$$

Therefore we proved (4.3) since $|Z(t,\xi)|$ is equivalent to $|V(t,\xi)|$. In correspondence to $y \in L^2(\mathbb{R},\mathbb{C})$ we take ϵ as in (2.20) and we derive

$$\|y\|^2_{L^2_\xi} = \frac{1+\kappa}{\kappa}\|y_\epsilon\|^2_{L^2_\xi}, \quad \text{where } y_\epsilon(\xi) := \begin{cases} y(\xi), & |\xi| \geqslant \epsilon, \\ 0, & |\xi| < \epsilon. \end{cases} \tag{4.7}$$

Taking into consideration (4.1) and (4.3), estimating $-m_2\nu_1(t_{|\xi|}) \geqslant -m_2\nu_1(t_\epsilon)$ for $|\xi| \geqslant \epsilon$, we get

$$\begin{aligned} \big\|\widehat{U}(t,\cdot)\big\|_{L^2} &\geqslant \left(\int_{|\xi|\geqslant\epsilon} \big|V(t,\xi)\big|^2\,d\xi\right)^{\frac{1}{2}} \\ &\geqslant C_1 \exp\big(m_1\nu_1(t) - m_2\nu_1(t_\epsilon)\big)\|y_\epsilon\|_{L^2} \\ &\geqslant C_1\sqrt{\frac{\kappa}{1+\kappa}}\exp\big(m_1\nu_1(t) - m_2\nu_1(t_\epsilon)\big)\|y\|_{L^2} \\ &\geqslant C\exp\big(m_1\nu_1(t) - m_2\nu_1(t_\epsilon)\big)\|\widehat{U_0}\|_{L^2}, \quad t \geqslant t_\epsilon. \end{aligned} \tag{4.8}$$

This concludes the proof (see Remark 2.1). The case $\Re\psi \leqslant 0$ can be treated in an analogous way.

4.2. Proof of Theorem 4

In order to prove Theorem 4 some modifications to the proof of Theorem 3 are required. First of all we prove Lemma 2.1 by using the following statement.

Lemma 4.1. *Under Hypothesis* 4 *it holds*

$$\nu \circ \Theta^{-1}(N) = o(N) \quad \text{as } N \to +\infty. \tag{4.9}$$

Proof. From (1.17) we have $\nu(0)\Theta(t) \geqslant (\Lambda(t))^{1-\delta}$. We recall (see Definition 2) that Θ^{-1} is an increasing function from $[1/\nu(0), +\infty)$ to $[0, \infty)$. Let $N > 1/\nu(0)$ and $t = \Theta^{-1}(N)$. Then

$$\nu(0)N \geqslant \Lambda\big(\Theta^{-1}(N)\big)^{1-\delta},$$

that is, being Λ^{-1} increasing,

$$\Theta^{-1}(N) \leqslant \Lambda^{-1}\Big(\big(\nu(0)N\big)^{\frac{1}{1-\delta}}\Big).$$

Therefore, since $\delta/(1-\delta) < 1$, by using (1.17) again, we get

$$\nu \circ \Theta^{-1}(N) \leqslant \nu(0)\big(\nu(0)N\big)^{\frac{\delta}{1-\delta}} = \big(\nu(0)\big)^{\frac{1}{1-\delta}} N^{\frac{\delta}{1-\delta}} = o(N) \quad \text{as } N \to +\infty.$$

This completes the proof. □

Proof of Lemma 2.1. We fix $M > 0$ and $\epsilon > 0$ and define $M' := M/\epsilon$ and $N' := N/\epsilon$. Therefore we want to prove that there exists a constant $C_{M'} > 0$ such that

$$\frac{\nu(t_\epsilon)}{N'} \leqslant \frac{1}{M'} \quad \text{for any } N' \geqslant C_{M'}.$$

Thanks to (4.9) this holds for any positive M' since

$$\lim_{N' \to +\infty} \frac{\nu(t_\epsilon)}{N'} = \lim_{N' \to +\infty} \frac{\nu \circ \Theta^{-1}(N')}{N'} = 0.$$

So we can take $N(\epsilon, M) \geqslant \epsilon C_{M'}$ in Lemma 2.1. □

Now we fix $y \in L^2(\mathbb{R}, \mathbb{C})$ and $\epsilon > 0$ as in (2.20). Let $N \geqslant N_\epsilon$ with N_ϵ from (2.26). Analogously to the proof of Theorem 3 we can straight-forward estimate the fundamental solution $E(t,s,\xi)$ in $Z_{\text{pd}}(N)$, deriving, in particular, that

$$\big\| E_{\text{pd}}(t,s,\xi)\big\| \geqslant \exp\big(-N\nu(t_{|\xi|})\|A\|_{L^\infty}\big) = \exp\big(g\big(|\xi|\big)\big), \quad s,t \leqslant t_{|\xi|}. \tag{4.10}$$

In $Z_{\text{hyp}}(N)$ we can prove that

$$\big|V(t,\xi)\big| \geqslant C_1 \exp\big(m_1\nu_1(t) - m_2\nu_1(t_{|\xi|})\big)\big|y(\xi)\big|, \quad |\xi| \geqslant \epsilon,\ t \geqslant t_{|\xi|}. \tag{4.11}$$

We follow the proof of Theorem 3, but now for some $\rho = \rho(N,\epsilon) > 0$ that we will choose later. We introduce

$$T(t,\xi) := \begin{pmatrix} \exp(-\int_{t_{|\xi|}}^t (\theta(\tau) + \frac{\rho\lambda(\tau)\nu^2(\tau)}{|\xi|\Lambda^2(\tau)})\,d\tau) & 0 \\ 0 & \exp(\int_{t_{|\xi|}}^t (\theta(\tau) + \frac{\rho\lambda(\tau)\nu^2(\tau)}{|\xi|\Lambda^2(\tau)})\,d\tau) \end{pmatrix}.$$

Taking into consideration

$$\|T(t,\xi)\|, \|T^{-1}(t,\xi)\| \leqslant \exp\bigl(M_5 + \rho\nu(t_{|\xi|})/N\bigr), \quad t \geqslant t_{|\xi|},$$

the matrix $\widetilde{J} = T^{-1}JT$ verifies

$$\|\widetilde{J}(t,\xi)\| \leqslant \frac{M_4 \exp(2M_5 + 2\rho\nu(t_{|\xi|})/N)\lambda(t)\nu^2(t)}{|\xi|\Lambda^2(t)}.$$

We fix $\rho = N/(2\nu(t_\epsilon))$, that attains the maximum of the function $f(\rho) = \rho\exp(-2\rho\nu(t_\epsilon)/N)$. Thanks to (2.25) for such a choice of ρ it holds

$$\rho \geqslant M_4 \exp\bigl(2M_5 + 2\rho\nu(t_\epsilon)/N\bigr).$$

We remark that, with such a choice of ρ, the norms $\|T(t,\xi)\|$ and $\|T^{-1}(t,\xi)\|$ are uniformly bounded by $\exp(M_5 + 1/2)$. We follow the proof of Theorem 3 and we derive (4.11). Now, let y_ϵ be as in (4.7). Analogously to (4.8), estimating $-m_2\nu_1(t_{|\xi|}) \geqslant -m_2\nu_1(t_\epsilon)$ for $|\xi| \geqslant \epsilon$, it follows

$$\begin{aligned} \|\widehat{U}(t,\cdot)\|_{L^2} &\geqslant \biggl(\int_{|\xi|\geqslant\epsilon} |V(t,\xi)|^2\,d\xi\biggr)^{\frac{1}{2}} \\ &\geqslant C_1 \exp\bigl(m_1\nu_1(t) - m_2\nu_1(t_\epsilon)\bigr)\|y_\epsilon\|_{L^2} \\ &\geqslant C_1\sqrt{\frac{\kappa}{1+\kappa}}\exp\bigl(m_1\nu_1(t) - m_2\nu_1(t_\epsilon)\bigr)\|y\|_{L^2} \\ &\geqslant C\exp\bigl(m_1\nu_1(t) - m_2\nu_1(t_\epsilon)\bigr)\|U_0\|_g, \quad t \geqslant t_\epsilon, \end{aligned} \tag{4.12}$$

where in the last estimate, by virtue of (4.10), we used

$$\|y\|_{L^2}^2 \geqslant C_2 \int_{\mathbb{R}} |\widehat{U}(t_{|\xi|},\xi)|^2\,d\xi = C_2 \int_{\mathbb{R}} |E_{\mathrm{pd}}(t_{|\xi|},0,\xi)\widehat{U_0}(\xi)|^2\,d\xi \geqslant C_2\|U_0\|_g^2.$$

4.3. A corollary to Theorem 4

If we restrict in Theorem 4 the set to which y has to belong, that is, the choice of the initial data U_0 in (1.3), then we can improve our result. We define

$$G_\epsilon := \bigl\{V \in L^2\colon \operatorname{dist}(\operatorname{supp} V, 0) \geqslant \epsilon\bigr\}, \qquad F_\epsilon := \bigl\{U \in L^2\colon \widehat{U} \in G_\epsilon\bigr\},$$

and we remark that

$$F = \bigl\{U \in L^2\colon \operatorname{dist}(\operatorname{supp}\widehat{U}, 0) > 0\bigr\} = \bigcup_{\epsilon>0} F_\epsilon$$

is dense in L^2. For any $\epsilon > 0$ and $N \geqslant N_\epsilon$ let $M_\pm(N,\epsilon)$ be the set of initial data $U_0 \in F_\epsilon$ such that the solution of (1.3) with initial data U_0 verifies (2.15). Such a set is equipotent to F_ϵ.

Corollary 4.2. *We assume Hypotheses* 1, 2 *and* 4. *Moreover, we assume* (2.16) *and* (2.17) *and that* $\nu(t) = o(\nu_1(t))$ *as* $t \to \infty$. *We fix* $\epsilon > 0$. *Let* N_ϵ *be as in* (2.26) *and let* $N \geqslant N_\epsilon$ *and* $t_{|\xi|}$ *as in* (2.23). *Then, for any initial data* $U_0 \in M_+(N,\epsilon)$ (*resp.* $U_0 \in M_-(N,\epsilon)$) *the solution of* (1.3) *with initial data* U_0 *verifies*

$$\left\|U(t,\cdot)\right\|_{L^2} \geqslant C_\epsilon \exp\bigl(m_1\nu_1(t)\bigr)\|U_0\|_{L^2}, \quad t \geqslant t_\epsilon, \tag{4.13}$$

where $C_\epsilon = C\exp(-N\nu(t_\epsilon)\|A\|_{L^\infty} - m_2\nu_1(t_\epsilon))$ *with* $C > 0$ *independent of* ϵ.

In order to prove Corollary 4.2 it is sufficient to notice that if $y \in G_\epsilon$, then $U(t,\cdot) \in F_\epsilon$, therefore, we can directly glue (4.10) and (4.11) to derive

$$\begin{aligned}\left|\widehat{U}(t,\xi)\right| &\geqslant C_1\exp\bigl(m_1\nu_1(t) - m_2\nu_1(t_{|\xi|})\bigr)\left|y(\xi)\right| \\ &\geqslant C_1\exp\bigl(m_1\nu_1(t) - m_2\nu_1(t_{|\xi|})\bigr)\exp\bigl(g\bigl(|\xi|\bigr)\bigr)\left|\widehat{U_0}(\xi)\right|\end{aligned}$$

for any $t \geqslant t_\epsilon$. The proof follows from $-m_2\nu_1(t_{|\xi|}) \geqslant -m_2\nu_1(t_\epsilon)$ and $\exp(g(|\xi|)) \geqslant \exp(g(\epsilon))$.

4.4. Proof of Theorem 5

We divide the space $\mathbb{R}_\xi$ into the *pseudo-differential limited interval* $[-\rho_t,\rho_t]$ and the *hyperbolic complementary* $\mathbb{R}\setminus[-\rho_t,\rho_t]$ at any time $t \geqslant t_\epsilon$. This division is related to zones which we proposed in the previous sections. Here $\epsilon > 0$ is fixed and $N \geqslant N_\epsilon$. We recall that $\rho_t \leqslant \epsilon$ for any $t \geqslant t_\epsilon$. In the expected estimates we have to replace (4.10) by

$$\left\|E_{\mathrm{pd}}(t,0,\xi)\right\| \geqslant \exp\Bigl(-N\nu(t)\max_{s\leqslant t}\left\|A(s)\right\|\Bigr), \quad |\xi| \leqslant \rho_t, \tag{4.14}$$

that is,

$$\left|\widehat{U}(t,\xi)\right| \geqslant \exp\Bigl(-N\nu(t)\max_{s\leqslant t}\left\|A(s)\right\|\Bigr)\left|\widehat{U_0}(\xi)\right|, \quad |\xi| \leqslant \rho_t,$$

whereas we replace (4.11) by

$$\begin{aligned}\left|\widehat{U}(t,\xi)\right| &\geqslant C_1\exp\bigl(m_1\nu_1(t) - m_2\nu_1(t_{|\xi|})\bigr)\left|y(\xi)\right| \\ &\geqslant C\exp\bigl(m_1\nu_1(t) - m_2\nu_1(t_{|\xi|})\bigr)\left|E_{\mathrm{pd}}(t_{|\xi|},0,\xi)\widehat{U_0}(\xi)\right| \\ &\geqslant C\exp\Bigl(m_1\nu_1(t) - m_2\nu_1(t_{|\xi|}) - N\nu(t_{|\xi|})\max_{s\leqslant t_{|\xi|}}\left\|A(s)\right\|\Bigr)\left|\widehat{U_0}(\xi)\right|, \quad |\xi| \geqslant \rho_t,\ t \geqslant t_\epsilon.\end{aligned} \tag{4.15}$$

Gluing together (4.14) and (4.15), estimating $-m_2\nu_1(t_{|\xi|}) \geqslant -m_2\nu_1(t_\epsilon)$ and $-\nu(t_{|\xi|}) \geqslant -\nu(t)$ in (4.15), and integrating with respect to ξ we conclude the proof.

5. Concluding remarks

Concerning the problem of *generalized energy conservation*, that is, to derive (1.2), one can in general not expect that this holds in the case of *oscillations which are not very slow*. Nevertheless, we can get some benefit of higher regularity of the coefficients by assuming a so-called *stabilization condition*. The case of $\mathcal{C}^2$ *stabilization condition* together with (1.8) has been considered in [1]. Here we propose a $\mathcal{C}^m$ *stabilization condition*, $m \geqslant 2$, for a special class of systems (1.3) with $\lambda \in \mathcal{C}^{m-1}$ and

$$A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & a(t) \end{pmatrix}, \tag{5.1}$$

where $a \in \mathcal{C}^0$ and $b, c \in \mathcal{C}^m$ are real-valued and bounded. Condition (1.4) in Hypothesis 1 reads as $\Delta(t) = 4b(t)c(t) \geqslant m_0 > 0$. In particular, $b(t)$ and $c(t)$ have the same sign and this is constant. Let $H(t)$ be as in (2.5). We remark that the eigenvalues of $\widetilde{A}(t)$ are $\mu_\pm(t) = \pm\mu(t)$, where $\mu(t) := \sqrt{b(t)c(t)}$. System (2.6) reads as

$$\partial_t V - \lambda(t) \begin{pmatrix} \mu + a & 0 \\ 0 & -\mu + a \end{pmatrix} i\xi V + h(t) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} V = 0, \tag{5.2}$$

where

$$h(t) = \frac{\det H(t)}{2\Delta(t)} \left(\frac{\sqrt{\Delta(t)}(b(t) + c(t))}{\det H(t)} \right)' = i \frac{c'b - b'c}{2bc} \in i\mathbb{R}.$$

We remark that $\Re h = -ih$ is the derivative of the function $\log(\sqrt{c(t)/b(t)}\,)$.

We assume Hypotheses 2 and 4 in correspondence with a $\mathcal{C}^m$ regularity rather than only $\mathcal{C}^2$, that is, we replace (1.5) and (1.14) by

$$\left|\lambda^{(k)}(t)\right| \leqslant M' \frac{\lambda^{k+1}(t)}{\Lambda^k(t)}, \quad t \geqslant 0,\ k = 1, \ldots, m-1, \tag{5.3}$$

$$\left|b^{(k)}(t)\right| + \left|c^{(k)}(t)\right| \leqslant M' \left(\frac{\lambda(t)\nu(t)}{\Lambda(t)} \right)^k, \quad t \geqslant 0,\ k = 1, \ldots, m, \tag{5.4}$$

for some $M' \geqslant 0$. Moreover, we assume the following condition:

Hypothesis 5 *($\mathcal{C}^m$-stabilization condition).* We assume that there exists a positive, strictly increasing, continuous function $\Theta_1(t)$ such that:

- $\lim_{t\to\infty} \Theta_1(t) = +\infty$;
- there exists a constant $C_1 > 0$ such that $\Theta_1(t) \leqslant C_1 \Theta(t)$ for $t \geqslant 0$;
- there exists a constant $C_2 > 0$ such that

$$\int_t^\infty \left(\Theta(\tau)\right)^{-m} \lambda(\tau)\, d\tau \leqslant C_2 \left(\Theta_1(t)\right)^{-(m-1)}, \quad t \geqslant 0; \tag{5.5}$$

- there exist two constants b_∞ and c_∞ and a constant $C_3 > 0$ such that

$$\int_0^t \lambda(\tau) \left(\left|b(\tau) - b_\infty\right| + \left|c(\tau) - c_\infty\right| \right) d\tau \leqslant C_3 \Theta_1(t), \quad t \geqslant 0. \tag{5.6}$$

If (5.6) holds, then b_∞ and c_∞ are uniquely determined and $b_\infty c_\infty > 0$. We are ready to state the following result about *generalized energy conservation*.

Theorem 6. *Let $A(t)$ be as in* (5.1)*, and we assume Hypotheses* 2 *and* 4 (*in absence of* (1.16)) *in correspondence with C^m regularity, that is,* (5.3), (5.4)*. If we assume Hypothesis* 5*, then the solution to* (1.3) *satisfies* (1.2)*.*

Remark 5.1. If we directly consider the system (5.2) with no assumption about C^m regularity or stabilization and we define its *Lyapunov functional* as

$$S(t,\xi) = |v_1|^2 - |v_2|^2,$$

where $V = (v_1, v_2)$ is the solution to (5.2), then

$$\partial_t S(t,\xi) = 2\Re\big((\partial_t v_1)\overline{v_1} - (\partial_t v_2)\overline{v_2}\big) = 2\Re(-hv_2\overline{v_1} - hv_1\overline{v_2}) = 2\Re\big(-h2\Re(v_1\overline{v_2})\big) = 0.$$

This proves, in particular, that if $S(0,\xi) \not\equiv 0$, that is, the two components of U_0 do not coincide in L^2, then $\|U(t,\cdot)\|_{L^2} \geqslant C > 0$, that is, the energy cannot vanish for $t \to \infty$.

One can find more details and some examples about *stabilization condition* in [2–4]; in particular, for systems, see [1].

5.1. Examples

Example 5.2 *(Polynomial growth).* Let $\lambda(t) = (1+t)^{p-1}$ with $p > 0$, that is, $\Lambda(t) \approx (1+t)^p$, and let $\nu(t) = (1+t)^q$ with $0 < q < p$, that is, $\Theta(t) \approx (1+t)^{p-q}$. It follows

$$\int_t^\infty \big(\Theta(\tau)\big)^{-m} \lambda(\tau)\, d\tau \leqslant C(1+t)^{-((p-q)m-p)},$$

provided that $q < p(m-1)/m$. Hence, we may choose $\Theta_1(t) = (1+t)^r$ with $r = p - qm/(m-1)$. Therefore, (5.6) holds if

$$\int_0^t (1+\tau)^{p-1}\big(\big|b(\tau) - b_\infty\big| + \big|c(\tau) - c_\infty\big|\big)\, d\tau \leqslant C_3(1+t)^r. \tag{5.7}$$

Example 5.3 *(Exponential growth).* Let $\lambda(t) = e^{pt}$ with $p > 0$, that is $\Lambda(t) \approx e^{pt}$, and let $\nu(t) = e^{qt}$ with $0 < q < p$, that is, $\Theta(t) = e^{(p-q)t}$. It follows

$$\int_t^\infty \big(\Theta(\tau)\big)^{-m} \lambda(\tau)\, d\tau \leqslant Ce^{-((p-q)m-p)t},$$

provided that $q < p(m-1)/m$. Hence, we may take $\Theta_1(t) = e^{rt}$ with $r = p - qm/(m-1)$. Therefore (5.6) holds if

$$\int_0^t e^{p\tau}\big(\big|b(\tau) - b_\infty\big| + \big|c(\tau) - c_\infty\big|\big)\, d\tau \leqslant C_3 e^{rt}. \tag{5.8}$$

Now we show how to construct *explicitly* coefficients $b(t)$ and $c(t)$ in the polynomial case (resp. exponential case) satisfying (5.7) (resp. (5.8)).

Example 5.4. Let $b_\infty, c_\infty \in \mathbb{R}$ with $b_\infty c_\infty > 0$. For the sake of simplicity we assume $b_\infty, c_\infty \geqslant 1$. For each of these, say b_∞, we construct a not identically vanishing function $\varphi \in \mathcal{C}^m$ with $\operatorname{supp}\varphi \subset [0,1]$ and

$$-1 < \varphi^{(k)}(t) < 1, \quad k = 0, \dots, m,$$

and we look for a sequence $\{t_j, \delta_j, \eta_j\}_{j\geqslant 1}$ such that

$$t_j \nearrow \infty, \qquad 0 \leqslant \delta_j \leqslant t_{j+1} - t_j, \qquad 0 \leqslant \eta_j \leqslant 1.$$

If we put

$$b(t) = b_\infty + \sum_{j=1}^{\infty} \eta_j \varphi\bigl((t - t_j)/\delta_j\bigr),$$

then

$$\bigl|b^{(k)}(t)\bigr| \leqslant \eta_j \delta_j^{-k} \leqslant \bigl(\eta_j^{-1/m}\delta_j\bigr)^{-k}, \quad t \in [t_j, t_{j+1}]$$

for any $k = 1, \dots, m$. For an opportune choice of $\{t_j\}_{j\geqslant 1}$ let

$$\lambda_j = \lambda(t_j), \qquad \Lambda_j = \sum_{l=1}^{j} (t_{l+1} - t_l)\lambda_l, \qquad \nu_j = \frac{\Lambda_j \eta_j^{1/m}}{\delta_j \lambda_j}.$$

We have to choose $\{t_j, \delta_j, \eta_j\}_{j\geqslant 1}$ in a such way that $\nu_j \to +\infty$ and $\nu_j = o(\Lambda_j)$.

Via the change of variables $\sigma = (\tau - t_l)/\delta_l$ for $t \in [t_l, t_{l+1}]$ we are able to estimate

$$\int_{t_1}^{t} \lambda(\tau)\bigl|b(\tau) - b_\infty\bigr|\,d\tau \leqslant C \sum_{l=1}^{j} \eta_l \lambda_l \int_{t_l}^{t_{l+1}} \bigl|\varphi\bigl((\tau - t_l)/\delta_l\bigr)\bigr|\,d\tau = C\|\varphi\|_{L^1} \sum_{l=1}^{j} \eta_l \lambda_l \delta_l.$$

In order to derive (5.7) we choose $\{t_j, \delta_j, \eta_j\}_{j\geqslant 1}$ in a such way that

$$\sum_{l=1}^{j} \eta_l \lambda_l \delta_l \approx \Theta_j = \Theta_1(t_j). \tag{5.9}$$

In the polynomial case let $t_j = e^j$ so that $\lambda_j = e^{j(p-1)}$ and $\Theta_j = e^{jr}$ with $r = p - qm/(m-1)$ as in Example 5.2, where $\nu_j \approx e^{jq}$. Let $\delta_j = e^{j\alpha}$ with $\alpha \leqslant 1$ and $\eta_j = e^{-j\beta}$ with $\beta \geqslant 0$. Then the condition (5.9) is satisfied if we take

$$-\beta + \alpha + p - 1 = p - qm/(m-1), \quad \text{that is, } \alpha = 1 + \beta - qm/(m-1). \tag{5.10}$$

By definition of ν_j and from (5.10) we derive

$$q = 1 - \beta/m - \alpha = -\beta(m+1)/m + qm/(m-1), \quad \text{that is, } \beta = qm/\big(m^2-1\big).$$

It follows $\alpha = 1 - qm^2/(m^2-1)$.

In the exponential case, let $t_j = j$ so that $\lambda_j = e^{jp}$ and $\Theta_{1,j} = e^{jr}$ with $r = p - qm/(m-1)$ as in Example 5.3, where $\nu_j \approx e^{jq}$. Let $\delta_j = e^{j\alpha}$ with $\alpha \leqslant 0$ and $\eta_j = e^{-j\beta}$ with $\beta \geqslant 0$. Analogously to (5.10) to obtain (5.9) we take

$$-\beta + \alpha + p = p - qm/(m-1), \quad \text{that is, } \alpha = \beta - qm/(m-1). \tag{5.11}$$

By definition of ν_j and from (5.11) we derive

$$q = -\beta/m - \alpha = -\beta(m+1)/m + qm/(m-1).$$

Therefore we get again $\beta = qm/(m^2-1)$ and $\alpha = 1 - qm^2/(m^2-1)$.

5.2. *Proof of Theorem 6*

In the proof of Theorem 6 we will use a *refined diagonalization*. The system in (5.2) has a special structure since the *lower order term* is anti-diagonal and its entries are anti-conjugate. Such a special structure is preserved by successive steps of diagonalization and this property is fundamental to derive suitable energy estimates if (5.5) is satisfied for $m > 2$. The procedure of *refined diagonalization* itself can be applied to ordinary differential equations with parameter, in general. For the ease of readiness we use a different notation for the system's entries.

Lemma 5.1. *Let ξ be a parameter; for any ξ, let $\phi(t,\xi)$ be continuous and complex-valued, let $\alpha_1(t,\xi)$ be continuous and real-valued and let $\beta_1(t,\xi) \in C^1$ complex-valued, with respect to the t variable, and let I_ξ be an interval such that $\sup_{t\in I_\xi} |\beta_1(t,\xi)|^2 < 1$. If we define*

$$\alpha_2 := \frac{(1-3|\beta_1|^2)\alpha_1 + [(\Re\beta_1)'(\Im\beta_1) - (\Re\beta_1)(\Im\beta_1)']}{1-|\beta_1|^2} \in \mathbb{R}, \qquad \beta_2 := \frac{\alpha_1\beta_1|\beta_1|^2 - i\beta_1'/2}{\alpha_2(1-|\beta_1|^2)}$$

then the system $\partial_t U_1 = \phi(t,\xi)U_1 + \mathcal{A}_1(t,\xi)U_1$ is equivalent to $\partial_t U_2 = \phi(t,\xi)U_2 + \mathcal{A}_2(t,\xi)U_2$ in I_ξ, where

$$\mathcal{A}_j = i\alpha_j\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - 2\begin{pmatrix} 0 & \beta_j \\ -\overline{\beta_j} & 0 \end{pmatrix}\right], \quad j = 1, 2,$$

via the change of variables $U_1 = M(\det M)^{-1/2}U_2$ with

$$M = \begin{pmatrix} 1 & \beta_1 \\ \overline{\beta_1} & 1 \end{pmatrix}.$$

Proof. Straight-forward calculations imply the statement. Indeed, applying the change of variable $U_1 = M(\det M)^{-1/2}U_2$ we derive

$$\begin{aligned}\partial_t U_2 &= \big(\phi + (\det M)'/(2\det M) + M^{-1}\mathcal{A}_1 M - M^{-1}M'\big)U_2 \\ &= \phi U_2 + (\det M)^{-1}\big((\det M)'/2 + M^{\mathrm{adj}}(\mathcal{A}_1 M - M')\big)U_2,\end{aligned}$$

where $\det M = 1 - |\beta_1|^2$ and M^{adj} is the *adjoint* of M, given by

$$M^{\mathrm{adj}} = \begin{pmatrix} 1 & -\beta_1 \\ -\overline{\beta_1} & 1 \end{pmatrix}.$$

Therefore, this gives

$$\begin{aligned}
\mathcal{A}_2 &:= (\det M)^{-1}\big((\det M)'/2 + M^{\mathrm{adj}}(\mathcal{A}_1 M - M')\big) \\
&= \frac{1}{1-|\beta_1|^2}\left(-\frac{1}{2}\big(\beta_1\overline{\beta_1}' + \beta_1'\overline{\beta_1}\big) + \begin{pmatrix} 1 & -\beta_1 \\ -\overline{\beta_1} & 1 \end{pmatrix}\left[i\alpha_1\begin{pmatrix} 1-2|\beta_1|^2 & -\beta_1 \\ \overline{\beta_1} & -(1-2|\beta_1|^2) \end{pmatrix}\right.\right. \\
&\quad \left.\left. - \begin{pmatrix} 0 & \beta_1' \\ \overline{\beta_1}' & 0 \end{pmatrix}\right]\right) \\
&= \frac{1}{1-|\beta_1|^2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\left(i\alpha_1\big(1-3|\beta_1|^2\big) + \frac{1}{2}\big(\beta_1\overline{\beta_1}' - \beta_1'\overline{\beta_1}\big)\right) \\
&\quad - \frac{1}{1-|\beta_1|^2}\begin{pmatrix} 0 & 2i\alpha_1\beta_1|\beta_1|^2 + \beta_1' \\ -2i\alpha_1\overline{\beta_1}|\beta_1|^2 + \overline{\beta_1}' & 0 \end{pmatrix}.
\end{aligned}$$

The conclusion of the proof follows from the following property:

$$\beta_1\overline{\beta_1}' - \beta_1'\overline{\beta_1} = 2i\big[(\Re\beta_1)'(\Im\beta_1) - (\Re\beta_1)(\Im\beta_1)'\big]. \qquad \square$$

Proof of Theorem 6. For some $N > 0$, that we will fix later, we define

$$t_{|\xi|} = \begin{cases} \Theta_1^{-1}(N/|\xi|) & \text{if } \Theta_1(0)|\xi| \leqslant N, \\ 0 & \text{otherwise}, \end{cases} \tag{5.12}$$

where $\Theta_1(t)$ is the function in Hypothesis 5. We need the zones $Z_{\mathrm{pd}}(N)$ and $Z_{\mathrm{hyp}}(N)$ as in (2.10). In $Z_{\mathrm{pd}}(N)$ we consider the Cauchy problem

$$\begin{cases} \partial_t \widehat{U} - \lambda(t)\big(a(t)\mathrm{I}_2 + A_\infty\big)i\xi\widehat{U} = 0, & t \leqslant t_{|\xi|}, \\ \widehat{U}(0,\xi) = \widehat{U_0}(\xi), \end{cases} \tag{5.13}$$

where we denoted

$$A_\infty := \begin{pmatrix} 0 & b_\infty \\ c_\infty & 0 \end{pmatrix}.$$

The matrix A_∞ is strictly hyperbolic due to $b_\infty c_\infty > 0$. Hence, it admits a (constant) diagonalizer H_∞. Therefore the fundamental solution $E_\infty(t,s,\xi)$ for (5.13) satisfies the estimate $\|E_\infty(t,s,\xi)\| \leqslant C$ and $E_\infty^{-1}(t,s,\xi) = E_\infty(s,t,\xi)$. Coming back to the Cauchy problem (2.4) for any $(t,\xi), (s,\xi) \in Z_{\mathrm{pd}}(N)$ we write its fundamental solution in the form

$$E_{\mathrm{pd}}(t,s,\xi) = E_\infty(t,s,\xi)Q_\infty(t,s,\xi),$$

that is, the matrix $Q_\infty(t,s,\xi)$ has to solve the following Cauchy problem:

$$\begin{cases} \partial_t Q_\infty(t,s,\xi) = \lambda(t)i\xi R(t,s,\xi)Q_\infty(t,s,\xi), & t \leqslant t_{|\xi|}, \\ Q_\infty(s,s,\xi) = \mathrm{I}_2, \end{cases}$$

where

$$R(t,s,\xi) = E_\infty(s,t,\xi)\big(\widetilde{A}(t) - A_\infty\big)E_\infty(t,s,\xi).$$

Thanks to the boundedness of $E_\infty(t,s,\xi)$ we derive $\|R(t,s,\xi)\| \leqslant C^2\|\widetilde{A}(t) - A_\infty\|$. From (5.6) it follows that

$$\big\|Q_\infty(t,s,\xi)\big\| \leqslant \exp\left(C^2|\xi|\int_0^{\tilde{t}_{|\xi|}} \lambda(\tau)\big\|\widetilde{A}(\tau) - A_\infty\big\|\,d\tau\right) \leqslant \exp\big(C'|\xi|\Theta_1(\tilde{t}_{|\xi|})\big) \leqslant \exp\big(C'N\big) = C_1.$$

By Liouville's formula we derive such an estimate from below, too. Therefore we proved that the fundamental solution is bounded both from above and from below in $Z_{\mathrm{pd}}(N)$. In $Z_{\mathrm{hyp}}(N)$ we use the procedure of *refined diagonalization* presented in Lemma 5.1.

Coming back to (5.2) we put

$$\mu_1(t,\xi) := \mu(t) \equiv \sqrt{bc}, \qquad h_1(t,\xi) := h(t) \equiv i\frac{c'b - b'c}{2bc}.$$

By (finite) induction, we define for any $j = 1,\dots,m-1$

$$g_j(t,\xi) := \frac{h_j}{2i\xi\lambda\mu_j}, \qquad K_j(t,\xi) := \begin{pmatrix} 1 & g_j \\ \overline{g_j} & 1 \end{pmatrix},$$

and

$$\mu_{j+1}(t,\xi) := \frac{(1-3|g_j|^2)\mu_j + [(\Re g_j)'(\Im g_j) - (\Re g_j)(\Im g_j)']}{1-|g_j|^2}, \qquad h_{j+1}(t,\xi) := \frac{h_j|g_j|^2 - i\xi\lambda g_j'}{1-|g_j|^2}.$$

We remark that μ_j is real-valued for any j, whereas g_j and h_j are, in general, complex-valued. We claim that (5.2) is equivalent to the system

$$\partial_t W_j - i\xi\lambda(t)a(t)W_j - i\xi\lambda(t)\mu_j(t,\xi)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}W_j + \begin{pmatrix} 0 & h_j \\ \overline{h_j} & 0 \end{pmatrix}W_j = 0,$$

that is,

$$\partial_t W_j = i\xi\lambda(t)a(t)W_j + i\xi\lambda(t)\mu_j(t,\xi)\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - 2\begin{pmatrix} 0 & g_j \\ -\overline{g_j} & 0 \end{pmatrix}\right]W_j \tag{5.14}$$

for any $j = 1,\dots,m$, provided that $N > \overline{N}(m)$ with $\overline{N}(m)$ sufficiently large. It is clear that (5.2) is (5.14) for $j = 1$, where we put $V = W_1$. By the principle of induction it is sufficient to prove that the system (5.14) in correspondence with $j = k$ is equivalent to (5.14) in correspondence with $j = k+1$. To prove this, it is sufficient to apply Lemma 5.1 with

$$U_1 = W_k, \quad U_2 = W_{k+1}, \quad \phi = i\xi\lambda(t)a(t), \quad \alpha_1 = \xi\lambda(t)\mu_k(t,\xi), \quad \beta_1 = g_k(t,\xi), \quad M = K_k,$$

after taking $\overline{N}(m)$ sufficiently large to have $|g_j| \leqslant 1/2$ for any $j = 1,\dots,m$. Indeed, by (finite) induction we can prove that $|g_j(t,\xi)|$ can be taken arbitrarily small in correspondence of sufficiently large N. This can be easily proved by having in mind the related symbol classes for g_j, μ_{j+1}, h_{j+1}

(see, for instance, Lemma 3 in [2]), which we did not introduced in this paper for the sake of brevity. By using the properties of the hyperbolic zone $Z_{\mathrm{hyp}}(N)$ this leads to the above estimate for g_j. Then it is clear that K_j is invertible.

We write the fundamental solution to (5.14) for $j = m$ in the form $E_m(t,s,\xi)Q_m(t,s,\xi)$ for $t,s \geqslant t_{|\xi|}$, where

$$E_m(t,s,\xi) = \begin{pmatrix} \exp(i\xi \int_s^t (a(\tau) + \mu_m(\tau,\xi))\,d\tau) & 0 \\ 0 & \exp(i\xi \int_s^t (a(\tau) - \mu_m(\tau,\xi))\,d\tau) \end{pmatrix}.$$

We have $\|E_m\| = 1$ and $E_m^{-1}(t,s,\xi) = E_m(s,t,\xi)$, whereas Q_m is bounded both from above and from below since it solves

$$\partial_t Q_m = -E_m^{-1} \begin{pmatrix} 0 & h_m \\ \overline{h_m} & 0 \end{pmatrix} E_m Q_m, \qquad Q_m(s,s,\xi) = \mathrm{I}_2.$$

Here we haven taken into consideration that (5.5) implies

$$\int_{t_{|\xi|}}^{\infty} \left|h_m(\tau,\xi)\right| d\tau \leqslant \frac{C'}{\Theta_1^{m-1}(t_{|\xi|})|\xi|^{m-1}} = \frac{C'}{N^{m-1}}.$$

Therefore, the fundamental solution $E(t,s,\xi)$ is bounded both from above and from below in $Z_{\mathrm{hyp}}(N)$ too. This completes the proof. □

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