An analogue of Beurling’s theorem for the Laguerre hypergroup

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Abstract

In this paper, we prove an analogue of Beurling’s theorem for the Laguerre hypergroup, then we derive some other versions of uncertainty principle.

Keywords: Laguerre hypergroup; Uncertainty principle; Beurling’s theorem

1. Introduction

The uncertainty principle states that a function and its Fourier transform cannot simultaneously decay very rapidly. This principle has several versions which were proved by Hardy, Morgan and Gelfand–Shilov, etc. (cf. [4,13]). A more general version of uncertainty principle, which is called Beurling’s theorem, has been proved by Hörmander [5] and generalized by Bonami et al. [2] as follows:

Theorem 1. Let \( f \in L^2(\mathbb{R}^d) \) and \( N \geq 0 \). Then

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x)||\hat{f}(y)|}{(1 + |x| + |y|)^N} e^{[x||y]} \, dx \, dy < \infty
\]

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implies that
\[ f(x) = P(x)e^{-\langle Ax, x \rangle}, \]
where \( A \) is a real positive definite symmetric matrix and \( P \) is a polynomial of degree \( < N - \frac{d}{2} \). In particular, \( f = 0 \) when \( N \leq d \).

**Remark 1.** We can easily derive from Theorem 1 the \( L^2 \)-version of Beurling’s theorem. For example, in case \( n = 1 \), if
\[
\int \int_{\mathbb{R}} |f(x)|^2 |\hat{f}(y)|^2 \frac{e^{2|x||y|}}{(1 + |x| + |y|)^{(N-1)/2}} \, dx \, dy < \infty,
\]
then \( f(x) = P(x)e^{-ax^2} \) where \( P(x) \) is a polynomial of degree \( < \frac{N-1}{4} \). In particular, \( f = 0 \) when \( N \leq 1 \).

The Beurling’s theorem has been extended to different settings (cf. [3,8,10]). The goal of this paper is to prove an analogue of Beurling’s theorem for the Laguerre hypergroup. In next section we state some basic knowledge about Laguerre hypergroup. The main result is proved in Section 3. In Section 4 we derive some other versions of uncertainty principle.

### 2. Preliminaries

In this section, we set some notations and collect some basic facts about the Laguerre hypergroup. We refer the reader to [9,16] and [11] for detail.

Given \( \alpha \geq 0 \), let \( \mathbb{K} = [0, \infty) \times \mathbb{R} \) equipped with the measure
\[
dm_{\alpha}(x, t) = \frac{1}{\pi \Gamma(\alpha + 1)} x^{2\alpha+1} \, dx \, dt.
\]
We simply write \( L^p(\mathbb{K}) \) instead of \( L^p(\mathbb{K}, dm_{\alpha}) \).

For \( (x, t) \in \mathbb{K} \), the generalized translation operators \( T_{(x,t)}^{(\alpha)} \) are defined by
\[
T_{(x,t)}^{(\alpha)} f(y, s) = \begin{cases} 
\frac{1}{2\pi} \int_0^{2\pi} f(\sqrt{x^2 + y^2 + 2xy \cos \theta}, s + t + xy \sin \theta) \, d\theta, & \text{if } \alpha = 0, \\
\frac{\alpha}{\pi} \int_0^{2\pi} \int_0^1 f(\sqrt{x^2 + y^2 + 2xyr \cos \theta}, s + t + xyr \sin \theta)r(1 - r^2)^{\alpha-1} \, dr \, d\theta, & \text{if } \alpha > 0.
\end{cases}
\]
Let \( M_b(\mathbb{K}) \) denote the space of bounded Radon measures on \( \mathbb{K} \). The convolution on \( M_b(\mathbb{K}) \) is defined by
\[
(\mu * \nu)(f) = \int_{\mathbb{K} \times \mathbb{K}} T_{(x,t)}^{(\alpha)} f(y, s) \, d\mu(x, t) \, d\nu(y, s).
\]
Then \( (\mathbb{K}, *, i) \) is a hypergroup in the sense of Jewett (cf. [1,7]), where \( i \) denotes the involution defined by \( i(x, t) = (x, -t) \). If \( \alpha = n - 1 \) is a nonnegative integer, then the Laguerre hypergroup \( \mathbb{K} \) can be identified with the hypergroup of radial functions on the Heisenberg group \( \mathbb{H}^n \).

We consider the partial differential operator
\[
L = -\left( \frac{\partial^2}{\partial x^2} + \frac{2\alpha + 1}{x} \frac{\partial}{\partial x} + x^2 \frac{\partial^2}{\partial t^2} \right).
\]
Let \( L \) be positive and symmetric in \( L_2^2(\mathbb{K}) \). When \( \alpha = n - 1 \), \( L \) is the radial part of the sublaplacian on the Heisenberg group \( \mathbb{H}^n \). We call \( L \) the generalized sublaplacian.

Let \( L_m^\alpha(x) \) be the Laguerre polynomial of degree \( m \) and order \( \alpha \) given by

\[
L_m^\alpha(x) = \sum_{j=0}^{m} \frac{\Gamma(m+\alpha+1)}{\Gamma(m-j+1)\Gamma(j+\alpha+1)} \frac{(-x)^j}{j!}.
\]

For \( (\lambda, m) \in \mathbb{R} \times \mathbb{N} \), we put

\[
\varphi(\lambda, m)(x, t) = \frac{m! \Gamma(\alpha+1)}{\Gamma(m+\alpha+1)} e^{i\lambda t} e^{-\frac{|\lambda|^2}{2}} L_m(\alpha)(|\lambda|^2).
\]

**Lemma 1.** The function \( \varphi(\lambda, m) \) satisfies that

(a) \( \|\varphi(\lambda, m)\|_{\alpha, \infty} = \varphi(\lambda, m)(0, 0) = 1 \),

(b) \( T_{(x,t)}(\alpha) \varphi(\lambda, m)(y, s) = \varphi(\lambda, m)(x, t) \varphi(\lambda, m)(y, s) \),

(c) \( L \varphi(\lambda, m) = 4|\lambda|(m + \frac{\alpha+1}{2}) \varphi(\lambda, m) \).

Let \( f \in L^1(\mathbb{K}) \), the generalized Fourier transform of \( f \) is defined by

\[
\hat{f}(\lambda, m) = \int_{\mathbb{K}} f(x, t) \varphi_{-\lambda, m}(x, t) \, dm_\alpha(x, t).
\]

Let \( d\gamma_\alpha \) be the positive measure defined on \( \mathbb{R} \times \mathbb{N} \) by

\[
\int_{\mathbb{R} \times \mathbb{N}} g(\lambda, m) \, d\gamma_\alpha(\lambda, m) = \sum_{m=0}^{\infty} \frac{\Gamma(m+\alpha+1)}{m! \Gamma(\alpha+1)} \int_{\mathbb{R}} g(\lambda, m) |\lambda|^\alpha \, d\lambda.
\]

We write \( L^p(\mathbb{K}) \) instead of \( L^p(\mathbb{R} \times \mathbb{N}, d\gamma_\alpha) \). The generalized Fourier transform extends to an isometric isomorphism from \( L^2(\mathbb{K}) \) onto \( L^2(\mathbb{K}) \). We also have the inverse formula of the generalized Fourier transform.

\[
f(x, t) = \int_{\mathbb{R} \times \mathbb{N}} \hat{f}(\lambda, m) \varphi_{-\lambda, m}(x, t) \, d\gamma_\alpha(\lambda, m)
\]

provided \( \hat{f} \in L^1(\mathbb{K}) \).

We also need the following lemma (cf. [12]).

**Lemma 2.** Let \( \varphi_m(x) \) be the Laguerre function defined by

\[
\varphi_m(x) = \left( \frac{2m!}{\Gamma(m+\alpha+1)} \right)^\frac{1}{2} e^{-\frac{x^2}{2}} L_m^\alpha(x^2).
\]

For any \( r > 0 \), the system \( \{ r^{\alpha+1} \varphi_m(rx) : m \in \mathbb{N} \} \) forms an orthonormal basis of the space \( L^2([0, \infty), x^{2\alpha+1} \, dx) \).

**3. An analogue of Beurling’s theorem**

In this section, we prove the following theorem of Beurling type.
Theorem 2. Let $f \in L^2(K)$ and $N \geq 0$. Suppose that
\[
\int_{K} \int_{\mathbb{R} \times N} |f(x, t)|^2 |\hat{f}(\lambda, m)|^2 \frac{e^{2t|\lambda|}}{(1 + |t| + |\lambda|)^N} \, dm_\alpha(x, t) \, d\gamma_\alpha(\lambda, m) < \infty.
\] (1)

Then
\[
f(x, t) = e^{-at} \left( \sum_{j=0}^{k} \psi_j(x) t^j \right)
\]
where $a > 0$, $k < \frac{N-1}{4}$, and $\psi_j(x) \in L^2([0, \infty), x^{2\alpha+1} \, dx)$. In particular, when $N \leq 5$,
\[
f(x, t) = e^{-at} \psi(x)
\]
where $\psi \in L^2([0, \infty), x^{2\alpha+1} \, dx)$. If $N \leq 1$, then $f = 0$.

Proof. Write
\[
f^\lambda(x) = \int_{\mathbb{R}} f(x, t) e^{-it\lambda} \, dt.
\]

Then
\[
\hat{f}(\lambda, m) = \int_{K} f(x, t) \varphi_{-\lambda, m}(x, t) \, dm_\alpha(x, t)
\]
\[
= \left( \frac{m!}{2\pi^2 |\lambda|^{\alpha+1} \Gamma(m + \alpha + 1)} \right)^{\frac{1}{2}} \int_0^\infty f^\lambda(x) |\lambda|^{\frac{\alpha+1}{2}} \varphi_m(\sqrt{\lambda} x) x^{2\alpha+1} \, dx.
\]

By Lemma 2,
\[
\int_0^\infty |f^\lambda(x)|^2 x^{2\alpha+1} \, dx = 2\pi^2 |\lambda|^{\alpha+1} \sum_{m=0}^{\infty} \frac{\Gamma(m + \alpha + 1)}{m!} |\hat{f}(\lambda, m)|^2.
\] (2)

Substituting (2) in (1), we get
\[
\int_{K} \int_{\mathbb{R} \times N} \frac{|f(x, t)|^2 |\hat{f}(\lambda, m)|^2}{(1 + |t| + |\lambda|)^N} \, e^{2t|\lambda|} \, dm_\alpha(x, t) \, d\gamma_\alpha(\lambda, m)
\]
\[
= \frac{1}{2\pi^3 \Gamma(\alpha + 1)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} G(t, \lambda) \frac{e^{2t|\lambda|}}{(1 + |t| + |\lambda|)^N} \, dt \, d\lambda < \infty,
\] (3)

where
\[
G(t, \lambda) = \int_0^\infty \int_0^\infty |f(x, t)|^2 |f^\lambda(y)|^2 x^{2\alpha+1} y^{2\alpha+1} \, dx \, dy.
\]

Let
\[
g_m(t) = \int_0^\infty f(x, t) \varphi_m(x) x^{2\alpha+1} \, dx.
\] (4)
Then
\[ \hat{g}_m(\lambda) = \int_0^\infty f^\lambda(x)\varphi_m(x)x^{2\alpha+1} \, dx. \]

By Lemma 2,
\[ G(t,\lambda) = \sum_{m=0}^\infty \sum_{n=0}^\infty |g_m(t)|^2 |\hat{g}_n(\lambda)|^2. \]

It follows from (3) and (5) that, for any \( m, n \in \mathbb{N} \),
\[ \int \int \frac{|g_m(t)|^2 |\hat{g}_n(\lambda)|^2}{(1 + |t| + |\lambda|)^N} e^{2|t||\lambda|} \, dt \, d\lambda < \infty. \] (6)

Specifically,
\[ \int \int \frac{|g_m(t)|^2 |\hat{g}_n(\lambda)|^2}{(1 + |t| + |\lambda|)^N} e^{2|t||\lambda|} \, dt \, d\lambda < \infty. \]

By Remark 1, when \( N \leq 1 \), \( g_m(t) = 0 \) for any \( m \in \mathbb{N} \). Therefore \( f = 0 \). When \( N > 1 \),
\[ g_m(t) = P_m(t)e^{-a_m t^2} \]
where \( a_m > 0 \) and each \( P_m(t) \) is a polynomial of degree \( k < \frac{N-1}{4} \). Hence
\[ f(x,t) = \sum_{m=0}^\infty P_m(t)e^{-a_m t^2} \varphi_m(x). \]

If \( a_m \neq a_n \) for some \( m, n \in \mathbb{N} \), then (6) would be false. So \( a_m = a \) are independent of \( m \). Let
\[ P_m(t) = \sum_{j=0}^k c_{m,j}t^j. \]

Then
\[ f(x,t) = e^{-at^2} \left( \sum_{j=0}^k \psi_j(x)t^j \right), \]
where
\[ \psi_j(x) = \sum_{m=0}^\infty c_{m,j}\varphi_m(x) \in L^2([0, \infty), x^{2\alpha+1} \, dx). \]

The proof of Theorem 2 is completed. \( \Box \)

**Remark 2.** We note that Theorem 2 is essentially about the central variable as the most results on the nilpotent Lie groups. Thangavelu [14] gives a heat kernel version of Hardy’s theorem on the Heisenberg group. A heat kernel version of Hardy’s theorem for the Laguerre hypergroup is also true (cf. [6]). As pointed in [15], there are some difficulties to extend the Beurling’s theorem to a general setting. The heat kernel version of Beurling’s theorem for the Laguerre hypergroup is still open.
4. Some other versions of uncertainty principle

In this section, we derive some other versions of uncertainty principle on the Laguerre hyper-group.

The following uncertainty principle of Gelfand–Shilov type is a direct consequence of Theorem 2.

Corollary 1. Let \( N \geq 0 \) and assume that \( f \in L^2(\mathbb{K}) \) satisfies

\[
\int_\mathbb{K} |f(x,t)|^2 e^{\frac{2}{p}a|t|^p} \, dm_\alpha(x,t) < \infty,
\]

\[
\int_{\mathbb{R} \times \mathbb{N}} |\hat{f}(\lambda,m)|^2 e^{\frac{2}{q}b|\lambda|^q} \, d\gamma_\alpha(\lambda,m) < \infty,
\]

where \( 1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1 \) and \( a, b \) are positive numbers such that \( ab \geq 1 \). Then \( f = 0 \) unless \( p = q = 2, ab = 1 \) and \( N > 1 \), in which case

\[
f(x,t) = e^{-at^2} \left( \sum_{j=0}^{k} \psi_j(x)t^j \right),
\]

where \( k < \frac{N-1}{2} \) and \( \psi_j(x) \in L^2([0, \infty), x^{2\alpha+1} \, dx) \). In particular, when \( N \leq 3 \),

\[
f(x,t) = e^{-at^2} \psi(x),
\]

where \( \psi \in L^2([0, \infty), x^{2\alpha+1} \, dx) \).

The lower bound 1 in Corollary 1 is not sharp for \( p \neq 2 \). The critical low bound should be \( |\cos(\frac{p\pi}{2})|^{1/p} \) as proved by Bonami et al. [2] on Euclidean spaces.

Theorem 3. Let \( f \in L^2(\mathbb{K}) \) satisfies

\[
\int_\mathbb{K} |f(x,t)|^2 e^{\frac{2}{p}a|t|^p} \, dm_\alpha(x,t) < \infty, \quad (7)
\]

\[
\int_{\mathbb{R} \times \mathbb{N}} |\hat{f}(\lambda,m)|^2 e^{\frac{2}{q}b|\lambda|^q} \, d\gamma_\alpha(\lambda,m) < \infty, \quad (8)
\]

where \( 1 < p < 2, \frac{1}{p} + \frac{1}{q} = 1 \) and \( a, b \) are positive numbers. Then \( f = 0 \) if \( ab > |\cos(\frac{p\pi}{2})|^{1/p} \).

Proof. We can choose \( \epsilon > 0 \) such that \((a - \epsilon)(b - \epsilon) > |\cos(\frac{p\pi}{2})|^{1/p}\). By same argument of Theorem 2, for any \( m \in \mathbb{N} \), we get from (7) and (8) that

\[
\int_\mathbb{R} |g_m(t)|^2 e^{\frac{2}{p}a|t|^p} \, dt < \infty,
\]

\[
\int_\mathbb{R} |\hat{g}_m(\lambda)|^2 e^{\frac{2}{q}b|\lambda|^q} \, d\lambda < \infty,
\]
where \( g_m \) is given by (4). It follows that
\[
\int_{\mathbb{R}} |g_m(t)| e^{\frac{1}{p}(a-\epsilon)p|t|^p} \, dt < \infty,
\]
\[
\int_{\mathbb{R}} \left| \hat{g}_m(\lambda) \right| e^{\frac{1}{q}(b-\epsilon)q|\lambda|^q} \, d\lambda < \infty.
\]
By Theorem 1.4 in [2], \( g_m = 0 \). Theorem 3 is proved. \( \square \)

It is easy to derive from above some other versions of uncertainty principle. For example, we have

**Corollary 2 (Morgan type).** Suppose \( f \in L^2(\mathbb{K}) \) satisfies
\[
\int_0^\infty \left| f(x,t) \right|^2 x^{2\alpha+1} \, dx \leq C_1 e^{-2a|\lambda|^p},
\]
\[
\sum_{m=0}^\infty \frac{\Gamma(m + \alpha + 1) |\lambda|^{\alpha+1}}{m!\Gamma(\alpha+1)} \left| \hat{f}(\lambda, m) \right|^2 \leq C_2 e^{-2b|\lambda|^q},
\]
where \( C_1, C_2 \) are all positive constants, \( 1 < p < 2, \frac{1}{p} + \frac{1}{q} = 1 \), and \( a, b \) are positive numbers. Then \( f = 0 \) if \( ab > \frac{1}{4} \left| \cos \left( \frac{p\pi}{2} \right) \right|^2 \).

**Corollary 3 (Hardy type).** Suppose \( f \in L^2(\mathbb{K}) \) satisfies
\[
\left| f(x,t) \right| \leq C_1 e^{-a(t^2+x^2)},
\]
\[
\left| \hat{f}(\lambda, m) \right| \leq C_2 e^{-b(\lambda^2+m^2)},
\]
where \( C_1, C_2 \) are positive constants, and \( a, b \) are positive numbers such that \( ab \geq \frac{1}{4} \). If \( ab > \frac{1}{4} \), then \( f = 0 \). When \( ab = \frac{1}{4} \),
\[
f(x,t) = e^{-at^2} \psi(x),
\]
where \( \psi \in L^2([0, \infty), x^{2\alpha+1} \, dx) \).

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**References**