ACADEMIC PRESS
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SCIENCE DIRECT®
Journal of Functional Analysis 200 (2003) 23–30
JOURNAL OF Functional Analysis
http://www.elsevier.com/locate/jfa

Meet-irreducible ideals and representations of limit algebras

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Received 19 March 2002; accepted 17 May 2002

Abstract

In this paper we give criteria for an ideal $\mathscr{J}$ of a TAF algebra $\mathscr{A}$ to be meet-irreducible. We show that $\mathscr{J}$ is meet-irreducible if and only if the C^*-envelope of $\mathscr{A}/\mathscr{J}$ is primitive. In that case, $\mathscr{A}/\mathscr{J}$ admits a faithful nest representation which extends to a $*$-representation of the C^*-envelope for $\mathscr{A}/\mathscr{J}$. We also characterize the meet-irreducible ideals as the kernels of bounded nest representations; this settles the question of whether the n-primitive and meet-irreducible ideals coincide.

MSC: 47L80

1. Introduction

Representation theory of operator algebras is still in its infancy. While for C^*-algebras the fundamentals of representation theory have long been known, for nonself-adjoint algebras there are hardly any results of a general nature. For 'triangular operator algebras' (a term which we leave undefined), intuition suggests that the fundamental building blocks for representation theory should be nest

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[1] Partially supported by an NSERC Grant.
[2] Research partially supported by a Grant from ECU.

PII: S0022-1236(02)00058-7

representations [5–7]. In the category of C^*-algebras, the nest representations are precisely the irreducible representations.

Recall that a *nest representation* is a representation for which the closed, invariant subspaces form a nest (i.e., are linearly ordered). In his study of nonself-adjoint crossed products, Lamoureux introduced the notion of *n-primitive ideal*. An ideal is n-primitive if it is the kernel of a nest representation. Lamoureux has shown that in various contexts in nonself-adjoint algebras the n-primitive ideals play a role analogous to the primitive ideals in C^*-algebras. Thus, one can give the set of n-primitive ideals the hull-kernel topology, and for every (closed, two-sided) ideal $\mathscr{I}$ in the algebra, $\mathscr{I}$ is the intersection of all n-primitive ideals containing $\mathscr{I}$; in other words, $\mathscr{I} = k(h(\mathscr{I}))$.

An ideal $\mathscr{J}$ of an algebra $\mathscr{A}$ is *meet-irreducible* if, for any ideals $\mathscr{I}_1$ and $\mathscr{I}_2$ containing $\mathscr{J}$, the relation $\mathscr{I}_1 \cap \mathscr{I}_2 = \mathscr{J}$ implies that either $\mathscr{I}_1 = \mathscr{J}$ or $\mathscr{I}_2 = \mathscr{J}$. In the case of T_n, the algebra of upper triangular $n \times n$ matrices, meet-irreducible ideals are obtained by 'cutting a wedge' from the algebra: let $1 \leqslant i_0 \leqslant j_0 \leqslant n$. The ideal

$$\mathscr{I} = \{(a_{ij}) : a_{ij} = 0, \ \ i_0 \leqslant i \leqslant j \leqslant j_0\}$$

is meet-irreducible, and every meet-irreducible ideal of T_n has this form.

The relationship between meet-irreducible and n-primitive ideals is studied in a variety of examples in [7], and in [3] meet-irreducible ideals in strongly maximal triangular AF-algebras are characterized by sequences of matrix units and also in terms of groupoids. In that paper it is shown that every meet-irreducible ideal is n-primitive. This is done by constructing a nest representation. The converse question, whether every n-primitive ideal is meet-irreducible, was left open.

In a recent work [2], the first two authors examined the C^*-envelope of a quotient $\mathscr{A}/\mathscr{J}$ of a strongly maximal TAF algebra by an ideal $\mathscr{J}$. They showed the C^*-envelope is an AF C^*-algebra, even though the quotient $\mathscr{A}/\mathscr{J}$ is not in general a TAF algebra. It turns out that the C^*-envelope of $\mathscr{A}/\mathscr{J}$ is sensitive enough to detect the meet irreducibility of $\mathscr{J}$. In Theorem 2.3 we show that $\mathscr{J}$ is meet-irreducible if and only if $C^*_{\text{env}}(\mathscr{A}/\mathscr{J})$, the C^*-envelope of $\mathscr{A}/\mathscr{J}$, is primitive. The theory of C^*-envelopes provides the natural framework for studying results of this type. In Theorem 2.4 we show that for a meet-irreducible ideal $\mathscr{J}$, there exists a faithful and irreducible $*$-representation of $C^*_{\text{env}}(\mathscr{A}/\mathscr{J})$, whose restriction on $\mathscr{A}/\mathscr{J}$ is a nest representation. Since the converse is easily seen to be true, Theorem 2.4 provides a characterization of meet-irreducible ideals in terms of the representation theory for $\mathscr{A}$.

The question of whether the kernel of a nest representation is a meet-irreducible ideal emerged at the Ambelside, UK conference in summer, 1997. Subsequently some progress was made. In [4] a partial result was obtained: if the TAF algebra $\mathscr{A}$ has totally ordered spectrum, or if the nest representation π has the property that the von Neumann algebra generated by $\pi(\mathscr{A} \cap \mathscr{A}^*)$ contains an atom, then $\ker(\pi)$ is meet-irreducible. The solution presented in Theorem 2.6 is self-contained and does not make use of the results of [3] or [4]. Thus the question is now settled for strongly maximal TAF algebras.

Despite the fact that evidence at hand is limited, it nonetheless seems worthwhile to ask.

Question 1.1. Are there any operator algebras for which the n-primitive ideals, and the meet-irreducible ideals do not coincide?

2. The main results

We begin by recalling a result of Lamoureux [7].

Lemma 2.1. *Let $\mathscr{I}$ be a closed, two-sided ideal in a separable C^*-algebra $\mathscr{A}$. Then the following are equivalent*:

(i) *$\mathscr{I}$ is n-primitive,*
(ii) *$\mathscr{I}$ is primitive,*
(iii) *$\mathscr{I}$ is prime,*
(iv) *$\mathscr{I}$ is meet-irreducible.*

One can actually characterize when an AF C^*-algebra is primitive in terms of its Bratteli diagram. Let $\mathfrak{A} = \varinjlim(\mathfrak{A}_{\mathrm{i}}, \varphi_{\mathrm{i}})$ be an AF C^*-algebra and assume that each $\mathfrak{A}_i$ decomposes as a direct sum $\mathfrak{A}_i = \oplus_j \mathfrak{A}_{ij}$ of finite-dimensional full matrix algebras $\mathfrak{A}_{ij}$. A *path* Γ for $\mathfrak{A} = \varinjlim(\mathfrak{A}_{\mathrm{i}}, \varphi_{\mathrm{i}})$ is a sequence $\{\mathfrak{A}_{ij_i}\}_{i=1}^{\infty}$ so that for each pair of nodes $((i,j_i),(i+1,j_{i+1}))$ there exist an arrow in the Bratteli diagram for $\mathfrak{A} = \varinjlim(\mathfrak{A}_{\mathrm{i}}, \varphi_{\mathrm{i}})$ which joins them. It is known that $\mathfrak{A}$ is primitive iff there is a path Γ for $\mathfrak{A} = \varinjlim(\mathfrak{A}_{\mathrm{i}}, \varphi_{\mathrm{i}})$ so that each summand of $\mathfrak{A}_i$ is eventually mapped into a member of Γ. We call such a path Γ an *essential path* for $\mathfrak{A}$.

Beyond C^*-algebras, a meet-irreducible ideal need not be primitive. In [3], a description of all meet-irreducible ideals of a TAF algebra was given in terms of matrix unit sequences.

Definition 2.2. Let $\mathscr{A} = \varinjlim(\mathscr{A}_{\mathrm{i}}, \varphi_{\mathrm{i}})$ be a TAF algebra. A sequence $(e_i)_{i \geqslant N}$ of matrix units from $\mathscr{A}$ will be called an *mi-chain* if the following two conditions are satisfied for all $i \geqslant N$:

(A) $e_i \in \mathscr{A}_i$.
(B) $e_{i+1} \in \mathrm{Id}_{i+1}(e_i)$,

where $\mathrm{Id}_{i+1}(e_i)$ denotes the ideal generated by e_i in $\mathscr{A}_{i+1}$.

If $(e_i)_{i \geqslant N}$ is an mi-chain for $\mathscr{A} = \varinjlim(\mathscr{A}_{\mathrm{i}}, \varphi_{\mathrm{i}})$, let $\mathscr{J}$ be the join of all ideals which do not contain any matrix unit e_i from the chain. In [3, Theorem 1.2] it is shown that for a TAF algebra $\mathscr{A} = \varinjlim(\mathscr{A}_{\mathrm{i}}, \varphi_{\mathrm{i}})$, given an mi-chain $(e_i)_{i \geqslant N}$, the ideal $\mathscr{J}$ associated with $(e_i)_{i \geqslant N}$ is meet-irreducible. Conversely, every proper meet-irreducible ideal in $\mathscr{A} = \varinjlim(\mathscr{A}_i, \varphi_i)$ is induced by some mi-chain, chosen from some contraction of this representation.

In this paper we give a characterization of the meet-irreducible ideals of TAF algebras in terms of C^*-envelopes of quotient algebras. We need to recall the notation and machinery from [2].

Let $\mathfrak{A} = \varinjlim(\mathfrak{A}_i, \varphi_i)$ be the enveloping C^*-algebra for a TAF algebra $\mathscr{A} = \varinjlim(\mathscr{A}_i, \varphi_i)$. Let $\mathscr{J} \subseteq \mathscr{A}$ be a closed ideal, and let $\mathscr{J}_i := \mathscr{J} \cap \mathscr{A}_i$. For each $i \geqslant 1$, $\mathscr{S}_i$ denotes the collection of all diagonal projections p which are semi-invariant for $\mathscr{A}_i$, are supported on a single summand of $\mathfrak{A}_i$ and satisfy $(p\mathscr{A}_i p) \cap \mathscr{J} = \{0\}$. We form finite dimensional C^*-algebras

$$\mathfrak{B}_i := \sum_{p \in S_i} \oplus \mathscr{B}(\operatorname{Ran} p),$$

where $\mathscr{B}(\operatorname{Ran} p)$ denotes the bounded operators on $\operatorname{Ran} p$. Of course, $\mathscr{B}(\operatorname{Ran} p)$ is isomorphic to $\mathfrak{M}_{\operatorname{rank} p}$. Let σ_i be the map from $\mathfrak{A}_i$ into $\mathfrak{B}_i$ given by $\sigma_i(a) = \sum_{p \in S_i}^{\oplus} pap|_{\operatorname{Ran} p}$. The map $\sigma_i|_{\mathscr{A}_i}$ factors as $\rho_i q_i$ where q_i is the quotient map of $\mathscr{A}_i$ onto $\mathscr{A}_i/\mathscr{J}_i$ and ρ_i is a completely isometric homomorphism of $\mathscr{A}_i/\mathscr{J}_i$ into $\mathfrak{B}_i$. Notice that $\mathfrak{B}_i$ equals the C^*-algebra generated by $\rho_i(\mathscr{A}_i/\mathscr{J}_i)$.

We then consider unital embeddings π_i of $\mathfrak{B}_i$ into $\mathfrak{B}_{i+1}$ defined as follows. For each $q \in \mathscr{S}_{i+1}$ we choose projections $p \in \mathscr{S}_i$ which maximally embed into q under the action of φ_i. This way, we determine multiplicity one embeddings of $\mathscr{B}(\operatorname{Ran} p)$ into $\mathscr{B}(\operatorname{Ran} q)$. Taking into account all such possible embeddings, we determine the embedding π_i of $\mathfrak{B}_i$ into $\mathfrak{B}_{i+1}$.

Finally we form the subsystem of the directed limit $\mathfrak{B} = \varinjlim(\mathfrak{B}_i, \pi_i)$ corresponding to all summands which are *never* mapped into a summand $\mathscr{B}(\operatorname{Ran} p)$ where p is a maximal element of some $\mathscr{S}_i$. Evidently, this system is directed upwards. It is also hereditary in the sense that if every image of a summand lies in one of the selected blocks, then it clearly does not map into a maximal summand and thus already lies in our system. By Davidson [1, Theorem III.4.2], this system determines an ideal $\mathfrak{I}$ of $\mathfrak{B}$. The quotient $\mathfrak{B}' = \mathfrak{B}/\mathfrak{I}$ is the AF algebra corresponding to the remaining summands and the remaining embeddings; it can be expressed as a direct limit $\mathfrak{B}' = \varinjlim(\mathfrak{B}'_i, \pi'_i)$, with the understanding that $\mathfrak{B}'_i = \oplus_j \mathfrak{B}_{ij}$ for these remaining summands $\mathfrak{B}_{ij}$ of $\mathfrak{B}_i$. It can be seen that the quotient map is isometric on $\mathscr{A}/\mathscr{J}$ and that $\mathfrak{B}'$ is the C^*-envelope of $A/\mathscr{J}$.

Theorem 2.3. *Let $\mathscr{A}$ be a TAF algebra and let $\mathscr{J} \subseteq \mathscr{A}$ be an ideal. Then $\mathscr{J}$ is meet-irreducible if and only if the algebra $C^*_{\text{env}}(\mathscr{A}/\mathscr{J})$ is primitive.*

Proof. Assume that $\mathfrak{B}' = C^*_{\text{env}}(\mathscr{A}/\mathscr{J})$ is primitive and let $\Gamma = (\mathfrak{B}_{ij_i})_{i=1}^{\infty}$ an essential path for $\mathfrak{B}'$. Let e_i in $\mathfrak{B}_{ij_i}$ be the *characteristic matrix units* for $\mathfrak{B}_{ij_i}$, i.e., the ones on the top right corner of $\mathfrak{B}_{ij_i}$.

Assume that there exist ideals $\mathscr{I}_1$ and $\mathscr{I}_2$, properly containing $\mathscr{J}$. Since $\mathscr{I}_1$ and $\mathscr{I}_2$ properly contain $\mathscr{J}$, there exist matrix units $f_k \in \mathscr{I}_k$ with $f_k \notin \mathscr{J}$, $k = 1, 2$. So the images of the f_k appear in the presentation for the C^*-envelope in perhaps different summands. However, the existence of an essential path Γ implies that some subordinates for the f_k will appear in some member of Γ, say $\mathfrak{B}_{ij_i}$, for i sufficiently large, and so $e_i \in \mathscr{I}_1 \cap \mathscr{I}_2$. However, $e_i \notin \mathscr{J}$ and so $\mathscr{J}$ is properly contained in $\mathscr{I}_1 \cap \mathscr{I}_2$. It follows $\mathscr{J}$ is meet-irreducible.

Conversely, assume that $\mathscr{J}$ is meet-irreducible. In light of Lemma 2.1 and the subsequent comments, it suffices to show that the trivial ideal $\{0\}$ is meet-irreducible in the C^*-envelope $\mathfrak{B}'$.

By way of contradiction assume that there are nontrivial ideals $\mathscr{I}_1$ and $\mathscr{I}_2$ of $\mathfrak{B}'$ so that $\mathscr{I}_1 \cap \mathscr{I}_2 = \{0\}$. We claim that $(\mathscr{A}/\mathscr{J}) \cap \mathscr{I}_k \neq \{0\}$, $k = 1, 2$. Indeed, any nontrivial summand of $\mathscr{I}_k$ will eventually be mapped into a direct summand $\mathfrak{B}_{ij_i}$ of $\mathfrak{B}'$ corresponding to some maximal element of $\mathscr{S}_i$. Hence all matrix units in $\mathfrak{B}_{ij_i}$ belong to $\mathscr{I}_k$, including the characteristic one. This one however also belongs to $\mathscr{A}/\mathscr{J}$ and therefore in the intersection $(\mathscr{A}/\mathscr{J}) \cap \mathscr{I}_k$.

The claim shows that the zero ideal is not meet-irreducible in $\mathscr{A}/\mathscr{J}$. By considering the pullback, this implies that $\mathscr{J}$ is not meet-irreducible in $\mathscr{A}$, which is the desired contradiction.

Notice that the sequence $(e_i)_{i=1}^{\infty}$ associated with the path Γ in the proof above satisfies conditions (A) and (B) of Definition 2.2 and is therefore an mi-chain for the ideal $\mathscr{J}$.

Theorem 2.4. *If $\mathscr{A}$ is a TAF algebra and $\mathscr{J}$ an ideal of $\mathscr{A}$, then the following are equivalent*:

(i) *There exists a faithful representation $\tau : C^*_{\text{env}}(\mathscr{A}/\mathscr{J}) \to \mathscr{B}(\mathscr{H})$ so that $\tau(\mathscr{A}/\mathscr{J})$ is weakly dense in some nest algebra.*

(ii) *$\mathscr{J}$ is meet-irreducible.*

Proof. Assume that (i) is valid and let $\tau : C^*_{\text{env}}(\mathscr{A}/\mathscr{J}) \to \mathscr{B}(\mathscr{H})$ be a faithful representation so that $\tau(\mathscr{A}/\mathscr{J})$ weakly dense in some nest algebra $\operatorname{Alg} \mathscr{N}$. By way of contradiction assume that $\mathscr{J}$ is not meet-irreducible. Theorem 2.3 and Lemma 2.1 imply the existence of nonzero closed ideals $\mathscr{I}_1$ and $\mathscr{I}_2$ in $C^*_{\text{env}}(\mathscr{A}/\mathscr{J})$ so that $\mathscr{I}_1 \mathscr{I}_2 = \{0\}$. The nonzero subspaces $[\tau(\mathscr{I}_i)\mathscr{H}]$ must be mutually orthogonal. However they are both invariant under $\tau(\mathscr{A}/\mathscr{J})$, and therefore belong to $\mathscr{N}$, a contradiction.

Conversely, assume that (ii) is valid and so, by Theorem 2.3, we know that $C^*_{\text{env}}(\mathscr{A}/\mathscr{J})$ is primitive. Retain the notation established in the paragraphs preceding Theorem 2.3. Hence

$$C^*_{\text{env}}(\mathscr{A}/\mathscr{J}) = \mathfrak{B}' = \varinjlim(\mathfrak{B}'_{\mathrm{i}}, \pi'_{\mathrm{i}}),$$

where $\mathfrak{B}_i' = \oplus_j \mathfrak{B}_{ij}$ for the remaining summands $\mathfrak{B}_{ij}$ of $\mathfrak{B}_i$. Let $\Gamma = (\mathfrak{B}_{ij_i})_{i=1}^{\infty}$ be the essential path for $\mathfrak{B}'$.

Each $\mathfrak{B}_{ij}$ is a full matrix algebra and therefore contains the algebra $\mathcal{T}_{ij}$ of upper triangular matrices. Form the finite-dimensional algebras $\mathcal{T}_i' = \oplus_j \mathcal{T}_{ij}$ and consider the direct limit algebra

$$\mathcal{T}' = \varinjlim(\mathcal{T}_{\mathrm{i}}', \pi_{\mathrm{i}}'),$$

where π_i' is as earlier. Clearly, $\mathcal{T}'$ is a TAF algebra whose enveloping C^*-algebra is $\mathfrak{B}'$. Moreover, $\mathcal{T}'$ contains $\mathcal{A}/\mathcal{J}$.

We define a state ω on $\mathfrak{B}'$ as follows. Let $(p_i)_{i=1}^{\infty}$ be a sequence of diagonal projections with $p_i \in \mathcal{T}_{ij_i}$ so that p_{i+1} is a subordinate of p_i, $i \in \mathbb{N}$. We define $\omega_i : \mathfrak{B}_i' \to \mathbb{C}$ to be the compression on p_i and we let ω to be the direct limit $\omega = \varinjlim \omega_{\mathrm{i}}$. Consider the GNS triple $(\tau, \mathcal{H}, g)$ associated with the state ω, i.e., τ is a representation of $\mathfrak{B}'$ on $\mathcal{H}$ and $g \in \mathcal{H}$ so that $\omega(a) = \langle \tau(a)g, g \rangle$, $a \in \mathfrak{B}'$. Since ω is pure, τ is irreducible. Moreover, $p_i \in \mathfrak{B}_{ij_i}$, $i \in \mathbb{N}$ and so τ is also faithful.

An alternative presentation for $(\tau, \mathcal{H}, g)$ was given in [8, Proposition II.2.2]. Since ω is multiplicative on the diagonal $\mathcal{T}' \cap (\mathcal{T}')^*$, one considers $\mathcal{H}$ to be $L^2(\mathcal{X}, \mu)$, where $\mathcal{X}$ is the Gelfand spectrum of $\mathcal{T}' \cap (\mathcal{T}')^*$ and μ the counting measure on the orbit of ω in $\mathcal{X}$. With these identifications, given any matrix unit e, $\tau(e)$ is the translation operator on $\mathcal{X}$ defined in the paragraphs preceding [8, Theorem II.1.1].

In [8, Proposition II.2.2] it is shown that τ maps $\mathcal{T}'$ onto a weakly dense subset of some nest algebra. The proof of the theorem will follow if we show that the weak closure of $\tau(\mathcal{A}/\mathcal{J})$ contains $\tau(\mathcal{T}')$.

A moment's reflection shows that given any contraction $a \in \mathcal{T}_{ij}'$ and matrix units $e_1, e_2, \ldots, e_n$ and $f_1, f_2, \ldots, f_n$ in $\mathfrak{B}'$, there exists a contraction $\hat{a} \in \mathcal{A}/\mathcal{J}$ so that

$$\omega(f_k^* \hat{a} e_k) = \omega(f_k^* a e_k)$$

and therefore

$$\langle \tau(\hat{a})\tau(e_k)g, \tau(f_k)g \rangle = \langle \tau(a)\tau(e_k)g, \tau(f_k)g \rangle$$

for all $k = 1, 2, \ldots, n$. However the collection of all vectors of the form $\tau(e)g$, where e ranges over all matrix units of $\mathfrak{B}'$, forms a dense subset of $\mathcal{H}$ and so the desired density follows.

Remark 2.5. (1) Implication (i) $\Rightarrow$ (ii) also follows from Theorem 2.6.

(2) There exists a faithful representation τ of $C^*_{\mathrm{env}}(\mathcal{A}/\mathcal{J})$ in $\mathcal{B}(\mathcal{H})$ so that $\tau(\mathcal{A}/\mathcal{J})$ is weakly dense in a nest algebra if and only if there is a faithful *irreducible* representation τ of $C^*_{\mathrm{env}}(\mathcal{A}/\mathcal{J})$ in $\mathcal{B}(\mathcal{H})$ so that $\tau(\mathcal{A}/\mathcal{J})$ is weakly dense in a nest algebra.

Theorem 2.6. *Let $\mathscr{A}$ be a strongly maximal TAF algebra, and let π be a bounded nest representation of $\mathscr{A}$ on a Hilbert space $\mathscr{H}$. Then $\ker(\pi)$ is a meet-irreducible ideal.*

Proof. Since a bounded representation of the diagonal masa $\mathscr{A} \cap \mathscr{A}^*$ is completely bounded, cf. [9, Theorem 8.7], and a completely bounded representation is similar to a completely contractive representation by Paulsen [9, Theorem 8.1], we may assume that the restriction of π to the diagonal masa is completely contractive. It follows that the restriction of π to the diagonal masa is a star representation. Let $\mathscr{J} = \ker(\pi)$, and $\mathscr{J}_1, \mathscr{J}_2$ be ideals in $\mathscr{A}$ properly containing $\mathscr{J}$. We need to show that $\mathscr{J}_1 \cap \mathscr{J}_2$ properly contains $\mathscr{J}$.

Since π is a nest representation, we have (after possibly interchanging $\mathscr{J}_1$ and $\mathscr{J}_2$) that

$$(0) \neq [\pi(\mathscr{J}_1)\mathscr{H}] \subseteq [\pi(\mathscr{J}_2)\mathscr{H}],$$

where $[\mathscr{X}]$ denotes the closed subspace generated by $\mathscr{X} \subset \mathscr{H}$. Fix $n \in \mathbb{N}$ and let u be a matrix unit in $\mathscr{J}_1 \cap \mathscr{A}_n \backslash \mathscr{J}$. Choose a vector $h \in \mathscr{H}$ be such that $\|\pi(u)h\| = 1$. There exist $m \geqslant n$, $N \geqslant 1$, matrix units $v_t \in \mathscr{J}_2 \cap \mathscr{A}_m$ and vectors $h_t \in \mathscr{H}$ for $1 \leqslant t \leqslant N$ such that

$$\left\| \pi(u)h - \sum_{t=1}^{N} \pi(v_t)h_t \right\| < \frac{1}{4}.$$

In particular, $\|\sum_{t=1}^{N} \pi(v_t)h_t\| > 3/4$. We may assume that $\pi(v_t) \neq 0$ for all t.

Define a projection $E = \bigvee_{t=1}^{N} \pi(e_t) = \pi(\bigvee_{t=1}^{N} e_t)$, where $e_t = v_t v_t^*$ are diagonal matrix units (which need not be distinct). For all s, t,

$$\pi(e_s)\pi(v_t) = \pi(v_s v_s^* v_t) = \begin{cases} \pi(v_t) & \text{if } v_s v_s^* = v_t v_t^*, \\ 0 & \text{otherwise.} \end{cases}$$

So we have $E \sum \pi(v_t)h_t = \sum \pi(v_t)h_t$. Now

$$\begin{aligned} \left\| E\pi(u)h - \sum_{t=1}^{N} \pi(v_t)h_t \right\| &= \left\| E\left(\pi(u)h - \sum_{t=1}^{N} \pi(v_t)h_t \right) \right\| \\ &\leqslant \left\| \pi(u)h - \sum_{t=1}^{N} \pi(v_t)h_t \right\| < \frac{1}{4}. \end{aligned}$$

Therefore $\|E\pi(u)h\| > 1/2$. In particular, there exists at least one t, $1 \leqslant t \leqslant N$, such that $\pi(e_t)\pi(u) \neq 0$.

Embed $u \in \mathscr{A}_n \hookrightarrow \mathscr{A}_m$ and decompose it as a sum $u = \sum u_s$ of matrix units in $\mathscr{A}_m$. Then $e_t u = u_s$, for some s, and so it follows $\pi(u_s) \neq 0$, i.e., $u_s \notin \mathscr{J}$. Thus we have identified matrix units $u_s \in \mathscr{J}_1 \backslash \mathscr{J}$ and $v_t \in \mathscr{J}_2 \backslash \mathscr{J}$ of $\mathscr{A}_m$ with the same final projection.

Say $u_s = e_{ij}^{(m,r)}$ and $v_t = e_{ik}^{(m,r)}$. We now distinguish three cases:

$$\text{If } j = k, \text{ then } u_s = v_t \in \mathcal{J}_1 \cap \mathcal{J}_2 \backslash \mathcal{J},$$

$$\text{If } j < k, \text{ then } v_t = u_s e_{jk}^{(m,r)} \in \mathcal{J}_1 \cap \mathcal{J}_2 \backslash \mathcal{J},$$

$$\text{If } j > k, \text{ then } u_s = v_t e_{kj}^{(m,r)} \in \mathcal{J}_1 \cap \mathcal{J}_2 \backslash \mathcal{J}.$$

It follows that in all three cases $\mathcal{J}_1 \cap \mathcal{J}_2$ properly contains $\mathcal{J}$. Thus $\mathcal{J}$ is meet-irreducible.

Acknowledgments

We would like to thank Alan Hopenwasser for several suggestions, including one leading to the final version of Theorem 2.6.

References

[1] K. Davidson, C^*-algebras by example. Fields Institute Monographs, American Mathematical Society, Providence, RI, 1996.
[2] K. Davidson, E. Katsoulis, Primitive limit algebras and C^*-envelopes, Adv. Math., to appear.
[3] A. Donsig, A. Hopenwasser, T. Hudson, M. Lamoureux, B. Solel, Meet irreducible ideals in direct limit algebras, Math. Scand. 87 (2000) 27–63.
[4] A. Hopenwasser, J. Peters, S. Power, Nest representations of TAF algebras, Canad. J. Math. 52 (2000) 1221–1234.
[5] M. Lamoureux, Nest representations and dynamical systems, J. Funct. Anal. 114 (1993) 345–376.
[6] M. Lamoureux, Ideals in some continuous nonselfadjoint crossed product algebras, J. Funct. Anal. 142 (1996) 211–248.
[7] M. Lamoureux, Some triangular AF algebras, J. Operator Theory 37 (1997) 91–109.
[8] J. Orr, J. Peters, Some representations of TAF algebras, Pacific. J. Math. 167 (1995) 129–161.
[9] V.I. Paulsen, Completely Bounded Maps and Dilations, Longman Scientific, New York, Wiley, 1986.