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Diffusive mixing of periodic wave trains in reaction–diffusion systems

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ABSTRACT

We consider reaction–diffusion systems on the infinite line that exhibit a family of spectrally stable spatially periodic wave trains $u_0(kx - \omega t; k)$ that are parameterized by the wave number k. We prove stable diffusive mixing of the asymptotic states $u_0(kx + \phi_\pm; k)$ as $x \to \pm\infty$ with different phases $\phi_- \neq \phi_+$ at infinity for solutions that initially converge to these states as $x \to \pm\infty$. The proof is based on Bloch wave analysis, renormalization theory, and a rigorous decomposition of the perturbations of these wave solutions into a phase mode, which shows diffusive behavior, and an exponentially damped remainder. Depending on the dispersion relation, the asymptotic states mix linearly with a Gaussian profile at lowest order or with a nonsymmetric non-Gaussian profile given by Burgers equation, which is the amplitude equation of the diffusive modes in the case of a nontrivial dispersion relation.

1. Introduction

We consider spatially extended pattern-forming systems that exhibit periodic traveling-wave solutions $u(x,t) = u_0(kx - \omega t; k)$ for a certain range of wave numbers $k \in (k_l, k_r)$. The profile $u_0(\theta; k)$ is assumed to be 2π-periodic in $\theta = kx - \omega t$, where the wave number k and the temporal frequency ω are assumed to be related via a nonlinear dispersion relation $\omega = \omega(k)$. Examples are the Taylor

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vortices in the Taylor–Couette problem, roll solutions in convection problems, or periodic wave trains in reaction–diffusion systems.

We are interested in the dynamics of perturbations of wave-train solutions of the above form. Since the linearization around a wave train always possesses essential spectrum up to the imaginary axis, we cannot expect exponential relaxation towards the original profile even for spectrally stable wave trains. Moreover, the periodic nature of the underlying wave train suggests that we should allow perturbations that change the phase or the wave number of the underlying profile. In these cases, we expect that diffusive decay or diffusive mixing of phases or wave numbers dominate the dynamics. More precisely, consider an initial condition of the form

$$u(x,0) = u_0\big(q_0(x)x + \phi_0(x); q_0(x)\big), \quad q_0(x) \to k_\pm,\ \phi_0(x) \to \phi_\pm \quad \text{as } x \to \pm\infty, \tag{1.1}$$

where the functions $q_0(x)$ and $\phi_0(x)$ are bounded and small in an appropriate norm. We may then expect that the solution $u(x,t)$ can, to leading order, be written in the form

$$u(x,t) \approx u_0\big(q(x,t)x + \phi(x,t) - \omega_0 t; q(x,t)\big),$$

and the issue is to determine the behavior of the phase $\phi(x,t)$ and the local wave number $q(x,t)$ as $t \to \infty$. We distinguish three different classes of initial data, namely

(a) constant wave number $q_0(x) \equiv k_0$ and equal phases $\phi_+ = \phi_-$ at infinity for non-zero phase perturbations $\phi_0(x) \not\equiv 0$, which correspond to localized perturbations of the underlying wave train;
(b) constant wave number $q_0(x) \equiv k_0$ but different phases $\phi_+ \neq \phi_-$ at infinity, which correspond to a relative phase shift of the wave train at $\pm\infty$;
(c) different wave numbers $k_- \neq k_+$ at infinity, which correspond to interface dynamics between two different wave trains.

In this paper, we address the cases (a) and (b) for general reaction–diffusion systems

$$\partial_t u = D\partial_x^2 u + f(u) \tag{1.2}$$

with $x \in \mathbb{R}$, $t \geqslant 0$, and $u(x,t) \in \mathbb{R}^d$, where $D \in \mathbb{R}^{d\times d}$ is symmetric and positive definite, and f is smooth. We now outline our results and refer to Theorems 1 and 2 for the precise statements, and to Fig. 1 for illustrations.

For localized perturbations of a single wave train (a) we transfer existing stability results from specific systems [15,17,16,19] to general reaction–diffusion systems. In lowest order, the dynamics near a wave train can be described by the evolution of the local wave number $q(t,x)$, and we prove that the renormalized wave number difference $t[q(t^{1/2}x,t) - k_0]$ converges towards a multiple of the x-derivative of the Gaussian $\frac{1}{\sqrt{4\pi\alpha}}\exp(-\frac{x^2}{4\alpha})$ for an appropriate constant $\alpha > 0$. This yields the asymptotics

$$\sup_{x\in\mathbb{R}}\big|u(x,t) - \phi_{\lim}\phi^*(x - c_g t, t)\partial_\theta u_0(\theta;k)\big| \leqslant C_2 t^{-1+b} \quad \text{as } t \to \infty,$$

with $\phi^*(x,t) = \frac{1}{\sqrt{4\pi\alpha t}}\mathrm{e}^{-x^2/(4\alpha t)}$, where $\phi_{\lim} \in \mathbb{R}$ depends on the initial data, where $\alpha > 0$ and $c_g \in \mathbb{R}$ are constants determined by the spectral properties of $u_0(\cdot,k_0)$, and where $b > 0$ is a small, but arbitrary, correction coefficient. Thus we have Gaussian decay, where throughout this paper we say that a function decays (in time) like a Gaussian if it can be bounded by a constant times the heat kernel $\frac{1}{\sqrt{4\pi t}}\exp(-\frac{x^2}{4t})$; in particular, the terms Gaussian or Gaussian profiles will always refer to functions of the form $\frac{1}{\sqrt{4\pi\alpha}}\exp(-\frac{x^2}{4\alpha})$.

For perturbations that induce a global phase shift (b), that is, for $q_0(x) \equiv k_0$ and $\phi_- \neq \phi_+$ but with $|\phi_d| := |\phi_+ - \phi_-|$ small, we establish diffusive decay of wave-number perturbations. Specifically, the

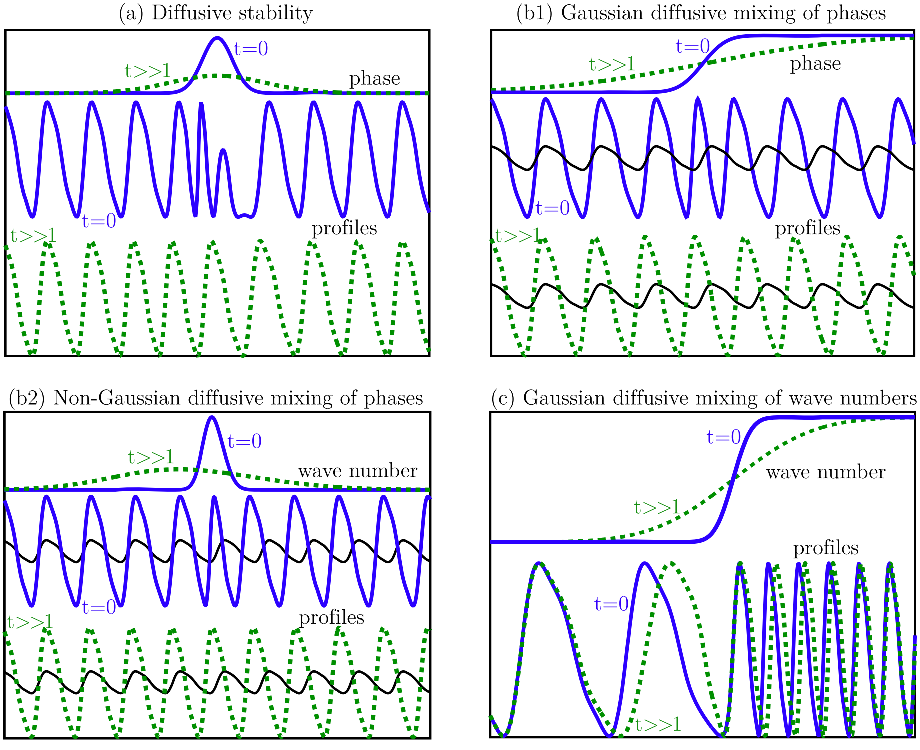

Fig. 1. The panels illustrate different types of diffusive behavior in the frame that moves with the speed of the group velocity c_g. (a) Localized perturbations of wave trains decay diffusively like Gaussians. (b1) If $\omega'' = 0$, then phase fronts develop when $\phi_- \neq \phi_+$, and the wave-number perturbation (i.e., the derivative of the phase plotted in (b1)) decays diffusively like a Gaussian. (b2) If $\omega'' \neq 0$, then phase fronts develop, and the wave-number perturbation decays as determined by the Burgers equation. (c) Shown is the expected formal diffusive mixing of wave-number fronts in case $\omega'' = 0$. Solid and dashed lines indicate solutions at $t = 0$ and for $t \gg 1$, respectively, while the small solid curves in (b1)–(b2) indicate the amplitude-scaled spatially periodic wave train to visualize the phase shifts.

renormalized wave number converges to a Gaussian profile when $\omega''(k_0) = 0$, while it converges to a nonsymmetric non-Gaussian profile when $\omega''(k_0) \neq 0$. This latter case is the major result of this paper.

The case (c) where $q_0(x) \to k_\pm$ as $x \to \pm\infty$ with $k_- \neq k_+$ is more difficult and depends crucially on the sign of $\omega''(k_0)$. If $\omega''(k_0) \neq 0$, diffusive mixing of the local wave number cannot be expected: instead, depending on the sign of $\omega''(k_0)(k_+ - k_-)$, we expect that $q(x,t)$ evolves either as a stable viscous shock or as an approximate rarefaction wave [3]. If $\omega''(k_0) = 0$, nonlinear diffusive mixing can be expected, but, for some technical issues that we explain below, a rigorous proof remains open and is left for future research.

The proof of diffusive mixing of phases of wave trains in systems with no S^1-symmetry has resisted many attempts. With the rigorous separation of the phase variable ϕ from remaining modes found in [3], a new technique is now available to treat this question. This method combined with the renormalization group method [1,2], which has been applied for instance in [15,17,4,6,18,19] to a variety of pattern-forming and hydrodynamic systems, finally yields our results. Diffusive mixing results for the real Ginzburg–Landau equation, which has a natural decomposition into phase and amplitude variables due to its gauge symmetry, have been obtained for instance in [1,5].

The results in this paper were presented by the last author at the Snowbird meeting in 2007. Meanwhile, similar results on the diffusive stability of wave trains have been established in [9,7,8] using pointwise estimates.

Notation. Throughout this paper, we denote many different constants that are independent of the Burgers parameters α, β and the rescaling parameter $L>0$ by the same symbol C. For $m_1, m_2 \in \mathbb{N}$, we define the weighted spaces $H^{m_2}(m_1) = \{u \in L^2(\mathbb{R})\colon \|u\|_{H^{m_2}(m_1)} < \infty\}$ with norm $\|u\|_{H^{m_2}(m_1)} = \|u\rho^{m_1}\|_{H^{m_2}(\mathbb{R})}$, where $\rho(x) = (1+x^2)^{1/2}$ and $H^{m_2}(\mathbb{R})$ is the Sobolev space of functions with weak derivatives up to order m_2 in $L^2(\mathbb{R})$. With an abuse of notation, we sometimes write $\|u(x,t)\|_{H^{m_2}(m_1)}$ for the $H^{m_2}(m_1)$-norm of the function $x \mapsto u(x,t)$. The Fourier transform is denoted by $\mathcal{F}$ so that $\hat{u}(k) := \mathcal{F}(u)(k) = \frac{1}{2\pi}\int e^{-ikx} u(x)\,dx$ for $u \in L^2(\mathbb{R})$. Parseval's identity and $\mathcal{F}(\partial_x u)(k) = ik\hat{u}(k)$ imply that $\mathcal{F}$ is an isomorphism between $H^{m_2}(m_1)$ and $H^{m_1}(m_2)$, that is, the weight in physical space yields smoothness in Fourier space and vice versa. To indicate functions in Fourier space, we also write $\hat{u} \in \hat{H}^{m_1}(m_2)$ instead of $\hat{u} \in H^{m_1}(m_2)$.

2. Statement of results

2.1. Wave trains and their dispersion relations

We assume that there are numbers $k_0 \neq 0$ and $\omega_0 \in \mathbb{R}$ such that (1.2) has a solution of the form $u(x,t) = u_0(k_0 x - \omega_0 t)$, where $u_0(\theta)$ is 2π-periodic in its argument. Thus, u_0 is a 2π-periodic solution of the boundary-value problem

$$k^2 D\partial_\theta^2 u + \omega\partial_\theta u + f(u) = 0 \tag{2.1}$$

with $k = k_0$ and $\omega = \omega_0$. Linearizing (2.1) at u_0 yields the linear operator

$$\mathcal{L}_0 = \mathcal{L}(k_0) = k_0^2 D\partial_\theta^2 + \omega_0\partial_\theta + f'\big(u_0(\theta)\big), \tag{2.2}$$

which is closed and densely defined on $L^2_{\text{per}}(0,2\pi)$ with domain $\mathcal{D}(\mathcal{L}_0) = H^2_{\text{per}}(0,2\pi)$. We assume that $\lambda = 0$ is a simple eigenvalue of $\mathcal{L}_0$ on $L^2_{\text{per}}(0,2\pi)$, so that its null space is one-dimensional and therefore spanned by the derivative $\partial_\theta u_0$ of the wave train.

We may now vary the parameter k in (2.1) near $k = k_0$ and again seek 2π-periodic solutions of (2.1). The derivative of the boundary-value problem (2.1) with respect to ω, evaluated at $k = k_0$ in the solution u_0, is given by $\partial_\theta u_0$. Since $\lambda = 0$ is a simple eigenvalue of $\mathcal{L}_0$ on $L^2_{\text{per}}(0,2\pi)$, we see that $\partial_\theta u_0$ does not lie in the range of $\mathcal{L}_0$, and the linearization of the boundary-value problem (2.1) with respect to (u,ω) is therefore onto. Thus, exploiting the translation symmetry of (2.1) we can solve (2.1) uniquely, up to translations in θ, for (u,ω) as functions of k and obtain the wave trains

$$u(x,t) = u_0\big(kx - \omega(k)t; k\big), \quad k \in (k_l, k_r), \tag{2.3}$$

where $\omega(k_0) = \omega_0$ and $k_l < k_0 < k_r$. In particular, wave trains exist for wave numbers k in an open interval centered around k_0. We call the function $k \mapsto \omega(k)$ the nonlinear dispersion relation and define the phase speed of the wave train with wave number k by $c_\text{p} := \omega(k)/k$ and its group velocity by

$$c_\text{g} = \frac{d\omega}{dk}(k). \tag{2.4}$$

To state our assumptions on the spectral stability of the wave train u_0 as a solution to the reaction–diffusion system (1.2), we consider the linearization

$$\partial_t v = \mathcal{L}_0 v \tag{2.5}$$

of (1.2) in the frame $\theta = k_0 x - \omega_0 t$ that moves with the phase speed $c_\text{p} = \omega_0/k_0$. Particular solutions to this problem can be found through the Bloch-wave ansatz

$$u(\theta,t) = e^{\lambda(\ell)t + i\ell\theta/k_0}\tilde{v}(\theta,\ell), \tag{2.6}$$

where $\ell \in \mathbb{R}$ and $\tilde{v}(\theta, \ell)$ is 2π-periodic in θ for each ℓ. In fact, since $\tilde{v}(\vartheta, \ell + k_0) = e^{i\vartheta}\tilde{v}(\vartheta, \ell)$, we can restrict ℓ to the interval $[-k_0/2, k_0/2)$. Substituting (2.6) into (2.5), we obtain

$$\tilde{\mathcal{L}}(\ell)\tilde{v} = \lambda(\ell)\tilde{v} \tag{2.7}$$

with a family of operators $\tilde{\mathcal{L}}$ given by

$$\tilde{\mathcal{L}}(\ell)\tilde{v} = e^{-i\ell\theta/k_0}\mathcal{L}\big(e^{i\ell\theta/k_0}\tilde{v}(\theta, \ell)\big) = k_0^2 D(\partial_\theta + i\ell/k_0)^2\tilde{v} + \omega(\partial_\theta + i\ell/k_0)\tilde{v} + f'\big(u_0(\theta)\big)\tilde{v}, \tag{2.8}$$

each of which is a closed operator on $L^2_{\mathrm{per}}(0, 2\pi)$ with dense domain $H^2_{\mathrm{per}}(0, 2\pi)$. In particular, $\tilde{\mathcal{L}}(\ell)$ has compact resolvent, and its spectrum is therefore discrete. We can label the eigenvalues of $\tilde{\mathcal{L}}(\ell)$ by indices $j \in \mathbb{N}$ and write them as continuous functions $\lambda_j(\ell)$ of ℓ. In addition, we can order these eigenvalues so that $\operatorname{Re}\lambda_{j+1}(0) \leqslant \operatorname{Re}\lambda_j(0)$ for all j. In fact, the curves $\ell \mapsto \lambda_j(\ell)$ are analytic except possibly near a discrete set of values of ℓ where the values of two or more curves $\lambda_j(\ell)$ for different indices j coincide.

Next, we assume that $\lambda_1(0)$ is the rightmost element in the spectrum for $\ell = 0$. Since we assumed that $\lambda = 0$ is algebraically simple as an eigenvalue of $\mathcal{L}$, there is a curve $\lambda_1(\ell)$ of eigenvalues with $\lambda_1(0) = 0$, and this curve is analytic in ℓ for ℓ close to zero. We call the curve $\lambda_1(\ell)$ the linear dispersion relation and denote the associated eigenfunctions of $\tilde{\mathcal{L}}(\ell)$ by $\tilde{v}_1(\theta, \ell)$. We shall compute the derivative $d\lambda_1/d\ell$ and recover the group velocity as defined via the nonlinear dispersion relation, namely

$$-\operatorname{Im}\partial_\ell\lambda_1|_{\ell=0} = -c_{\mathrm{p}} + \partial_k\omega(k_0) = -c_{\mathrm{p}} + c_{\mathrm{g}}. \tag{2.9}$$

We remark that the phase velocity c_{p} appears in this formula solely because we computed λ_1 in the frame moving with speed c_{p}, while ω was computed in the steady frame. We also note that the signs of the second derivatives of λ_1 and ω are, in general, not related. Finally, we assume that $\operatorname{Re}\lambda_1''(0) < 0$ and that all other eigenvalues $\lambda_j(\ell)$ satisfy $\operatorname{Re}\lambda_j(\ell) < -\sigma_0$. The following hypothesis summarizes the assumptions we made so far.

Hypothesis 2.1 *(Existence of spectrally stable wave trains).* Eq. (1.2) admits a spectrally stable wave-train solution $u(x, t) = u_0(\theta)$ with $\theta = k_0 x - \omega_0 t$ for appropriate numbers $k_0 \neq 0$ and $\omega_0 \in \mathbb{R}$, where u_0 is 2π-periodic. Spectral stability entails the following properties. First, the linearization $\mathcal{L}_0$ of (1.2) about u_0 has a simple eigenvalue at $\lambda = 0$. Furthermore, the linear dispersion relation $\lambda_1(\ell)$ with $\lambda_1(0) = 0$ is dissipative so that $\lambda_1''(0) < 0$, and there exist constants $\sigma_0, \ell_0, \alpha_0 > 0$ such that $\operatorname{Re}\lambda_1(\ell) < -\sigma_0$ for $|\ell| > \ell_0$ and $\operatorname{Re}\lambda_1(\ell) < -\alpha_0\ell^2$ for $|\ell| < \ell_0$, while all other eigenvalues $\lambda_j(\ell)$ with $j \geqslant 2$ have $\operatorname{Re}\lambda_j(\ell) \leqslant -\sigma_0$ for all $\ell \in [-k/2, k/2)$.

Standard perturbation theory yields that the wave trains $u_0(kx - \omega(k)t; k)$ are also spectrally stable, possibly for a smaller interval $k_l \leqslant \tilde{k}_l < k < \tilde{k}_r \leqslant k_r$ of wave numbers than the interval of existence. By changing k_l, k_r accordingly, we shall assume from now on that the wave trains $u_0(\cdot; k)$ with $k \in (k_l, k_r)$ are spectrally stable with uniform constants ℓ_0, σ_0, α_0.

For later use, we collect a few properties of the linear dispersion relation and refer to [3, §4.2] for their derivation. We denote by

$$\mathcal{L}_{\mathrm{ad}}u = k_0^2 D\partial_\theta^2 u - \omega_0\partial_\theta u + f'\big(u_0(\theta)\big)^T u$$

the $L^2_{\mathrm{per}}((0, 2\pi))$-adjoint of $\mathcal{L}_0 = \mathcal{L}(k_0)$ and let u_{ad} be a nontrivial function in its null space with the normalization

$$\langle u_{\mathrm{ad}}, \partial_\theta u_0\rangle_{L^2(0,2\pi)} = 1. \tag{2.10}$$

Using the adjoint eigenfunction, we have

$$\lambda_1'(0) = \mathrm{i}\langle u_{\mathrm{ad}}, c_{\mathrm{p}}\partial_\theta u_0 + 2k_0 D\partial_\theta^2 u_0\rangle_{L^2} = \mathrm{i}(c_{\mathrm{p}} - c_{\mathrm{g}}) \in \mathrm{i}\mathbb{R}, \tag{2.11}$$

$$\lambda_1''(0) = -\langle u_{\mathrm{ad}}, 4k_0 D\partial_k\partial_\theta u_0 + 2D\partial_\theta u_0\rangle_{L^2} \in \mathbb{R}. \tag{2.12}$$

We shall also use the identity

$$\partial_\ell \tilde{v}_1(\cdot, 0) = \mathrm{i}\partial_k u_0 \tag{2.13}$$

that was established in [3, §4.2].

2.2. Statement of results

Throughout this section, we fix the wave number k_0 of a wave train $u_0(k_0x - \omega_0 t; k_0)$ of the reaction–diffusion system (1.2) that satisfies Hypothesis 2.1. We then set

$$\omega_0 = \omega(k_0), \qquad c_p = \omega_0/k_0, \qquad c_g = \omega'(k_0), \qquad \beta = -\frac{1}{2}\omega''(k_0), \qquad \theta = k_0 x - \omega_0 t$$

and write

$$\lambda_1(\ell) = \mathrm{i}(c_p - c_g)\ell - \alpha\ell^2 + \mathcal{O}(\ell^3) \tag{2.14}$$

for the expansion of the linear dispersion relation of $u_0(\cdot; k_0)$. For convenience henceforth we write $k = k_0$. Before we state our result, we remark that the decomposition of the initial data that we shall use below in the statements of our theorems is not unique. This non-uniqueness will be removed in the proofs but does not affect the conclusions made in the results below.

Our first result states that u_0 is diffusively stable with respect to localized perturbations and extracts the leading-order behavior of the displacement for large times. For notational convenience, in the following we consider initial conditions at $t = 1$.

Theorem 1 *(Diffusive stability). Let $u_0(\cdot; k)$ be a spectrally stable wave train that satisfies Hypothesis* 2.1 *and pick $b \in (0, 1/2)$; then there are $\varepsilon, C > 0$ such that the following holds. If, for some $\theta_0 \in [0, 2\pi)$,*

$$u(x,t)|_{t=1} = u_0\big(\theta - \theta_0 + \phi_0(x); k\big) + v_0(x) \quad \textit{with } \|\phi_0\|_{H^3(3)}, \|v_0\|_{H^2(3)} \leqslant \varepsilon, \tag{2.15}$$

then the solution $u(x,t)$ of (1.2) *exists for all times $t \geqslant 1$, it can be written as*

$$u(x,t) = u_0\big(\theta - \theta_0 + \phi(x,t); k\big) + v(x,t),$$

and there is a constant $\phi_{\lim} \in \mathbb{R}$ depending only on the initial condition so that

$$\sup_{x\in\mathbb{R}}\big|\phi(x,t) - \phi_{\lim} G\big(x - c_g(t-1), t\big)\big| + \big|v(x,t)\big| \leqslant Ct^{-1+b}, \tag{2.16}$$

where

$$G(x,t) = \frac{1}{\sqrt{4\alpha\pi t}}\mathrm{e}^{-x^2/(4\alpha t)}. \tag{2.17}$$

In particular, we have

$$\sup_{x\in\mathbb{R}}\left|u(x,t)-u_0\big(\theta-\theta_0+\phi_{\lim}G\big(x-c_g(t-1),t\big);k\big)\right|\leqslant C_1 t^{-1+b}.$$

Next, we discuss diffusive mixing of phases for non-localized phase perturbations. In this situation, the precise asymptotics of perturbations depends on $\beta=-\frac{1}{2}\omega''(k)$.

Theorem 2 *(Diffusive mixing of phases). Let $u_0(\cdot;k)$ be a spectrally stable wave train that satisfies Hypothesis* 2.1 *and pick $b\in(0,1/2)$; then there are constants $\varepsilon, C>0$ such that the following holds.*

(i) *Assume that $\beta=-\frac{1}{2}\omega''(k)=0$ and $u(x,t)|_{t=1}=u_0(\theta-\theta_0+\phi_0(x);k)+v_0(x)$ with $\phi_0(x)\to\phi_\pm$ for $x\to\pm\infty$, $|\phi_d|=|\phi_+-\phi_-|\leqslant\varepsilon$, and*

$$\left\|\phi_0'(\cdot)\right\|_{H^2(2)},\ \|v_0\|_{H^2(2)}\leqslant\varepsilon. \tag{2.18}$$

Then the solution $u(x,t)$ to (1.2) *exists for all $t\geqslant 1$, and can be written as*

$$u(x,t)=u_0\big(\theta-\theta_0+\phi^*\big(x-c_g(t-1),t\big);k\big)+v(x,t),$$

where

$$\phi^*(x,t)=\phi_-+(\phi_+-\phi_-)\,\mathrm{erf}(x/\sqrt{\alpha t}),\quad \textit{with}\ \mathrm{erf}(x)=\frac{1}{\sqrt{4\pi}}\int_{-\infty}^{x}\mathrm{e}^{-\xi^2/4}\,\mathrm{d}\xi \tag{2.19}$$

and $\sup_{x\in\mathbb{R}}|v(x,t)|\leqslant Ct^{-1/2+b}$.

(ii) *The same result holds if $\beta=-\frac{1}{2}\omega''(k)\neq 0$, with $\phi^*(x,t)$ replaced by*

$$\phi^*(x,t)=\frac{\alpha}{\beta}\ln\big(1+z\,\mathrm{erf}(x/\sqrt{\alpha t})\big),\quad \ln(1+z)=\phi_+-\phi_-. \tag{2.20}$$

Remark 2.2. Clearly, the decompositions (2.15) and (2.18) are not unique. For instance, $\phi_0\equiv 0$ would be one possibility in (2.15), but we may shift perturbations between ϕ_0 and v_0. In the proof we shall fix this non-uniqueness via mode filters.

The higher weight in the initial conditions in Theorem 1 vs. 2 is due to the fact that in Theorem 1 we want to extract higher-order asymptotics, i.e., faster decay. The asymptotic phase-profiles in (2.17), (2.19) and (2.20) only depend on k via α from (2.14) (and on β for (2.20)). In particular, they are independent of the phase speed c_p and therefore are formulated in x and t.

Remark 2.3. Formally, we may as well describe the diffusive mixing of wave numbers in case $\omega''=0$, see Remark 2.7. However, then the rigorous separation of the (then unbounded) phase, see Section 3, becomes more difficult. Therefore, we will not consider this case here.

2.3. *The idea*

The translation invariance of (1.2) and the fact that by assumption we have periodic wave trains $u_0(\theta;k)$ for wave numbers k in a whole interval (k_l,k_r) suggest to consider initial conditions for (1.2) of the form

$$\left|u_{\mathrm{ic}}(x)-u_0(k_\pm x+\phi_\pm;k_\pm)\right|\to 0\quad \text{as } x\to\pm\infty. \tag{2.21}$$

The behavior of the corresponding solutions can be discussed formally if we assume that the initial phase shift $\phi_+ - \phi_-$ or the initial wave number shift $q_+ - q_-$ happens on a long spatial scale. We make the ansatz

$$u(x,t) = u_0\big(k_0x - \omega_0 t + \Phi(X,T); k_0 + \delta\partial_X\Phi(X,T)\big) \tag{2.22}$$

where $0 < \delta \ll 1$ is a small perturbation parameter that determines the length scale over which the wave number is modulated by the function $\partial_X\Phi$, and where X and T are long spatial and temporal scales. Plugging (2.22) into (1.2) and comparing equal powers in δ it turns out (see [3, §4.3]) that

$$(X,T) = \big(\delta(x - c_g t), \delta^2 t\big), \quad \text{where } c_g = \omega'(k), \tag{2.23}$$

are the correct spatial and temporal scales, and that $q(X,T) := \partial_X\Phi(X,T)$ should satisfy the Burgers equation

$$\partial_T q = \alpha\partial_X^2 q + \beta\partial_X\big(q^2\big), \quad \alpha = -\frac{1}{2}\lambda_1''(0),\ \beta = -\frac{1}{2}\omega''(k), \tag{2.24}$$

while the phase $\Phi(X,T)$ itself satisfies the integrated Burgers equation

$$\partial_T\Phi = \alpha\partial_X^2\Phi + \beta(\partial_X\Phi)^2. \tag{2.25}$$

Note again that in case $\beta = 0$ (Theorem 2(i)) (2.24) resp. (2.25) are the linear diffusion equations.

Two questions arise: (a) What do we (formally) learn from (2.24) resp. (2.25)? (b) In what sense, i.e. in what spaces and over what time scales, do solutions of (2.24) via (2.22) approximate solutions of (1.2), and can we give rigorous proofs for that?

To answer (a) we briefly review some well-known results about dynamics and stability in the Burgers equation in the following section. With this in mind we turn to (b). One way to translate the formal analysis into rigorous results is to give estimates for the difference between the formal approximation

$$U_{\text{approx}}(x,t) = u_0\big(\theta + \Phi(X,T;\delta); k + \delta\partial_X\Phi(X,T;\delta)\big)$$

and a true solution $u(x,t)$ of (1.2), on sufficiently long time scales. In [3] this has been achieved for a variety of cases using a separation of the critical mode (the phase mode) from the exponentially damped remaining modes by Bloch wave analysis. Here, for special initial data we obtain the diffusive stability results and the mixing results from Theorems 1 and 2. The proofs heavily rely on the coordinates from [3], which are introduced in Section 3.

2.4. Dynamics in the perturbed Burgers equation

In the spectrally stable case the amplitude equation for long wave modulations of the local wave number is given by the Burgers equation. For given classes of initial conditions the behavior of solutions of the Burgers equation is well understood, and, moreover, this behavior is stable under perturbations of the Burgers equation. Thus, before proving our results for (1.2) with initial data (2.21) we briefly review some well-known results about the (perturbed) Burgers equation, cf. [2,11] and [19, Section 3]. This also motivates the ideas and methods of the proof. To keep track of α and β we do not rescale (2.24) to the standard form $\partial_\tau q = \partial_\xi^2 q + \partial_\xi(q^2)$.

The Burgers equation (2.24) has Galilean invariance: if q solves (2.24), then $v = q + c$ solves $\partial_T v = \alpha\partial_X^2 v - 2c\beta\partial_X v + \beta\partial_X(v)^2$ which can be transformed back to (2.24) via $X \mapsto X + 2c\beta T$. Thus, concerning the stability of constant solutions of (2.24) we can restrict to $q \equiv 0$.

We add a higher-order perturbation in the form of a total derivative to (2.24) and for notational convenience we take initial conditions at time $T=1$. Thus we consider

$$\partial_T q = \alpha \partial_X^2 q + \beta \partial_X \big(q^2\big) + \gamma \partial_X h(q, \partial_X q), \quad q|_{T=1} = q_0, \tag{2.26}$$

where for simplicity $h(a,b) = a^{d_1} b^{d_2}$ is a monomial. For $\gamma = 0$ we again have the Burgers equation. The perturbation is assumed to be of higher order. To make this precise we define the degree

$$d_h = d_1 + 2d_2 - 3, \quad \text{and assume that} \quad d_2 \leqslant 1 \quad \text{and} \quad d_h \geqslant 0. \tag{2.27}$$

The mean $\int_{\mathbb{R}} q(X,T)\,\mathrm{d}X$ is conserved also by the perturbed Burgers equation (2.26). Diffusive stability of $q=0$ in (2.26) is based on the fact that solutions to the linear diffusion equation in Fourier space concentrate at wave number $\kappa = 0$. Roughly speaking, for initial data in $L^1(\mathbb{R})$ that decay like $|X|^{-n}$, the solutions of $\partial_T q = \alpha \partial_X^2 q$ fulfill

$$q(X,T) = \sum_{j=0}^{n-1} T^{-(j+1)/2} \hat{q}_0^{(j)}(0) H_j(X/\sqrt{T}) + O\big(T^{-n/2}\big) \quad \text{for } T \to \infty, \tag{2.28}$$

where H_j is a multiple of the (scaled) jth Hermite function $H(x) = (-1)^j \partial_x^j \exp(-x^2/(4\alpha))$. Thus, if $\hat{q}_0(0) = \frac{1}{2\pi}\int_{\mathbb{R}} q_0(X)\,\mathrm{d}X \neq 0$ then $\|q(\cdot,T)\|_{L^\infty} \leqslant CT^{-1/2}\|q_0\|_{L^1}$, while for $\hat{q}(0) = 0$ we have $\|q(\cdot,T)\|_{L^\infty} \leqslant CT^{-1}\|q_0\|_{L^1}$. In the second case it turns out that solutions to the nonlinear equation (2.26) with zero mean have the same asymptotics as solutions to the linearization with zero mean. Thus, both nonlinear terms $\beta\partial_X(q^2)$ and $\gamma\partial_X h(q, \partial_X q)$ are called asymptotically irrelevant.

For $\hat{q}(0) \neq 0$ only $\gamma \partial_X h(q, \partial_X q)$ is irrelevant, and there is a nonlinear correction to the dynamics for (2.26) compared to (2.28). To derive this we use the Cole–Hopf transformation

$$Q(X,T) = \exp\left(\frac{\beta}{\alpha}\int_{-\infty}^{\sqrt{\alpha}X} q(Y,T)\,\mathrm{d}Y\right), \qquad q(X,T) = \frac{\sqrt{a}}{\beta}\frac{\partial_Y Q(Y,T)}{Q(Y,T)}, \quad Y = X/\sqrt{\alpha},$$

which transforms (2.26) with $\gamma = 0$ into the linear heat equation $\partial_T Q = \partial_X^2 Q$, $Q|_{T=1} = Q_0$, with $\lim_{X\to-\infty} Q_0(X) = 1$ and $\lim_{X\to\infty} Q_0(X) = 1 + z > 0$, i.e.

$$\ln(1+z) = \frac{\beta}{\alpha}\int_{-\infty}^{\infty} q(Y,1)\,\mathrm{d}Y.$$

Since $\lim_{T\to\infty} Q(\sqrt{T}X, T) = 1 + z\,\mathrm{erf}(X) + \mathcal{O}(1/\sqrt{T})$ we find that the solution q to the Burgers equation (2.26) with $\gamma = 0$ satisfies

$$\lim_{T\to\infty} \sqrt{T} q(\sqrt{T}X, T) = \frac{\alpha}{\beta}\frac{\mathrm{d}}{\mathrm{d}X}\ln\big(1 + z\,\mathrm{erf}(X/\sqrt{\alpha})\big) =: f_z^*(X), \tag{2.29}$$

with rate $\mathcal{O}(1/\sqrt{T})$. Therefore, if $\beta \neq 0$, then the renormalized solutions converge toward a non-Gaussian limit $f_z^*(X)$. Again, the same behavior can be shown for (2.26) with $\gamma \neq 0$. We summarize these results as follows:

Proposition 2.4. *For each $b \in (0, 1/2)$, there exist $C_1, C_2, T_0 > 0$ such that for solutions q of the perturbed Burgers equation* (2.26) *the following holds.*

(i) *Assume that* $\|q_0\|_{H^2(3)} \leqslant C_1$ *and* $\int_{-\infty}^{\infty} q_0(X)\,\mathrm{d}X = 0$. *Then there exists a* $q_{\lim} \in \mathbb{R}$ *such that*

$$\left\| Tq(\sqrt{T}X, T) - q_{\lim} X \mathrm{e}^{-X^2/4\alpha} \right\|_{H^2(3)} \leqslant C_2 T^{-\frac{1}{2}+b}. \tag{2.30}$$

Thus, $\|q(X,T)\|_{L^1} \leqslant C_2 T^{-1/2+b}$ *and* $\|q(X,T)\|_{L^\infty} \leqslant C_2 T^{-1+b}$.

(ii) *Assume that* $A = \int_{-\infty}^{\infty} q_0(X)\,\mathrm{d}X \neq 0$, $\beta = 0$, *and* $\|q_0\|_{H^2(2)} \leqslant C_1$. *Then*

$$\left\| T^{1/2} q(\sqrt{T}X, T) - \frac{A}{\sqrt{4\pi\alpha}} \mathrm{e}^{-X^2/(4\alpha)} \right\|_{H^2(2)} \leqslant C_2 T^{-\frac{1}{2}+b}, \tag{2.31}$$

and consequently $\|q(X,T)\|_{L^\infty} \leqslant C_2 T^{-1/2}$.

(iii) *Assume that* $A = \int_{-\infty}^{\infty} q_0(X)\,\mathrm{d}X \neq 0$, $\beta \neq 0$, *and* $\|q_0\|_{H^2(2)} \leqslant C_1$. *Then*

$$\left\| T^{1/2} q\big(T^{1/2}X, T\big) - f_z^*(X) \right\|_{H^2(2)} \leqslant C_2 T^{-1/2+b}, \quad \text{where } f_z^*(X) = \frac{\sqrt{\alpha}}{\beta\sqrt{4\pi}} \frac{z\mathrm{e}^{-X^2/\alpha}}{1 + z\,\mathrm{erf}(X/\sqrt{\alpha})}, \tag{2.32}$$

and $\ln(1+z) = \frac{\beta}{\alpha}\int_{-\infty}^{\infty} q_0(Y)\,\mathrm{d}Y$. *In particular, again* $\|q(X,T)\|_{L^\infty} \leqslant C_2 T^{-1/2}$.

Remark 2.5. (a) By translation invariance of (2.26), we can replace q_0 in Proposition 2.4 by $q_0(\cdot - X_0)$ for some $X_0 \in \mathbb{R}$ and obtain the corresponding results for $q(X - X_0, T)$; w.l.o.g. we set $X_0 = 0$.

(b) The higher weight for q_0 in Proposition 2.4(i) compared to (ii), (iii) is due to the fact that we want to isolate higher-order asymptotics (with faster decay). For this we need $\mathcal{F}(q_0) \in H^3(2) \hookrightarrow C^2$.

(c) The profiles in (ii), (iii) are explicitly given in terms of A due to the conservation of $\int q\,\mathrm{d}x$, i.e., since the right-hand side of (2.26) is a total derivative. On the other hand, the constant $q_{\lim}$ in (i) in general depends on q_0 in a complicated way.

(d) The local phase Φ, which is related to the wave number q by $q = \partial_X \Phi$, satisfies the (perturbed) integrated Burgers equation

$$\partial_T \Phi = \alpha \partial_X^2 \Phi + \beta(\partial_X \Phi)^2 + \gamma h\big(\partial_X \Phi, \partial_X^2 \Phi\big), \quad \Phi(X,1) = \Phi_0(X). \tag{2.33}$$

For (2.33) there exist $C_1, C_2 > 0$ such that we have the following asymptotics.

(i) If $\|\Phi_0\|_{H^3(3)} \leqslant C_1$ then there exists a $\phi_{\lim} = -2\alpha q_{\lim} \in \mathbb{R}$ such that

$$\left\| T^{1/2} \Phi(\sqrt{T}X, T) - \phi_{\lim} \mathrm{e}^{-X^2/4\alpha} \right\|_{H^3(3)} \leqslant C_2 T^{-1/2+b}. \tag{2.34}$$

Thus, the renormalized phase converges toward a Gaussian.

(ii) If $\beta = 0$ and $\Phi_0(X) \to \Phi_\pm$ as $X \to \pm\infty$ with $|\Phi_+ - \Phi_-| \leqslant C_1$ and $\|\Phi_0'\|_{H^2(2)} \leqslant C_1$, then

$$\left\| \Phi(\sqrt{T}X, T) - \Phi^*(X) \right\|_{H^3(2)} \leqslant C_2 T^{-1/2+b}$$

where $\Phi^*(X) = \Phi_- + (\Phi_+ - \Phi_-)\,\mathrm{erf}(X/\sqrt{\alpha})$.

(iii) If $\beta \neq 0$ and $\Phi_0(X) \to \Phi_\pm$ as $X \to \pm\infty$ with $|\Phi_+ - \Phi_-| \leqslant C_1$ and $\|\Phi_0'\|_{H^2(2)} \leqslant C_1$, then

$$\left\| \Phi(\sqrt{T}X, T) - \Phi_z^*(X) \right\|_{H^3(2)} \leqslant C_2 T^{-1/2+b}$$

where $\Phi_z^*(X) = \Phi_- + \frac{\alpha}{\beta}\ln(1 + z\,\mathrm{erf}(X/\sqrt{\alpha}))$, $\ln(1+z) = \frac{\beta}{\alpha}(\Phi_+ - \Phi_-)$.

Remark 2.6. We briefly want to explain the reason for (2.27) and the idea of (discrete) renormalization. If $\int q_0(X)\,\mathrm{d}X \neq 0$, then, for $L > 1$ chosen sufficiently large, we let

$$q_n(\xi,\tau) = L^n q\big(L^n\xi, L^{2n}\tau\big). \tag{2.35}$$

Then q_n satisfies

$$\partial_\tau q_n = \alpha\partial_\xi^2 q_n + \beta\partial_\xi\big(q_n^2\big) + \gamma L^{-n}\partial_\xi h_n(q_n, \partial_\xi q_n), \tag{2.36}$$

with

$$h_n(q_n, \partial_\xi q_n) = L^{-(d_1+2d_2-3)n} q_n^{d_1}(\partial_\xi q_n)^{d_2}, \tag{2.37}$$

where $d_h = d_1 + 2d_2 - 3 \geqslant 0$ due to (2.27). Next, solving $\partial_T q = \alpha\partial_X^2 q + \beta\partial_X(v^2) + \gamma\partial_X h(q, \partial_X q)$ for $T \in [1,\infty)$ is equivalent to iterating the renormalization process

$$\text{solve (2.36) for } \tau \in \big[L^{-2}, 1\big] \text{ with initial data } q_n\big(\xi, L^{-2}\big) = Lq_{n-1}(L\xi, 1) \in \mathcal{X}, \tag{2.38}$$

where $\mathcal{X}$ is a suitable Banach space. Since (formally) $L^{-n}\partial_\xi h_n$ in (2.36) goes to zero, in the limit $n \to \infty$ we recover the linear diffusion equation (if $\beta = 0$) respectively the Burgers equation (if $\beta \neq 0$) for q_n, with the known asymptotics (2.28) respectively (2.29). Similarly, if $\int q_0\,\mathrm{d}X = 0$, then we scale

$$q_n(\xi,\tau) = L^{2n} q\big(L^n\xi, L^{2n}\tau\big), \tag{2.39}$$

and (independent of whether β is zero or not) end up with the linear diffusion equation in the respective renormalization process. To make this rigorous we need a suitable Banach spaces $\mathcal{X}$ and rigorous control of the iterative process (2.38), and again we refer to [2] and [19, Section 3] for details. However, two observations are most important. (a) In (2.37) we see that each derivative in x gives an additional L^{-1} in the rescaling. (b) The diffusive spreading in physical space corresponds to concentration at $\kappa = 0$ in Fourier space according to $\mathcal{F}(Lu(L\cdot))(k) = \hat{u}(\kappa/L)$. Thus, for the linear part, only the parabolic shape of the spectrum $\lambda(\kappa) = -\alpha\kappa^2$ of $\alpha\partial_x^2$ near $\kappa = 0$ is relevant.

Remark 2.7. If $q_0(X) \to q_\pm$ for $X \to \pm\infty$, then $q(\sqrt{T}X, T) = Q_0^*(X) + \mathcal{O}(T^{-1/2})$ as $T \to \infty$ for the solutions of $q_T = \alpha\partial_X^2 q$, where $Q_0^*(X) = q_- + (q_+ - q_-)\operatorname{erf}(X/\sqrt{\alpha})$. Thus we have diffusive mixing of the wave numbers. Then, for $\beta = 0$ and for suitable q_0, we have the asymptotics

$$q(\sqrt{T}X, T) = Q^*(X) + \mathcal{O}\big(T^{-1/2}\big) \quad \text{as } T \to \infty \tag{2.40}$$

for (2.26), where $|Q^*(X) - Q_0^*(X)| \leqslant C\mathrm{e}^{-X^2/4}$, i.e., we have essentially the same asymptotics as in the linear case, with a small localized correction, see [2]. On the other hand, for $\beta \neq 0$ a front is created, see [3]. However, here we do not further comment on this case since below we focus on diffusive mixing of phases.

3. The separation of the wave numbers

3.1. The ansatz

Only special systems such as the cGL have an $\mathcal{S}^1$-symmetry and therefore a natural decomposition into amplitude and phase. Hence, the first step is to extract from a general reaction–diffusion system an equation for the phase, and then out of this for the wave number. We follow the formal derivation made in [3] which uses a multi-scale expansion which however we cannot assume a priori. Thus, here we proceed as follows for the reaction–diffusion system (1.2). As above we change coordinates via $\theta = kx - \omega t$, and obtain

$$\partial_t u = k^2 D \partial_\theta^2 u + \omega \partial_\theta u + f(u). \tag{3.1}$$

A stationary wave train $u_0(\theta; k)$ of (3.1) with period 2π satisfies

$$k^2 D \partial_\theta^2 u_0 + \omega \partial_\theta u_0 + f(u_0) = 0. \tag{3.2}$$

Given a smooth phase function $\phi(\vartheta, t)$ we seek solutions of the form

$$u(\theta, t) = u_0\big(\vartheta; k\big(1 + \partial_\vartheta \phi(\vartheta, t)\big)\big) + w(\vartheta, t), \tag{3.3}$$

where the phase $\phi(\vartheta, t)$ and the coordinates θ and ϑ are related by

$$\theta = \vartheta - \phi(\vartheta, t). \tag{3.4}$$

Roughly speaking we require that $\partial_\vartheta \phi$ is small, uniformly in ϑ, and that $\phi(\vartheta, t)$ is close to the asymptotic profile we want to extract. Still, (3.3) adds an additional degree of freedom by introducing ϕ; we later add additional conditions on ϕ and w, via mode filters, to remove this additional degree of freedom again.

Remark 3.1. It might seem more natural to make the ansatz

$$u(\theta, t) = u_0\big(\theta + \phi(\theta, t); k\big(1 + \partial_\theta \phi(\theta, t)\big)\big) + w(\theta, t) \tag{3.5}$$

instead of (3.3). However, we need to be able to relate the dynamics of $u(\theta, t)$ back to properties of the wave train $u_0(\theta; k)$. Thus, we would need to express $u(\theta, t)$ in terms of $\vartheta = \theta + \phi(\theta, t)$, i.e.,

$$u_0\big(\theta + \phi(\theta, t); k\big(1 + \partial_\theta \phi(\theta, t)\big)\big) \mapsto u_0\big(\vartheta; k\big(1 + \partial_\theta \phi\big(\theta(\vartheta, t), t\big)\big)\big)$$

which involves the inverse $\theta(\vartheta, t)$ of the function $\vartheta = \theta + \phi(\theta, t)$. The occurrence of this inverse would have made the forthcoming analysis much more complicated.

Remark 3.2. Suppose that we found a phase function $\phi(\vartheta, t)$ with small derivative $\partial_\vartheta \phi(\vartheta, t)$ so that (3.3) satisfies (3.1). Using the implicit function theorem, we can then, a posteriori, solve (3.4) for ϑ as a function of θ which is of the form $\vartheta = \theta + \check{\phi}(\theta, t)$, where

$$\check{\phi}(\theta, t) = \phi(\vartheta, t) = \phi\big(\theta + \check{\phi}(\theta, t), t\big).$$

In particular, we see that

$$u_0\big(\vartheta; k\big(1 + \partial_\theta \phi(\vartheta, t)\big)\big) = u_0\big(\theta + \phi\big(\theta + \check{\phi}(\theta, t), t\big); k\big(1 + \partial_\theta \phi\big(\theta + \check{\phi}(\theta, t), t\big)\big)\big) \tag{3.6}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta} \phi\big(\theta + \check{\phi}(\theta, t), t\big) &= \big(1 + \partial_\theta \check{\phi}(\theta, t)\big) \partial_\theta \phi\big(\theta + \check{\phi}(\theta, t), t\big) \\ &= \partial_\theta \phi\big(\theta + \check{\phi}(\theta, t), t\big) + \mathcal{O}\big(\big|\partial_\theta \phi\big(\theta + \check{\phi}(\theta, t), t\big)\big|^2\big).\end{aligned}$$

Thus, to leading order, the solution (3.6) is of the desired form (3.5) with $\phi(\theta, t)$ replaced by $\phi(\theta + \check{\phi}(\theta, t), t)$.

We now substitute the ansatz (3.3) into (3.1) and derive the resulting PDE in ϑ. We use the notation

$$u_0^\phi := u_0\big(\vartheta; k(1+\partial_\vartheta\phi)\big), \qquad \partial_j u_0^\phi := (\partial_j u_0)^\phi := (\partial_j u_0)\big(\vartheta; k(1+\partial_\vartheta\phi)\big), \quad j=\vartheta, k. \tag{3.7}$$

Assuming that $\partial_\vartheta\phi$ is small, we obtain

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}\theta} = \frac{1}{1-\partial_\vartheta\phi}, \qquad \frac{\mathrm{d}\vartheta}{\mathrm{d}t} = \frac{-\partial_t\phi}{1-\partial_\vartheta\phi},$$
$$\frac{\mathrm{d}}{\mathrm{d}\theta} = \frac{1}{1-\partial_\vartheta\phi}\frac{\mathrm{d}}{\mathrm{d}\vartheta}, \qquad \frac{\mathrm{d}^2}{\mathrm{d}\theta^2} = \left(\frac{1}{1-\partial_\vartheta\phi}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\right)^2,$$

and therefore

$$\frac{\mathrm{d}u}{\mathrm{d}\theta} = \frac{1}{1-\partial_\vartheta\phi}\partial_\vartheta u_0^\phi + \frac{k\partial_\vartheta^2\phi}{1-\partial_\vartheta\phi}\partial_k u_0^\phi,$$
$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = \left(\frac{1}{1-\partial_\vartheta\phi}\frac{\mathrm{d}}{\mathrm{d}\vartheta} + \frac{k\partial_\vartheta^2\phi}{1-\partial_\vartheta\phi}\frac{\mathrm{d}}{\mathrm{d}k}\right)^2 u_0^\phi,$$
$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{-\partial_t\phi}{1-\partial_\vartheta\phi}\partial_\vartheta u_0^\phi + k\left(-\frac{\partial_\vartheta^2\phi\,\partial_t\phi}{1-\partial_\vartheta\phi} + \partial_\vartheta\partial_t\phi\right)\partial_k u_0^\phi,$$

and

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial w}{\partial t} - \frac{\partial w}{\partial\vartheta}\frac{\partial_t\phi}{1-\partial_\vartheta\phi}, \qquad \frac{\mathrm{d}w}{\mathrm{d}\theta} = \frac{1}{1-\partial_\vartheta\phi}\frac{\partial w}{\partial\vartheta}, \qquad \frac{\mathrm{d}^2w}{\mathrm{d}\theta^2} = \left(\frac{1}{1-\partial_\vartheta\phi}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\right)^2 w.$$

Thus

$$\begin{aligned}
-&\frac{\partial_t\phi}{1-\partial_\vartheta\phi}\partial_\vartheta u_0^\phi - k\left(\partial_\vartheta^2\phi\frac{\partial_t\phi}{1-\partial_\vartheta\phi} - \partial_\vartheta\partial_t\phi\right)\partial_k u_0^\phi + \partial_t w - \frac{\partial_t\phi}{1-\partial_\vartheta\phi}\partial_\vartheta w \\
&= k^2 D\left(\left(\frac{1}{1-\partial_\vartheta\phi}\frac{\partial}{\partial\vartheta} + \frac{k\partial_\vartheta^2\phi}{1-\partial_\vartheta\phi}\frac{\partial}{\partial k}\right)^2 u_0^\phi + \left(\frac{1}{1-\partial_\vartheta\phi}\frac{\partial}{\partial\vartheta}\right)^2 w\right) \\
&\quad + \omega\frac{1}{1-\partial_\vartheta\phi}\big(\partial_\vartheta u_0^\phi + k\big(\partial_\vartheta^2\phi\big)\partial_k u_0^\phi + \partial_\vartheta w\big) \\
&\quad - \big(k^2 D\partial_\vartheta^2 u_0 - \omega\partial_\vartheta u_0 + f(u_0)\big) + f\big(u_0^\phi + w\big)
\end{aligned} \tag{3.8}$$

where we used (3.2) in the last equation.

Our goal is to separate the critical modes, which involve the dynamics of ϕ, from the damped noncritical modes using the eigenfunctions of the linearization $\mathcal{L}(k)$. This is done via Bloch waves which we introduce next.

3.2. Bloch wave analysis

Bloch wave transform $\mathcal{J}$ is a generalization of Fourier transform $\mathcal{F}$. We briefly review the main properties and refer to [12,17,13,3] for proofs and further details. From now on we use a slightly rescaled Fourier transform, namely

$$\hat{w}(\ell) = (\mathcal{F}w)(\ell) = \frac{1}{2\pi k}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\ell\vartheta/k} w(\vartheta)\,\mathrm{d}\vartheta, \qquad w(\vartheta) = \left(\mathcal{F}^{-1}\hat{w}\right)(\vartheta) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\ell\vartheta/k}\hat{w}(\ell)\,\mathrm{d}\ell, \tag{3.9}$$

and thus, denoting the classical Fourier transform (where $k = 1$) by $\mathcal{F}_1$,

$$(\mathcal{F}w)(\ell) = \frac{1}{k}(\mathcal{F}_1 w)(\ell/k) \quad \text{and} \quad (\mathcal{F}\hat{w})(\vartheta) = \left(\mathcal{F}_1^{-1}\hat{w}\right)(\vartheta/k). \tag{3.10}$$

Then, for sufficiently smooth and rapidly enough decaying functions w, we have

$$\begin{aligned}\left(\mathcal{J}^{-1}\tilde{w}\right)(\vartheta) := w(\vartheta) &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\ell\vartheta/k}\hat{w}(\ell)\,\mathrm{d}\ell = \sum_{j\in\mathbb{Z}}\int_{-k/2}^{k/2} \mathrm{e}^{\mathrm{i}\vartheta(\ell+jk)/k}\hat{w}(\ell+jk)\,\mathrm{d}\ell \\ &= \int_{-k/2}^{k/2} \mathrm{e}^{\mathrm{i}\ell\vartheta/k}\left[\sum_{j\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}j\vartheta}\,\hat{w}(\ell+jk)\right]\mathrm{d}\ell = \int_{-k/2}^{k/2} \mathrm{e}^{\mathrm{i}\ell\vartheta/k}\tilde{w}(\vartheta,\ell)\,\mathrm{d}\ell \end{aligned} \tag{3.11}$$

where

$$\tilde{w}(\vartheta,\ell) = (\mathcal{J}w)(\vartheta,\ell) = \sum_{j\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}j\vartheta}\,\hat{w}(\ell+jk). \tag{3.12}$$

Similar to the Fourier transform, the Bloch transform can be defined for tempered distributions. By construction,

$$\tilde{w}(\vartheta+2\pi,\ell) = \tilde{w}(\vartheta,\ell) \quad \text{and} \quad \tilde{w}(\vartheta,\ell+k) = \mathrm{e}^{\mathrm{i}\vartheta}\,\tilde{w}(\vartheta,\ell), \tag{3.13}$$

such that we can restrict ourselves to $\ell \in [-k/2, k/2)$. The Bloch transform of the product of two functions w_1 and w_2 in ϑ-space is given by the convolution

$$\mathcal{J}[w_1\cdot w_2](\vartheta,\ell) = [\tilde{w}_1 * \tilde{w}_2](\vartheta,\ell) = \int_{-k/2}^{k/2} \tilde{w}_1(\vartheta,\ell-\tilde{\ell})\tilde{w}_2(\vartheta,\tilde{\ell})\,\mathrm{d}\tilde{\ell} \tag{3.14}$$

of their Bloch transforms $\tilde{w}_1$ and $\tilde{w}_2$ in Bloch space, where (3.13) is used for $|\ell - \tilde{\ell}| > k/2$. The analytic properties of the Bloch transform are based on a generalization of Parseval's identity

$$\int_{-\infty}^{\infty} \left|u(\vartheta)\right|^2 \mathrm{d}\vartheta = 2\pi k\int_0^{2\pi}\int_{-k/2}^{k/2} \left|\tilde{u}(\vartheta,\ell)\right|^2 \mathrm{d}\ell\,\mathrm{d}\vartheta.$$

As a consequence, Bloch wave transform is an isomorphism between $H^{m_2}(m_1)$, and the space $B^{m_1}(m_2)$ of functions $\tilde{u}(\vartheta,\ell)$ that are 2π-periodic w.r.t. ϑ, satisfy (3.13), and whose norm

$$\|\tilde{u}\|_{B^{m_1}(m_2)} = \sum_{j=0}^{m_1}\sum_{i=0}^{m_2}\int_0^{2\pi}\int_{-k/2}^{k/2} \left|\partial_\ell^j\partial_\vartheta^i\tilde{u}(\vartheta,\ell)\right|^2 \mathrm{d}\ell\,\mathrm{d}\vartheta$$

is finite. We now collect a few more properties; see, e.g., [17] or [3, §5.2] for more details and proofs.

Remark 3.3. (a) If $w_1(\vartheta)$ is 2π-periodic in ϑ and the support of the Fourier transform $\hat{w}_2$ of a complex-valued function $w_2(\vartheta)$ lies in $(-1/2, 1/2)$, then we have

$$\mathcal{J}[w_1 w_2](\vartheta, \ell) = w_1(\vartheta)\hat{w}_2(\ell). \tag{3.15}$$

Due to (3.15) Bloch transform is useful to analyze differential operators with spatially-periodic coefficients, which in Bloch space become multiplication operators.

(b) Since we are interested in functions which do not necessarily decay to zero at infinity, we employ a method already used in [14] to extend multiplication operators from the space L^2 of square-integrable functions to the space L^2_{ul} of uniformly locally square-integrable functions equipped with the norm $\|u\|_{L^2_{\mathrm{ul}}} = \sup_{x\in\mathbb{R}} \int_x^{x+1} |u(y)|^2\,\mathrm{d}y$. We recall that

$$H^m_{\mathrm{ul}} = \Big\{u : \mathbb{R} \to \mathbb{R};\ \|u\|_{H^m_{\mathrm{ul}}} = \|u\|_{H^m(x,x+1)} < \infty \text{ with } \lim_{y\to 0} \|u - T_y u\|_{H^m_{\mathrm{ul}}} \to 0\Big\}$$

where $[T_y u](x) = u(x+y)$. Now let $m, s \in \mathbb{Z}$ with $m+s \geqslant 0$ and $m \geqslant 0$, and consider a function

$$\tilde{\mathcal{M}} : \mathbb{R} \to L\big(H^{m+s}_{\mathrm{per}}(0, 2\pi), H^m_{\mathrm{per}}(0, 2\pi)\big), \quad \ell \mapsto \tilde{\mathcal{M}}(\ell)$$

which is $\mathcal{C}^2$ in the Bloch wave number ℓ. Then $\tilde{\mathcal{M}}$ defines a bounded operator $\mathcal{M} : H^{m+s}_{\mathrm{ul}} \to H^m_{\mathrm{ul}}$ with

$$\|\mathcal{M}\|_{L(H^{m+s}_{\mathrm{ul}}, H^m_{\mathrm{ul}})} \leqslant C(m,s)\|\tilde{\mathcal{M}}\|_{\mathcal{C}^2_{\mathrm{b}}((-k/2,k/2), L(H^{m+s}_{\mathrm{per}}, H^m_{\mathrm{per}}))}. \tag{3.16}$$

Clearly, this can be extended to multi-linear operators.

3.3. *Mode filters, and separation into critical and noncritical modes*

Our goal is to separate the dynamics of the eigenmodes $\tilde{v}_1(\vartheta, \ell)$ associated with the critical eigenvalues $\lambda_1(\ell)$ from the remaining modes, which are linearly exponentially damped and therefore called noncritical. We use mode filters to obtain this splitting.

Due to Hypothesis 2.1 there exists a number ℓ_1 with $0 < \ell_1 \ll 1$ so that the eigenvalue $\lambda_1(\ell)$ of $\tilde{\mathcal{L}}(\ell)$ is bounded away from the rest of the spectrum for $|\ell| < \ell_1$. Therefore, there exists an $\tilde{\mathcal{L}}(\ell)$-invariant projection

$$\tilde{Q}^{\mathrm{c}}(\ell) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \big[\lambda - \tilde{\mathcal{L}}(\ell)\big]^{-1}\,\mathrm{d}\lambda$$

onto the space spanned by $\tilde{v}_1(\vartheta, \ell)$, where $\Gamma \subset \mathbb{C}$ is a small circle that surrounds $\lambda_1(\ell)$ counterclockwise in the complex plane and does not intersect the rest of the spectrum of $\mathcal{L}(\ell)$ for this fixed ℓ. For $\ell = 0$ we have

$$\tilde{Q}^{\mathrm{c}}(0)\tilde{v}(\cdot, 0) = \langle u_{\mathrm{ad}}, \tilde{v}\rangle \tilde{v}_1(\cdot, 0),$$

and similarly $\tilde{Q}^{\mathrm{c}}(\ell)$ can be expressed by using the scalar product with the adjoint $\tilde{u}_{\mathrm{ad}}(\cdot, \ell)$ in Bloch space.

We choose a nonincreasing (for $\ell \geqslant 0$) $\mathcal{C}^\infty_0$-cutoff function $\chi : \mathbb{R} \to [0, 1]$ with

$$\chi(\ell) = \begin{cases} 1 & \text{for } |\ell| \leqslant 1, \\ 0 & \text{for } |\ell| \geqslant 2, \end{cases} \tag{3.17}$$

and define

$$\tilde{P}^{c}_{fs}(\ell) = \tilde{Q}^{c}(\ell)\chi\left(\frac{4\ell}{\ell_1}\right), \qquad \tilde{P}^{s}_{fs}(\ell) := 1 - \tilde{Q}^{c}(\ell)\chi\left(\frac{4\ell}{\ell_1}\right),$$
$$\tilde{P}^{c}_{mf}(\ell) = \tilde{Q}^{c}(\ell)\chi\left(\frac{8\ell}{\ell_1}\right), \qquad \tilde{P}^{s}_{mf}(\ell) := 1 - \tilde{Q}^{c}(\ell)\chi\left(\frac{8\ell}{\ell_1}\right),$$

and

$$\tilde{P}^{c}(\ell) = \tilde{Q}^{c}(\ell)\chi\left(\frac{2\ell}{\ell_1}\right), \qquad \tilde{P}^{s}(\ell) := 1 - \tilde{Q}^{c}(\ell)\chi\left(\frac{16\ell}{\ell_1}\right).$$

These operators commute for each fixed ℓ and satisfy

$$\big(1-\tilde{P}^{c}\big)\tilde{P}^{c}_{fs} = 0 = \big(1-\tilde{P}^{c}_{fs}\big)\tilde{P}^{c}_{mf}, \qquad \big(1-\tilde{P}^{s}\big)\tilde{P}^{s}_{fs} = 0 = \big(1-\tilde{P}^{s}\big)\tilde{P}^{s}_{mf},$$
$$\tilde{P}^{c}_{fs} + \tilde{P}^{s}_{fs} = 1, \qquad \tilde{P}^{c}_{mf} + \tilde{P}^{s}_{mf} = 1. \tag{3.18}$$

We define scalar-valued operators $\tilde{p}^{c}_{fs}(\ell)$ and $\tilde{p}^{c}_{mf}(\ell)$ implicitly by

$$\big[\tilde{p}^{c}_{fs}(\ell)\tilde{u}\big]\tilde{v}_1(\cdot,\ell) = \tilde{P}^{c}_{fs}(\ell)\tilde{u}, \qquad \big[\tilde{p}^{c}_{mf}(\ell)\tilde{u}\big]\tilde{v}_1(\cdot,\ell) = \tilde{P}^{c}_{mf}(\ell)\tilde{u}. \tag{3.19}$$

Remark 3.3(b) implies that each of the operators above extends to a bounded operator on H^{m+s}_{ul}. The resulting operators will be denoted by the same letter but with the superscript ˜ being dropped.

The mode filters p^{c}_{mf} and P^{s}_{mf} are now used to separate the critical and noncritical modes in (3.8), while p^{c}_{fs} and P^{s}_{fs} are used to limit the Fourier support of the critical modes. We write (3.8) given by

$$\begin{aligned}
&-\frac{\partial_t\phi}{1-\partial_\vartheta\phi}\partial_\vartheta u_0^\phi - k\left(\partial_\vartheta^2\phi\frac{\partial_t\phi}{1-\partial_\vartheta\phi} - \partial_\vartheta\partial_t\phi\right)\partial_k u_0^\phi + \partial_t w - \frac{\partial_t\phi}{1-\partial_\vartheta\phi}\partial_\vartheta w \\
&\quad = k^2 D\left(\left(\frac{1}{1-\partial_\vartheta\phi}\frac{\partial}{\partial\vartheta} + \frac{k\partial_\vartheta^2\phi}{1-\partial_\vartheta\phi}\frac{\partial}{\partial k}\right)^2 u_0^\phi + \left(\frac{1}{1-\partial_\vartheta\phi}\frac{\partial}{\partial\vartheta}\right)^2 w\right) \\
&\qquad - \omega\frac{1}{1-\partial_\vartheta\phi}\big(\partial_\vartheta u_0^\phi + k\big(\partial_\vartheta^2\phi\big)\partial_k u_0^\phi + \partial_\vartheta w\big) \\
&\qquad - \big(k^2 D\partial_\vartheta^2 u_0 - \omega\partial_\vartheta u_0 + f(u_0)\big) + f\big(u_0^\phi + w\big),
\end{aligned} \tag{3.20}$$

as

$$\big[-B_0 + B_1(\partial_\vartheta\phi, w)\big]\partial_t\phi + \partial_t w = -\mathcal{L}_i\partial_\vartheta\phi + \mathcal{L}w + G(\partial_\vartheta\phi, w), \tag{3.21}$$

where

$$B_0\partial_t\phi = (\partial_\vartheta u_0 - k\partial_k u_0\partial_\vartheta)\partial_t\phi,$$

$$\begin{aligned}
\mathcal{L}_i\partial_\vartheta\phi &= -\mathcal{L}(k\partial_\vartheta\phi\partial_k u_0) + k^2 D\big(2\partial_\vartheta\phi\partial_\vartheta^2 u_0 + \partial_\vartheta^2\phi\partial_\vartheta u_0\big) - \omega\partial_\vartheta\phi\partial_\vartheta u_0 \\
&= k\big[\mathcal{L}(\mathrm{i}\partial_\vartheta\phi\partial_\ell v_1) + kD\big(2\partial_\vartheta\phi\partial_\vartheta^2 u_0 + \partial_\vartheta^2\phi\partial_\vartheta u_0\big) - c_p\partial_\vartheta\phi\partial_\vartheta u_0\big],
\end{aligned}$$

$$B_1(\partial_\vartheta\phi, w)\partial_t\phi = \left(\partial_\vartheta u_0 - \frac{\partial_\vartheta u_0^\phi}{1-\partial_\vartheta\phi}\right)\partial_t\phi - k\left(\frac{\partial_\vartheta^2\phi\partial_k u_0^\phi}{1-\partial_\vartheta\phi}\partial_t\phi + \left(\partial_k u_0 - \partial_k u_0^\phi\right)\partial_\vartheta\partial_t\phi\right)$$
$$- \frac{\partial_\vartheta w}{1-\partial_\vartheta\phi}\partial_t\phi, \tag{3.22}$$

and where G contains the remaining terms. In the calculation above, we used that $\partial_\ell v_1 = \mathrm{i}\partial_k u_0$, see (2.13). The symbol $\mathcal{L}_i$ is used since in the critical modes $\partial_\vartheta\mathcal{L}_i$ corresponds to $\mathcal{L}$, see (3.33) below, i.e., $\mathcal{L}_i$ resembles an integration of $\mathcal{L}$. Clearly,

$$B_1(\partial_\vartheta\phi, w) = \mathcal{O}\left(|\partial_\vartheta\phi| + |w|\right), \qquad G(\partial_\vartheta\phi, w) = \mathcal{O}\left(|\partial_\vartheta\phi|^2 + |w|^2\right).$$

Our goal is to replace (3.21) with the system

$$\partial_t P_{\mathrm{fs}}^{\mathrm{c}} B_0\phi = P_{\mathrm{fs}}^{\mathrm{c}}\mathcal{L}_i\partial_\vartheta\phi + P_{\mathrm{mf}}^{\mathrm{c}} B_1(\partial_\vartheta\phi, w)\partial_t\phi - P_{\mathrm{mf}}^{\mathrm{c}} G(\partial_\vartheta\phi, w),$$
$$\partial_t w = \mathcal{L}w + P_{\mathrm{fs}}^{\mathrm{s}} B_0\partial_t\phi - P_{\mathrm{fs}}^{\mathrm{s}}\mathcal{L}_i\partial_\vartheta\phi - P_{\mathrm{mf}}^{\mathrm{s}} B_1(\partial_\vartheta\phi, w)\partial_t\phi + P_{\mathrm{mf}}^{\mathrm{s}} G(\partial_\vartheta\phi, w) \tag{3.23}$$

for (ϕ, w). Subtracting the first from the second equation and using (3.18), we see that solutions of (3.23) give solutions of (3.21). Alternatively, we may consider the system

$$\partial_t p_{\mathrm{fs}}^{\mathrm{c}} B_0\phi = p_{\mathrm{fs}}^{\mathrm{c}}\mathcal{L}_i\partial_\vartheta\phi + p_{\mathrm{mf}}^{\mathrm{c}} B_1(\partial_\vartheta\phi, w)\partial_t\phi - p_{\mathrm{mf}}^{\mathrm{c}} G(\partial_\vartheta\phi, w),$$
$$\partial_t w = \mathcal{L}w + P_{\mathrm{fs}}^{\mathrm{s}} B_0\partial_t\phi - P_{\mathrm{fs}}^{\mathrm{s}}\mathcal{L}_i\partial_\vartheta\phi - P_{\mathrm{mf}}^{\mathrm{s}} B_1(\partial_\vartheta\phi, w)\partial_t\phi + P_{\mathrm{mf}}^{\mathrm{s}} G(\partial_\vartheta\phi, w) \tag{3.24}$$

for (ϕ, w), where the first equation is now scalar-valued. Inspecting (3.19) we see that (3.23) and (3.24) are equivalent. We shall require that (ϕ, w) satisfy

$$\operatorname{supp}\mathcal{F}[\phi] \subset \mathcal{I} := \left\{\ell;\ \chi(4\ell/\ell_1) = 1\right\} \tag{3.25}$$

and

$$\left(1 - P^{\mathrm{s}}\right)w = 0 \tag{3.26}$$

for all $t \geqslant 1$. Since P^{s} commutes with $\mathcal{L}$, it follows from (3.18) and (3.24) that (3.26) holds for all $t > 1$ if it is true for $t = 1$.

It remains to check whether (3.25) is respected by (3.24) and to calculate the operator $p_{\mathrm{fs}}^{\mathrm{c}} B_0$ to see whether (3.24) is a proper evolution equation. Due to the properties of the multiplier $p_{\mathrm{mf}}^{\mathrm{c}}$, we know that

$$\operatorname{supp}\mathcal{F}\left[p_{\mathrm{mf}}^{\mathrm{c}}\left(B_1(\partial_\vartheta\phi, w)\partial_t\phi - G(\partial_\vartheta\phi, w)\right)\right] \Subset \mathcal{I}$$

for any sufficiently smooth function ϕ. From (3.22) we find that the operators B_0 and $\mathcal{T}_i$ have 2π-periodic coefficients in ϑ and are multipliers in Bloch space which allows us to use Remark 3.3. For any function ϕ that satisfies (3.25), we then obtain

$$\tilde{P}_{\mathrm{fs}}^{\mathrm{c}}\mathcal{J}[B_0\phi] = \tilde{P}_{\mathrm{fs}}^{\mathrm{c}}(\ell)\mathcal{J}[B_0\phi](\vartheta, \ell) \overset{(3.15)}{=} \hat{\phi}(\ell)\chi(4\ell/\ell_1)\tilde{Q}^{\mathrm{c}}(\ell)\left(\partial_\vartheta u_0(\vartheta) + \mathcal{O}(\ell)\right)$$
$$= \hat{\phi}(\ell)\chi(4\ell/\ell_1)\left(1 + \mathcal{O}(\ell)\right)\tilde{v}_1(\vartheta, \ell) \overset{(3.25)}{=} \left[\left(1 + \mathcal{O}(\ell_1)\right)\hat{\phi}(\ell)\right]\tilde{v}_1(\vartheta, \ell),$$

where the $\mathcal{O}(\ell_1)$-term is a multiplier and $[(1+\mathcal{O}(\ell_1))\hat{\phi}]$ has support in $\mathcal{I}$. Therefore, using the definition (3.19) of $p^{\mathrm{c}}_{\mathrm{fs}}$ and denoting the operator associated with the $\mathcal{O}(\ell_1)$-term by B_2, we get

$$p^{\mathrm{c}}_{\mathrm{fs}} B_0 \phi = (1+B_2)\phi \tag{3.27}$$

for all ϕ that satisfy (3.25), where B_2 has norm $\|B_2\| = \mathcal{O}(\ell_1)$ and respects (3.25), i.e. $\operatorname{supp} \mathcal{F}[B_2\phi] \subset \mathcal{I}$. Since similar arguments apply to the multiplier $\mathcal{L}_i$, (3.25) is indeed preserved by (3.24).

For all (ϕ, w) for which $(\partial_\vartheta \phi, w)$ is small and ϕ satisfies (3.25), the first equation of (3.24) can be written as

$$\partial_t \phi = \left[1 + B_2 + p^{\mathrm{c}}_{\mathrm{mf}} B_1(\partial_\vartheta \phi, w)\right]^{-1} \left[p^{\mathrm{c}}_{\mathrm{fs}} \mathcal{L}_i \partial_\vartheta \phi - p^{\mathrm{c}}_{\mathrm{mf}} G(\partial_\vartheta \phi, w)\right].$$

Substituting this expression for $\partial_t \phi$ into the second equation of (3.24) for w, we arrive at the system

$$\partial_t \phi = \left[1 + B_2 + p^{\mathrm{c}}_{\mathrm{mf}} B_1(\partial_\vartheta \phi, w)\right]^{-1} \left[p^{\mathrm{c}}_{\mathrm{fs}} \mathcal{L}_i \partial_\vartheta \phi - p^{\mathrm{c}}_{\mathrm{mf}} G(\partial_\vartheta \phi, w)\right], \tag{3.28}$$

$$\begin{aligned}\partial_t w = {} & \mathcal{L} w - P^{\mathrm{s}}_{\mathrm{fs}} \mathcal{L}_i \partial_\vartheta \phi + P^{\mathrm{s}}_{\mathrm{mf}} G(\partial_\vartheta \phi, w) + \left[P^{\mathrm{s}}_{\mathrm{fs}} B_0 - P^{\mathrm{s}}_{\mathrm{mf}} B_1(\partial_\vartheta \phi, w)\right] \\ & \times \left[1 + B_2 + p^{\mathrm{c}}_{\mathrm{mf}} B_1(\partial_\vartheta \phi, w)\right]^{-1} \left[p^{\mathrm{c}}_{\mathrm{fs}} \mathcal{L}_i \partial_\vartheta \phi - p^{\mathrm{c}}_{\mathrm{mf}} G(\partial_\vartheta \phi, w)\right].\end{aligned} \tag{3.29}$$

Thus we have a splitting of the critical modes ϕ and the noncritical modes w.

3.4. The system for wave numbers and damped modes

We now replace ϕ by $\psi = \partial_\vartheta \phi$ and obtain

$$\partial_t \psi = \partial_\vartheta \left[1 + B_2 + p^{\mathrm{c}}_{\mathrm{mf}} B_1(\psi, w)\right]^{-1} \left[p^{\mathrm{c}}_{\mathrm{fs}} \mathcal{L}_i \psi - p^{\mathrm{c}}_{\mathrm{mf}} G(\psi, w)\right], \tag{3.30}$$

$$\begin{aligned}\partial_t w = {} & \mathcal{L} w - P^{\mathrm{s}}_{\mathrm{fs}} \mathcal{L}_i \psi + P^{\mathrm{s}}_{\mathrm{mf}} G(\psi, w) + \left[P^{\mathrm{s}}_{\mathrm{fs}} B_0 - P^{\mathrm{s}}_{\mathrm{mf}} B_1(\psi, w)\right] \\ & \times \left[1 + B_2 + p^{\mathrm{c}}_{\mathrm{mf}} B_1(\psi, w)\right]^{-1} \left[p^{\mathrm{c}}_{\mathrm{fs}} \mathcal{L}_i \psi - p^{\mathrm{c}}_{\mathrm{mf}} G(\psi, w)\right],\end{aligned} \tag{3.31}$$

which we also write in short as

$$\partial_t \mathcal{V} = \Lambda \mathcal{V} + F(\mathcal{V}), \tag{3.32}$$

where $\mathcal{V} = (\psi, w)$, Λ is a linear operator, and $F(\mathcal{V}) = \mathcal{O}(|\mathcal{V}|^2)$. We now prove that the spectrum of the operator

$$\partial_\vartheta (1+B_2)^{-1} p^{\mathrm{c}}_{\mathrm{mf}} \mathcal{L}_i$$

near $\lambda = 0$ is approximately given by the linear dispersion curve $\lambda_1(\ell)$ with the associated eigenmodes given approximately by the Fourier modes $\exp(-\mathrm{i}\ell\vartheta/k)$. This follows from

$$\mathcal{L}_i\left(\mathrm{e}^{\mathrm{i}\ell\vartheta/k}\right) = k\left[\mathcal{L}\left(\mathrm{e}^{\mathrm{i}\ell\vartheta/k} \mathrm{i}\partial_\ell \tilde{v}_1\right) + \left(kD\left(2\partial_\vartheta^2 u_0 + \mathrm{i}(\ell/k)\partial_\vartheta u_0\right) - \mathrm{i}c_p \partial_\vartheta u_0\right)\mathrm{e}^{\mathrm{i}\ell\vartheta/k}\right],$$

and therefore $(\tilde{p}^{\mathrm{c}}_{\mathrm{mf}} \mathcal{J} \mathcal{L}_i(\mathrm{e}^{\mathrm{i}\ell\vartheta/k}))(\ell) = \chi(\frac{8\ell}{\ell_1}) \mathrm{i}k[\lambda_1'(0) - \mathrm{i}\ell(2kD\partial_\vartheta \partial_k \tilde{v}_1 + \partial_\vartheta u_0) + \mathcal{O}(\ell^2)]\mathrm{e}^{\mathrm{i}\ell\vartheta/k}$. Since $1 + B_2(\ell) = 1 + \mathcal{O}(\ell)$ as a multiplier and $\partial_\vartheta \mathrm{e}^{\mathrm{i}\ell\vartheta/k} = \mathrm{i}(\ell/k)\mathrm{e}^{\mathrm{i}\ell\vartheta/k}$ we find

$$\mathcal{J}\left(\partial_\vartheta (1+B_2)^{-1} p^{\mathrm{c}}_{\mathrm{mf}} \mathcal{L}_i \mathrm{e}^{\mathrm{i}\ell\vartheta/k}\right)(\ell) = \chi\left(\frac{8\ell}{\ell_1}\right)\left(\lambda_1(\ell) + \mathcal{O}\left(\ell^3\right)\right)\mathrm{e}^{\mathrm{i}\ell\vartheta/k}. \tag{3.33}$$

For notational convenience we diagonalize the linear part of (3.30), (3.31) by setting

$$\begin{pmatrix} v^c \\ v^s \end{pmatrix} = \mathcal{S}^{-1} \begin{pmatrix} \psi \\ w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\mathcal{S}_1 & 1 \end{pmatrix} \begin{pmatrix} \psi \\ w \end{pmatrix}, \tag{3.34}$$

where $\tilde{\mathcal{S}}_1 \in C^\infty([-k/2, k/2), L(\mathbb{C}, H^m(\mathcal{T}_{2\pi})))$ is a multiplier with $\operatorname{supp} \tilde{\mathcal{S}}_1 \subset \{\ell_1/8 < |\ell| < \ell_1/4\}$. Thus, $v^c = \psi$ and $P^s v^s = v^s$, and, by definition,

$$\mathcal{S}^{-1} \Lambda \mathcal{S} = \operatorname{diag}(\lambda^c, \Lambda^s), \tag{3.35}$$

with $\lambda^c(\ell) = \chi(\frac{8\ell}{\ell_1})(\lambda_1(\ell) + \mathcal{O}(\ell^3))$, cf. (3.33). In these coordinates, (3.32) becomes

$$\partial_t v^c = \lambda^c v^c + \partial_\vartheta p^c_{\mathrm{mf}} \mathcal{N}(v^c, v^s), \tag{3.36a}$$

$$\partial_t v^s = \Lambda^s v^s + P^s_{\mathrm{mf}} \mathcal{N}(v^c, v^s), \tag{3.36b}$$

where $\mathcal{N}$ is a smooth nonlinear map from $H^{m+2}_{\mathrm{ul}} \times H^{m+2}_{\mathrm{ul}}$ into H^m_{ul} for every $m \geqslant 1$.

3.5. The moving frame

To prove Theorems 1 and 2 we want to set up renormalization processes based on (3.36). For this we need to remove the $\mathcal{O}(\ell)$ terms in $\lambda_1(\ell) = \mathrm{i}(c_p - c_g)\ell - \alpha\ell^2 + \mathcal{O}(\ell^3)$. Therefore we define

$$\begin{pmatrix} u^c \\ u^s \end{pmatrix} = \mathcal{J}^{-1} \begin{pmatrix} v^c \\ v^s \end{pmatrix} \quad \text{via} \quad \begin{pmatrix} \tilde{u}^c \\ \tilde{u}^s \end{pmatrix} (\vartheta, \ell, t) = \mathrm{e}^{\mathrm{i}(c_p - c_g)\ell t} \begin{pmatrix} \tilde{v}^c \\ \tilde{v}^s \end{pmatrix} (\vartheta, \ell, t). \tag{3.37}$$

This yields

$$\partial_t \tilde{u}^c = \tilde{\lambda}_g(\ell) \tilde{u}^c + (\partial_\vartheta + \mathrm{i}\ell/k) \tilde{p}^c_{\mathrm{mf}} \tilde{\mathcal{N}}(\tilde{u}^c, \tilde{u}^s), \tag{3.38a}$$

$$\partial_t \tilde{u}^s = \tilde{\Lambda}^s_g(\ell) \tilde{u}^s + \tilde{P}^s_{\mathrm{mf}} \tilde{\mathcal{N}}(\tilde{u}^c, \tilde{u}^s), \tag{3.38b}$$

where $\tilde{\lambda}_g(\ell) = \lambda_1(\ell) - \mathrm{i}(c_p - c_g)\ell$ and $\tilde{\Lambda}^s_g(\ell) = \tilde{\Lambda}(\ell) - \mathrm{i}(c_p - c_g)\ell$. The factors $\mathrm{e}^{\pm\mathrm{i}c\ell t}$ drop out of the nonlinearities since as multipliers they commute with the mode filters and

$$(\tilde{u}^{*2})(\ell)\mathrm{e}^{-\mathrm{i}c\ell t} = \left(\left(\mathrm{e}^{\mathrm{i}c\ell t}\tilde{v}\right)^{*2}\right)(\ell)\mathrm{e}^{-\mathrm{i}c\ell t} = \int_m \mathrm{e}^{\mathrm{i}c(\ell - m)t}\tilde{v}(\ell - m)\mathrm{e}^{\mathrm{i}cmt}\tilde{v}(m)\,\mathrm{d}m\,\mathrm{e}^{-\mathrm{i}c\ell t} = (\tilde{v}^{*2})(\ell),$$

and similar for higher power convolutions.

In general, (3.37) does not correspond to a simple transform in ϑ-space. However, if $\tilde{u}$ has the special form $\tilde{u}(t, \ell, \vartheta) = \tilde{\alpha}(\ell, t)g(\vartheta)$ then, cf. (3.9),

$$\begin{aligned} v(\vartheta, t) &= \int_{-k/2}^{k/2} \mathrm{e}^{\mathrm{i}\ell(\vartheta/k + (c_p - c_g)t)} \tilde{\alpha}(t, \ell) g(\vartheta)\,\mathrm{d}\ell = \alpha\big(\vartheta/k + (c_p - c_g)t, t\big) g(\vartheta) \\ &= \big[\alpha(x - c_g t, t) + \partial_\vartheta \alpha(x - c_g t, t)\phi_\vartheta(\vartheta, t) + \text{h.o.t.}\big] g(\vartheta). \end{aligned} \tag{3.39}$$

Thus, (3.37) will be responsible for recovering the group speed in Theorems 1 and 2, which motivates the index g in (3.38). On the other hand, completely transforming (3.36) to a comoving frame would make the linear part spatially and temporally periodic, and the subsequent analysis would require Floquet theory in time and thus be more complicated.

The key features of (3.38) are the following. By construction,

$$\lambda_g(\ell) = \big(\lambda_1(\ell) - \mathrm{i}(c_p - c_g)\ell\big) = -\alpha\ell^2 + \mathcal{O}\big(\ell^3\big). \tag{3.40}$$

We have

$$(\partial_\vartheta + \mathrm{i}\ell/k)\tilde{p}^c_{\mathrm{mf}}\mathcal{N}\big(\tilde{u}^c + \tilde{u}^s\big) = \tilde{\eta}(\ell)\mathcal{N}^c\big(\tilde{u}^c, \tilde{u}^s\big), \tag{3.41}$$

where $|\tilde{\eta}(\ell)| = C\ell$ and $\mathcal{N}^c$ maps $H^{m+2}(n) \times H^{m+2}(n)$ into $H^s(n)$ for all $s \in \mathbb{N}$. In particular, by the calculations from [3],

$$\tilde{\eta}(\ell)\tilde{\mathcal{N}^c}\big(\tilde{u}^c, \tilde{u}^s\big) = \beta\mathrm{i}\ell\big(\tilde{u}^c\big)^{*2} + \text{h.o.t.}, \tag{3.42}$$

with $\beta = -\frac{1}{2}\omega''(k)$, and where the higher-order terms h.o.t. are discussed later. The spectrum of $\tilde{\Lambda}^s_g$ is left of $\operatorname{Re} z < -\sigma_0$, hence $\tilde{u}^s$ is linearly exponentially damped. Thus, heuristically, if for now we ignore $\tilde{u}^s$ and h.o.t. in (3.42), then, as explained in Section 2.4 we have the following situations: in Theorem 1 and in Theorem 2 case (i) (with $\omega'' = 0$), corresponding to Proposition 2.4 cases (i) and (ii), respectively, the whole nonlinearity is irrelevant and we obtain Gaussian diffusive behavior of $\tilde{u}^c$; for case (ii) of Theorem 2 ($\omega'' \neq 0$), corresponding to Proposition 2.4 case (iii), the dynamics are governed by the Burgers equation for $\tilde{u}^c$.

The (unavoidable) drawbacks of the coordinates (3.38) are their relatively complicated derivation, and that (3.38) is quasi-linear while the original system (1.2) is semi-linear.

4. The results in Bloch wave space

To prove Theorems 1 and 2, in Section 5 we set up renormalization processes for (3.38) in Bloch space. For this we need Bloch spaces with regularity and weights in ℓ. Thus we first collect a number of definitions and basic properties. We recall that $H^{m_2}(m_1) = \{u \in L^2(\mathbb{R})\colon \|u\|_{H^{m_2}(m_1)} < \infty\}$ with $\|u\|_{H^{m_2}(m_1)} = \|u\rho^{m_1}\|_{H^{m_2}(\mathbb{R})}$, where $\rho(x) = (1 + |x|^2)^{1/2}$, and that $\mathcal{F}$ is an isomorphism between $H^{m_2}(m_1)$ and $\hat{H}^{m_1}(m_2)$, where the notation $\hat{H}^{m_1}(m_2) = H^{m_1}(m_2)$ is used to indicate functions that live in Fourier space.

Similarly, for $L > 0$ and $m_1, m_2, b \geqslant 0$ define

$$\mathcal{B}^{m_1}_L(m_2, b) := \big\{\tilde{v} \in H^{m_1}\big((-Lk/2, Lk/2), H^{m_2}_{\mathrm{per}}\big((0, 2\pi)\big)\big)\colon \|\tilde{v}\|_{\mathcal{B}^{m_1}_L(m_2,b)} < \infty\big\},$$
$$\|\tilde{v}\|^2_{\mathcal{B}^{m_1}_L(m_2,b)} = \sum_{\alpha\leqslant m_1}\sum_{\beta\leqslant m_2}\big\|\big(\partial^\alpha_\ell\partial^\beta_\vartheta\tilde{v}\big)\rho^b\big\|^2_{L^2((-Lk/2,Lk/2),L^2(\mathcal{T}_{2\pi}))}.$$

Here $\rho = \rho(\ell)$, i.e., we introduce a weight in the Bloch wave number ℓ, and the subscript L indicates that the Bloch wave number varies in $[-kL/2, kL/2]$. For fixed $L > 0$ the weight ρ is irrelevant since, due to the bounded wave number domain, all norms $\|\cdot\|_{\mathcal{B}^{m_1}_L(m_2,b_1)}$ and $\|\cdot\|_{\mathcal{B}^{m_1}_L(m_2,b_2)}$ are equivalent, but the constants depend on b_1, b_2 and L. The purpose of the weights is to take advantage of the "derivative structure" of the nonlinearity in the equation for $\tilde{u}^c$, see (3.42), and Lemma 5.2 below.

Let $\mathcal{B}^{m_1}(m_2, b) := \mathcal{B}^{m_1}_1(m_2, b)$. Then $\mathcal{J}$ is an isomorphism between $H^{m_2}(m_1)$ and $\mathcal{B}^{m_1}(m_2, b)$, with arbitrary $b \geqslant 0$, see, e.g., [17, Lemma 5.4]. We define the scaling operators

$$\mathcal{R}_{1/L}\colon \mathcal{B}^{m_1}(m_2, b) \to \mathcal{B}^{m_1}_L(m_2, b), \qquad [\mathcal{R}_{1/L}\tilde{v}](\vartheta, \ell) = \tilde{v}(\vartheta, \ell/L).$$

Only ℓ is rescaled, and ϑ is not, and similar to (3.37) this does in general not correspond to a simple rescaling of v. However, note that $\tilde{u}^c = \tilde{u}^c(\ell, t)$ does not depend on ϑ, i.e., for $\tilde{u}^c$ Bloch space is identified with Fourier space, and in this case we have

$$\mathcal{J}^{-1}(\mathcal{R}_{1/L}\tilde{u}) = \mathcal{F}^{-1}(\mathcal{R}_{1/L}\tilde{u}) = L\mathcal{R}_L u, \tag{4.1}$$

i.e., concentration at $\ell = 0$ in Bloch space corresponds to spreading in ϑ. Finally,

$$\|\mathcal{R}_{1/L}\tilde{v}\|_{\mathcal{B}_L^{m_1}(2,b)} \leqslant CL^{b+1/2}\|\tilde{v}\|_{\mathcal{B}^{m_1}(2,b)}, \tag{4.2}$$

and, for $\tilde{u}, \tilde{v} \in \mathcal{B}_L^{m_1}(m_2, 0)$ with $m_1, m_2 \geqslant 1/2$ and $\ell \in (-L/2, L/2)$,

$$\begin{aligned} \mathcal{R}_{1/L}(\mathcal{R}_L\tilde{u} * \mathcal{R}_L\tilde{v})(\ell, x) &= \int_{-1/2}^{1/2} \tilde{u}(\ell - Lm, x) v(Lm, x)\,\mathrm{d}m \\ &= L^{-1}\int_{-L/2}^{L/2} \tilde{u}(\ell - m, x)\tilde{v}(m, x)\,\mathrm{d}m =: L^{-1}(\tilde{u} *_L \tilde{v})(\ell, x). \end{aligned} \tag{4.3}$$

This will be used to express the rescaled nonlinear terms, where henceforth we will drop the subscript $_L$ in $*_L$.

To recall the heuristics, as a model for Theorems 1 and 2(i) (in which the nonlinearities are completely irrelevant), consider the Fourier transformed version of $\partial_t u = \alpha\partial_x^2 u$, $u_{t=1} = u_0$, i.e., $\partial_t\tilde{u} = -\alpha\ell^2\tilde{u}$, which is solved by $\tilde{u}(\ell, t) = \mathrm{e}^{-(t-1)\alpha\ell^2}\tilde{u}(\ell, 1)$. Then, for any $c \in \mathbb{C}$, or more specifically $c \in \mathbb{R}$ since we consider real-valued functions u, $\tilde{f}_c(\ell) = c\mathrm{e}^{-\alpha\ell^2}$ is a fixed point of the renormalization map

$$\mathcal{G}^{(1)} : \tilde{u} \mapsto \mathrm{e}^{-\alpha\ell^2(1-1/L^2)}\mathcal{R}_{1/L}\tilde{u}. \tag{4.4}$$

Moreover, for $L > 1$ being sufficiently large, this line of fixed points is attractive in $H^2(2)$. To see this, write $\tilde{u}(\ell) = \tilde{f}_c(\ell) + \tilde{g}(\ell)$ with $\tilde{g}(0) = 0$. Then, using $|\tilde{g}(\ell)| \leqslant (\ell/L)\|\partial_\ell\tilde{g}\|_{C_b^0}$ (by the mean value theorem) and $H^2 \hookrightarrow C^1$ we obtain

$$\left\|\mathrm{e}^{-\alpha\ell^2(1-1/L^2)}\mathcal{R}_{1/L}\tilde{g}\right\|_{H^2(2)}^2 \leqslant CL^{-1}\|\tilde{g}\|_{H^2(2)}. \tag{4.5}$$

Thus, $\tilde{v}(\ell, t) = \tilde{u}(t^{-1/2}\ell, t) \to \tilde{f}_c(\ell)$ as $t \to \infty$ is the expected scaling for $\tilde{u}^c$ in Theorem 2(i). Theorem 2(ii) is also based on (4.4) but we have a nonlinear correction to the asymptotic profile as explained in Section 2.4.

Similarly, for any $c \in \mathbb{R}$, $\tilde{g}_c(\ell) = \mathrm{i}c\ell\mathrm{e}^{-\alpha\ell^2}$ is a fixed point of the renormalization map

$$\mathcal{G}^{(2)} : \tilde{u} \mapsto \mathrm{e}^{-\alpha\ell^2(1-1/L^2)}L\mathcal{R}_{1/L}\tilde{u}, \tag{4.6}$$

and again this line of fixed points is attractive in $H^3(2) \cap X_0$, where X_0 consists of functions with zero mean. For this write $\tilde{u}(\ell) = \tilde{g}_c(\ell) + \tilde{h}(\ell)$ with $\partial_\ell\tilde{h}(0) = 0$ and use $|\tilde{h}(\ell/L)| \leqslant (\ell/L)^2\|\partial_\ell^2\tilde{h}\|_{C_b^0}$ and $H^3 \hookrightarrow C^2$. Thus, $\tilde{v}(\ell, t) = t^{1/2}\tilde{u}(t^{-1/2}\ell, t) \to \tilde{g}_c(\ell)$ as $t \to \infty$ is the expected scaling for $\tilde{u}^c$ in Theorem 1. The need for $\tilde{u} \in C^2$ also explains the higher weight in x in Theorem 1.

Theorem 3 *(Diffusive stability). Let $u_0(\cdot;k)$ be a spectrally stable wave train and $b \in (0,1/2)$. There exist $\varepsilon, C > 0$ such that if $\|(\tilde{u}^c,\tilde{u}^s)|_{t=1}\|_{H^3(2)\times\mathcal{B}^3(2,2)} \leqslant \varepsilon_1$ and $\tilde{u}^c(0,1) = \frac{1}{2\pi k}\int u^c(\vartheta,1)\,\mathrm{d}\vartheta = 0$, then the solution $(\tilde{u}^c,\tilde{u}^s)$ to (3.38) exists for all $t \geqslant 1$, and there exists a $\tilde{\psi}_{\lim} \in \mathbb{R}$ such that*

$$\left\| t^{1/2}\tilde{u}^c\left(t^{-1/2}\ell,t\right) - \mathrm{i}\tilde{\psi}_{\lim}\ell \mathrm{e}^{-\alpha\ell^2}\right\|_{H^3(2)} \leqslant Ct^{-1/2+b}, \tag{4.7}$$

$$\left\| t^{1/2}\tilde{u}^s\left(t^{-1/2}\ell,t\right)\right\|_{\mathcal{B}^3_{\sqrt{t}}(2,2)} \leqslant Ct^{-1/2+b}. \tag{4.8}$$

Theorem 4 *(Diffusive mixing of phases). Let $u_0(\cdot;k)$ be a spectrally stable wave train and $b \in (0,1/2)$. There exist $\varepsilon, C > 0$ such that for $|\phi_d| \leqslant \varepsilon$ the following holds.*

(i) *Assume that $\beta = -\frac{1}{2}\omega''(k) = 0$, $\|\tilde{u}^c(\ell,1)\|_{H^2(2)} \leqslant \varepsilon$ with $\tilde{u}^c(0,1) = \phi_d/(2\pi k)$, $\|\tilde{u}^s(\cdot,1)\|_{\mathcal{B}^2(2,2)} \leqslant \varepsilon$ and $\tilde{P}^s\tilde{u}^s(\cdot,1) = \tilde{u}^s(\cdot,1)$. Then the solution $(\tilde{u}^c,\tilde{u}^s)$ to (3.38) exists for all $t \geqslant 1$, and*

$$\left\| \mathcal{R}_{1/\sqrt{t}}\tilde{u}^c(\ell,t) - \tilde{u}^c_*(\ell)\right\|_{H^2(2)} \leqslant Ct^{-1/2+b}, \tag{4.9}$$

$$\left\| \mathcal{R}_{1/\sqrt{t}}\tilde{u}^s(\ell,t)\right\|_{B^2_{\sqrt{t}}(2,2)} \leqslant Ct^{-1/2+b}\ , \tag{4.10}$$

where $\tilde{u}^c_*(\ell) = \phi_d \mathrm{e}^{-\alpha\ell^2}$.

(ii) *If $\beta = -\frac{1}{2}\omega''(k) \neq 0$ then the same result holds with $\tilde{u}^c_*(\ell)$ replaced by*

$$\tilde{u}^c_*(\ell) = \mathcal{F}\left(\frac{\sqrt{\alpha}}{\beta}\frac{z\mathrm{e}^{-\vartheta^2/(k^2\alpha)}}{1+z\,\mathrm{erf}(\vartheta/\sqrt{k\alpha})}\right)(\ell), \tag{4.11}$$

where $\ln(1+z) = \frac{\beta}{\alpha}\phi_d$.

Before proving Theorems 3 and 4 we show that they imply Theorems 1 and 2.

Proof of Theorem 1. Given initial data in the form (2.15) from Theorem 1, i.e.,

$$u(x,t)|_{t=0} = u_0\left(\theta-\theta_0+\phi_0(x);k\right) + v_0(x) \quad \text{with } \|\phi_0\|_{H^3(3)}, \|v_0\|_{H^3(2)} \leqslant \varepsilon,$$

we first need to extract $(\tilde{u}^c,\tilde{u}^s)|_{t=1}$ and show that they fulfill the assumptions of Theorem 3. Then we translate (4.7), (4.8) back into (ϕ, v) coordinates.

Thus, as explained in Remark 3.2, let $u(x,t)|_{t=1} = u_0(\vartheta;k(1+\partial_\vartheta\phi_0)) + w_0(\vartheta)$ with $\theta = \vartheta - \phi_0(\vartheta)$ and

$$w_0(\vartheta) := u_0(\vartheta;k) - u_0\left(\vartheta;k(1+\partial_\vartheta\phi_0)\right) + v_0(x).$$

W.l.o.g. assume that $(1-P^s)w_0 = 0$, otherwise redefine $\phi_0 = p^c_{\mathrm{mf}}\phi_0$. This fixes the non-uniqueness in (2.15). Also, $\phi_0 \in H^m(3)$ for all $m \in \mathbb{N}$ due to the compact support of $\tilde{\phi}_0$, and, with $\psi_0 = \partial_\vartheta\phi_0$, J from (3.12) and $\mathcal{S}$ from (3.34),

$$\left(\tilde{u}^c,\tilde{u}^s\right)\Big|_{t=1} = J\mathcal{S}^{-1}(\psi_0,w_0) = J(\psi_0, w_0 - S_1\psi_0) \tag{4.12}$$

is well defined and fulfills $\|(\tilde{u}^c,\tilde{u}^s)\|_{H^3(2)\times\mathcal{B}^3(2,2)} \leqslant C_1\varepsilon$ and $\tilde{u}^c(0,1) = \frac{1}{2\pi k}\int u^c(\vartheta,1)\,\mathrm{d}\vartheta = 0$.

We now use (4.7), (4.8) to recover Theorem 1. Using (3.10) and $\mathcal{F}_1^{-1}(\mathrm{i}\ell\mathrm{e}^{-\alpha\ell^2})(\vartheta) = -\frac{1}{\sqrt{4\pi\alpha}}\frac{\vartheta}{2\alpha}\mathrm{e}^{-\vartheta^2/(4\alpha)}$ we have

$$t\mathcal{R}_{t^{1/2}}u^c(\vartheta,t)-\psi_{\lim}\vartheta\mathrm{e}^{-\vartheta^2/(4k^2\alpha)}=\mathcal{F}^{-1}\big[t^{1/2}\mathcal{R}_{t^{-1/2}}\tilde{u}_c(\ell,t)-\mathrm{i}\tilde{\psi}_{\lim}\ell\mathrm{e}^{-\alpha\ell^2}\big](\vartheta),$$

where $\psi_{\lim}=-\frac{\tilde{\psi}_{\lim}}{\sqrt{4\pi\alpha}}\frac{1}{2\alpha k}$, and from $c_1\|\hat{u}\|_{H^n(m)}\leqslant\|u\|_{H^m(n)}\leqslant c_2\|\hat{u}\|_{H^n(m)}$ we obtain

$$\big\|t\mathcal{R}_{t^{1/2}}u^c(\vartheta,t)-\psi_{\lim}\vartheta\mathrm{e}^{-\vartheta^2/(4k^2\alpha)}\big\|_{H^2(3)}\leqslant Ct^{-1/2+b}.$$

Then, with

$$\psi(\vartheta,t)=u^c\big(\vartheta+k(c_p-c_g)t,t\big)\quad\text{and}\quad w(\vartheta,t)=\mathcal{S}_1u^c\big(\vartheta/k+(c_p-c_g)t,t\big)+u^s(\vartheta,t)$$

we obtain, in L^∞,

$$\begin{aligned}\phi(\vartheta,t):=\int_{-\infty}^{\vartheta}\psi(\xi,t)\,\mathrm{d}\xi&=-\frac{2k^2\psi_{\lim}\alpha}{\sqrt{t}}\int_{-\infty}^{\vartheta+k(c_p-c_g)t}\bigg(-\frac{\xi}{2\alpha k^2t}\mathrm{e}^{-\xi^2/(4k^2\alpha t)}\bigg)\,\mathrm{d}\xi+\mathcal{O}\big(t^{-1}\big)\\&=-2t^{-1/2}k^2\psi_{\lim}\alpha\mathrm{e}^{-(\vartheta+k(c_p-c_g)t)^2/(4\alpha k^2t)}+\mathcal{O}\big(t^{-1}\big),\end{aligned}\tag{4.13}$$

i.e., $\phi_{\lim}=-4k^2\sqrt{\alpha^3\pi}\psi_{\lim}$. Also $w(\vartheta,t)=\mathcal{O}(t^{-1})$ since $\operatorname{supp}\tilde{\mathcal{S}}_1\subset\{\ell_1/8<|\ell|<\ell_1/4\}$ and

$$\begin{aligned}\big\|u^s(\vartheta,t)\big\|_{L^\infty}=\big\|J^{-1}\tilde{u}^s(\cdot,\cdot,t)(\vartheta)\big\|_{L^\infty}&=\bigg\|\int_{-k/2}^{k/2}\mathrm{e}^{\mathrm{i}\ell\vartheta/k}\tilde{u}^s(\vartheta,\ell,t)\,\mathrm{d}\ell\bigg\|_{L^\infty}\\&=\bigg\|t^{-1/2}\int_{-kt^{1/2}/2}^{kt^{1/2}/2}\mathrm{e}^{\mathrm{i}\ell t^{-1/2}\vartheta/k}\tilde{u}^s\big(\vartheta,t^{-1/2}\ell,t\big)\big(1+\ell^2\big)^2\big(1+\ell^2\big)^{-2}\,\mathrm{d}\ell\bigg\|_{L^\infty}\\&\leqslant Ct^{-1/2}\big\|\mathcal{R}_{t^{-1/2}}\tilde{u}^s(\cdot,\cdot,t)\big\|_{B^3_{\sqrt{t}}(2,2)}\leqslant Ct^{-1+b}.\end{aligned}\tag{4.14}$$

Finally,

$$\begin{aligned}\vartheta&=\theta-\theta_0+\phi(\vartheta,t)=\theta-\theta_0-2k^2t^{-1/2}\psi_{\lim}\alpha\mathrm{e}^{-(\vartheta+k(c_p-c_g)t)^2/(4\alpha k^2t)}+\mathcal{O}\big(t^{-1}\big)\\&=\theta-\theta_0-2k^2t^{-1/2}\psi_{\lim}\alpha\mathrm{e}^{-(x-c_gt)^2/(4\alpha t)}+\mathcal{O}\big(t^{-1}\big)\end{aligned}$$

using $\theta=kx-\omega t=k(x-c_pt)$ and the implicit function theorem. □

Proof of Theorem 2. First, assume that $\beta=0$. As above we write $u(x,t)|_{t=0}=u_0(\vartheta;k(1+\partial_\vartheta\phi_0))+w_0(\vartheta)$ with $w_0(\vartheta):=u_0(\vartheta;k)-u_0(\vartheta;k(1+\partial_\vartheta\phi_0))+v_0(x)$, and where now $\|\phi_0'(\cdot)\|_{H^2(2)}\leqslant\varepsilon$ and $\phi_0(\vartheta)\to\phi_\pm$ as $\vartheta\to\pm\infty$. Again, w.l.o.g. assume that $(1-P^s)w_0=0$. Then, for

$$\psi_0(\vartheta)=\partial_\vartheta\phi_0(\vartheta)=u^c(\vartheta,1)$$

we obtain $2\pi k\tilde{u}^c(0,1)=\int u^c(\vartheta,1)\,\mathrm{d}\vartheta=\phi_d$, and Theorem 4 applies to

$$\big(\tilde{u}^c,\tilde{u}^s\big)\big|_{t=1}=J\mathcal{S}^{-1}(\psi_0,w_0)=J(\psi_0,w_0-S_1\psi_0).$$

Thus, with $\psi(\vartheta,t)=u^c(\vartheta+k(c_p-c_g)t,t)$ and $w(\vartheta,t)=\mathcal{S}_1u^c(\vartheta+k(c_p-c_g)t,t)+u^s(\vartheta,t)$ we obtain, in L^∞,

$$\phi(\vartheta,t) := \phi_- + \int_{-\infty}^{\vartheta} \psi(\xi,t)\,\mathrm{d}\xi = \phi_- + (\phi_+ - \phi_-)\frac{1}{\sqrt{4\pi k^2 t}} \int_{-\infty}^{\vartheta+k(c_p-c_g)t} \mathrm{e}^{-\xi^2/(4k^2\alpha t)}\,\mathrm{d}\xi + \mathcal{O}\big(t^{-1}\big)$$
$$= \phi_- + (\phi_+ - \phi_-)\,\mathrm{erf}\big((x-c_g t)/\sqrt{4\alpha t}\big) + \mathcal{O}\big(t^{-1}\big)$$

and $w(\vartheta,t) = \mathcal{O}(t^{-1/2+b})$ as in (4.13) and (4.14) above. Hence

$$\vartheta = \theta + \phi(\vartheta,t) = \theta + \phi_- + (\phi_+ - \phi_-)\,\mathrm{erf}\big((x-c_g t)/\sqrt{4\alpha t}\big) + \mathcal{O}\big(t^{-1/2}\big)$$

and by shifting the $\mathcal{O}(t^{-1/2})$-part to v we obtain part (i) in Theorem 2. Part (ii) with $\beta \neq 0$ works in the same way. □

5. Renormalization

We first prove Theorem 3; the minor modifications needed to prove Theorem 4(i) are then explained in Section 5.5, while the changes for the slightly more complicated proof of Theorem 4(ii) are explained in Section 5.6.

5.1. The rescaled systems

Based on the asserted behavior $t^{1/2}\tilde{u}^c(t^{-1/2}\ell,t) \to \mathrm{i}\psi_{\lim}\ell\mathrm{e}^{-\alpha\ell^2}$ we introduce, for $n \in \mathbb{N}$ and $L > 1$ chosen sufficiently large below, the variables

$$\tilde{u}_n^c(\kappa,\tau) := L^n \tilde{u}^c\big(\kappa/L^n, L^{2n}\tau\big) = L^n\big[\mathcal{R}_{L^{-n}}\tilde{u}^c\big]\big(\kappa, L^{2n}\tau\big), \tag{5.1}$$
$$\tilde{u}_n^s(\vartheta,\kappa,\tau) := L^n \tilde{u}^s\big(\vartheta,\kappa/L^n, L^{2n}\tau\big) = L^n\big[\mathcal{R}_{L^{-n}}\tilde{u}^s\big]\big(\vartheta,\kappa, L^{2n}\tau\big). \tag{5.2}$$

Then $(\tilde{u}_n^c, \tilde{u}_n^s)$ fulfill

$$\partial_\tau \tilde{u}_n^c(\kappa,\tau) - \tilde{\lambda}_{g,n}(\kappa)\tilde{u}_n^c(\kappa,\tau) = L^{3n}\tilde{\mathcal{N}}_n^c\big(\tilde{u}_n^c,\tilde{u}_n^s\big)(\kappa,\tau), \tag{5.3a}$$
$$\partial_\tau \tilde{u}_n^s(\vartheta,\kappa,\tau) - \tilde{\Lambda}_{g,n}\tilde{u}_n^s(\vartheta,\kappa,\tau) = L^{3n}\tilde{\mathcal{N}}_n^s\big(\tilde{u}_n^c,\tilde{u}_n^s\big)(\vartheta,\kappa,\tau), \tag{5.3b}$$

where

$$\tilde{\lambda}_{g,n}(\kappa) = L^{2n}\tilde{\lambda}_g\big(\kappa/L^n\big), \qquad \tilde{\Lambda}_{g,n} = L^{2n}\mathcal{R}_{L^{-n}}\tilde{\Lambda}_g\mathcal{R}_{L^n},$$
$$\tilde{\mathcal{N}}_n^c\big(\tilde{u}_n^c,\tilde{u}_n^s\big) = \tilde{\eta}\big(\kappa/L^n\big)\mathcal{R}_{L^{-n}}\tilde{\mathcal{N}}^c\big(L^{-n}\mathcal{R}_{L^n}\tilde{u}_n^c, L^{-n}\mathcal{R}_{L^n}\tilde{u}_n^s\big),$$
$$\tilde{\mathcal{N}}_n^s\big(\tilde{u}_n^c,\tilde{u}_n^s\big) = \mathcal{R}_{L^{-n}}\tilde{P}_{\mathrm{mf}}^s\tilde{\mathcal{N}}\big(L^{-n}\mathcal{R}_{L^n}\tilde{u}_n^c, L^{-n}\mathcal{R}_{L^n}\tilde{u}_n^s\big).$$

Except for the different scaling due to $\tilde{u}_n^c(0,\ell) = 0$, (5.3) has a very similar structure as, e.g., [15, Eq. (30)] or [18, Eq. (3.2)]. Thus, similar to (2.38), we shall consider the following iteration:

$$\text{solve (5.3) for } \tau \in I := \big[L^{-2}, 1\big] \text{ with initial data } \begin{pmatrix}\tilde{u}_n^c\\ \tilde{u}_n^s\end{pmatrix}\big(\vartheta,\kappa,L^{-2}\big) = L\begin{pmatrix}\tilde{u}_{n-1}^c\\ \tilde{u}_{n-1}^s\end{pmatrix}(\vartheta,\kappa/L,1). \tag{5.4}$$

Formally, (5.3) is solved by the variation of constant formula, i.e.

$$\tilde{u}_n^c(\kappa,\tau) = \mathrm{e}^{(\tau-1/L^2)\tilde{\lambda}_{g,n}(\kappa)}\tilde{u}_n^c\big(\kappa,1/L^2\big) + \int\limits_{1/L^2}^{\tau} \mathrm{e}^{(\tau-s)\tilde{\lambda}_{g,n}(\kappa)} L^{3n}\tilde{\mathcal{N}}_n^c\big(\tilde{u}_n^c,\tilde{u}_n^s\big)(\kappa,s)\,\mathrm{d}s, \tag{5.5a}$$

$$\tilde{u}_n^s(\vartheta,\kappa,\tau) = \mathrm{e}^{(\tau-1/L^2)\tilde{\Lambda}_g\mathcal{R}_{L^n}}\tilde{u}_n^s(\vartheta,\kappa,\tau) + \int\limits_{1/L^2}^{\tau} \mathrm{e}^{(\tau-s)\tilde{\Lambda}_{g,n}} L^{3n}\tilde{\mathcal{N}}_n^s\big(\tilde{u}_n^c,\tilde{u}_n^s\big)(\vartheta,\kappa,s)\,\mathrm{d}s. \tag{5.5b}$$

However, (5.5) cannot be used to construct the solution since (5.3) is a quasi-linear system, as it can be seen from $\mathcal{N}: H^{m_2}(m_1) \times H^{m_2}(m_1) \to H^{m_2-2}(m_1)$ in (3.36). To solve (5.3) we use maximal regularity methods [10] for parabolic equations in (weighted) Sobolev spaces as in [19]. A posteriori, (5.5) can then be used to estimate the solutions. Thus, we first note some properties of the linear semigroups and the nonlinearities in (5.5), and then explain how to obtain local existence for (5.3).

5.2. Estimates on the linear semigroups and the nonlinearities

We shall need some detailed estimates on the linear semigroups and the nonlinear terms in (5.5). The idea is to exploit the derivative-like structure in the Bloch wave number κ of $\tilde{\mathcal{N}}^c$ as expressed in (3.42) by relaxing the weight, and to regain the weight using $\mathrm{e}^{(\tau-\tau')\tilde{\lambda}_{g,n}}$. Thus, from this point on, the weights in κ become important.

Lemma 5.1. *There exists a $C > 0$ such that for all $L > 1$ we have*

$$\big\|\mathrm{e}^{(\tau-\tau')\tilde{\lambda}_{g,n}}\tilde{u}_n^c\big\|_{B^3_{L^n}(2,2)} \leqslant C\max\big\{1,\big(\tau-\tau'\big)^{-b/2}\big\}\big\|\tilde{u}_n^c\big\|_{B^3_{L^n}(2,2-b)}, \tag{5.6}$$

$$\big\|\mathrm{e}^{(\tau-\tau')\tilde{\Lambda}_{g,n}}\tilde{u}_n^s\big\|_{B^3_{L^n}(2,2)} \leqslant C\max\big\{1,\big(\tau-\tau'\big)^{-m_2/2}\big\}\mathrm{e}^{-\gamma_0 L^{2n}(\tau-\tau')}\big\|\tilde{u}_n^s\big\|_{B^3_{L^n}(2-m_2,2)}. \tag{5.7}$$

Proof. Eq. (5.6) holds since the real part of $\tilde{\lambda}_{g,n}(\kappa) = L^{2n}\tilde{\lambda}_g(\kappa/L^n) = -\alpha\kappa^2 + \mathcal{O}(\kappa^3)$ is bounded from above by the parabola $-\alpha_0\kappa^2$, while (5.7) holds since $\tilde{\Lambda}_{g,n}$ is a relatively bounded perturbation of $L^{2n}(\partial_\vartheta + \mathrm{i}\kappa/L^n)^2$ and by construction has spectrum left of $-L^{2n}\gamma_0$. □

The following lemma transfers the fact that derivatives give higher powers of L^{-1} upon rescaling to general convolution operators with a "derivative-like" structure.

Lemma 5.2. *Let $m_1 \in \mathbb{N}$, $\gamma \geqslant 0$, and $\tilde{K} \in C_b^{m_1}([-1/2,1/2)^2, H^2(\mathcal{T}_{2\pi}))$ with $\|\tilde{K}(\kappa-\ell,\ell)\|_{H^2(\mathcal{T}_{2\pi})} \leqslant C(|\kappa-\ell|+|\ell|)^\gamma$. Then*

$$(\tilde{v},\tilde{w}) \mapsto (\mathcal{M}_{1/L}K)(\tilde{v},\tilde{w})(\kappa,\vartheta) := \int\limits_{-L/2}^{L/2} [\mathcal{R}_{1/L}\tilde{K}](\kappa-\ell,\ell,\vartheta)\tilde{v}(\kappa,\vartheta)\tilde{w}(\kappa-\ell,\vartheta)\,\mathrm{d}\ell$$

defines a bilinear mapping $(\mathcal{M}_{1/L}K): \mathcal{B}_L^{m_1}(2,2) \times \mathcal{B}_L^{m_1}(2,2) \to \mathcal{B}_L^{m_1}(2,2)$, and there exists a $C > 0$ such that for all $L > 1$ we have

$$\big\|(\mathcal{M}_{1/L}K)(\tilde{v},\tilde{w})\big\|_{\mathcal{B}_L^{m_1}(2,2-\gamma)} \leqslant CL^{-\min\{\gamma,1\}}\|\tilde{v}\|_{\mathcal{B}_L^{m_1}(2,2)}\|\tilde{w}\|_{\mathcal{B}_L^{m_1}(2,2)}.$$

Proof. This holds due to $\sup_\ell |\frac{\ell^\gamma L^{-\gamma}}{(1+\ell^2)^{\gamma/2}}| \leqslant CL^{-\gamma}$. □

Lemma 5.3. *Let* $\|(\tilde{u}_n^c, \tilde{u}_n^s)\|_{[B^3_{L^n}(2,2)]^2} \leqslant R_n \leqslant 1$. *There exists a* $C > 0$ *such that*

$$L^{3n} \big\| \tilde{\mathcal{N}}_n^c \big(\tilde{u}_n^c, \tilde{u}_n^s\big) \big\|_{B^3(2,1)} \leqslant C L^{-n} R_n^2. \tag{5.8}$$

The term $L^{3n}\tilde{\mathcal{N}}_n^s$ *can be split according to the number of* ϑ *derivatives as* $L^{3n}\tilde{\mathcal{N}}_n^s = \tilde{\mathcal{N}}_{n,0}^s + \tilde{\mathcal{N}}_{n,1}^s + \tilde{\mathcal{N}}_{n,2}^s$ *such that*

$$\big\| \tilde{\mathcal{N}}_{n,i}^s \big\|_{B^3(2-i,2)} \leqslant C R_n^2. \tag{5.9}$$

Proof. We write $L^{3n}\tilde{\mathcal{N}}_n^c = s_1 + s_2$, where, as explained in Section 3.5, the lowest-order term s_1 in $L^{3n}\tilde{\mathcal{N}}_n^c(\tilde{u}_n^c, \tilde{u}_n^s)$ reads

$$s_1(\kappa) = L^{3n} \mathrm{i}\beta \frac{\kappa}{L^n} \mathcal{R}_{L^{-n}} \big(L^{-n} \mathcal{R}_{L^n} \tilde{u}_n^c \big)^{\star 2}(\kappa), \tag{5.10}$$

cf. (3.42). This yields $\|s_1\|_{\mathcal{B}^3(2,1)} \leqslant C L^{-n} R_n^2$ by direct calculation. The remaining terms s_2 can be estimated in a similar way using Lemma 5.2 and taking into account the finite support of $(\partial_\vartheta + \mathrm{i}\ell/k)\tilde{p}^c_{\mathrm{mf}} \tilde{\mathcal{N}}(\tilde{u}^c + \tilde{u}^s)$ in Fourier space.

This does not work for $L^{3n}\tilde{\mathcal{N}}_n^s(\tilde{u}_n^c, \tilde{u}_n^s)$. However, here we do not need an additional factor L^{-n}, and (5.9) simply follows by checking the number of derivatives in $\mathcal{N}$ and using (4.3). □

5.3. Local existence

Since (3.36) and hence (5.3) is quasi-linear we cannot combine Lemmas 5.1 and 5.3 to directly show local existence for (5.3) via (5.5). Instead we use maximal regularity theory from [10]. For $I = (\tau_0, \tau_1)$ and $r, s \geqslant 0$ let

$$H^{r,s}(I, m_1) = L^2\big(I, H^r(m_1)\big) \cap H^s\big(I, L^2(m_1)\big).$$

Since (3.36) is a parabolic problem these spaces only occur with $s = r/2$ and we set $K^{m_2}(I, m_1) = H^{m_2, m_2/2}(m_1)$. Then, for any given weight $b > 0$, Bloch transform is an isomorphism between $K^{m_2}(I, m_1)$ and

$$\tilde{K}^{m_1}(I, m_2, b) = L^2\big(I, B^{m_1}(m_2, b)\big) \cap H^{m_2/2}\big(I, B^{m_1}(0, b)\big).$$

Similarly, for every n, let

$$\tilde{K}^{m_1}_{L^n}(I, m_2, b) := \mathcal{R}_{1/L^n} \tilde{K}^{m_1}(I, m_2, b) := L^2\big(I, B^{m_1}_{L^n}(m_2, b)\big) \cap H^{m_2/2}\big(I, B^{m_1}_{L^n}(0, b)\big),$$

i.e., the subscript L^n again indicates that the Bloch wave number varies in $[-kL^n/2, kL^n/2)$. From (4.2) we have

$$\|\mathcal{R}_{1/L^n} \tilde{u}\|_{\tilde{K}^{m_1}_{L^n}(I, m_2, b)} \leqslant C L^{n(b+1/2)} \|\tilde{u}\|_{\tilde{K}^{m_1}(I, m_2, b)}. \tag{5.11}$$

Recall that for each n the weight b in κ gives an equivalent norm in $\tilde{K}^{m_1}_{L^n}(I, m_2, b)$, but the constants depend on n. We also need subspaces of functions that vanish sufficiently fast at τ_0, and define

$${}_0K^{m_2}(I, m_1) := \big\{ v \in K^{m_2}(I, m_1) \colon \partial_\tau^j v(\cdot, \tau_0) = 0 \text{ for } j \in \mathbb{N},\ j < m_2/2 - 1/2 \big\},$$
$${}_0\tilde{K}^{m_1}_{L^n}(I, m_2, b) := \big\{ v \in \tilde{K}^{m_1}_{L^n}(I, m_2, b) \colon \partial_\tau^j \tilde{v}(\cdot, \cdot, \tau_0) = 0 \text{ for } j \in \mathbb{N},\ j < m_2/2 - 1/2 \big\}.$$

We set

$$I = \big(L^{-2}, 1\big),$$

and for $(\tilde{u}_n^c, \tilde{u}_n^s)|_{\tau=L^{-2}} \in [B^3_{L^n}(2,2)]^2$ construct solutions $(\tilde{u}_n^c, \tilde{u}_n^s) \in [\tilde{K}^3_{L^n}(I,3,2)]^2$ to (5.3). Note again that for $\tilde{u}_n^c$ we can identify Bloch space with Fourier space such that in fact $\tilde{u}_n^c \in K^3(I,2)$ (in the Fourier sense) with $\operatorname{supp} \tilde{u}_n^c(\tau) \subset I_n = \{|\kappa| \leqslant L^n \ell_1/4\}$. We abbreviate (5.3) as $\mathcal{L}_n \tilde{U}_n = \tilde{\mathcal{N}}_n(\tilde{U}_n)$, where

$$\mathcal{L}_n \tilde{U}_n = \begin{pmatrix} \partial_\tau \tilde{u}_n^c(\kappa,\tau) - \tilde{\lambda}_{g,n}(\kappa)\tilde{u}_n^c(\kappa,\tau) \\ \partial_\tau \tilde{u}_n^s(\vartheta,\kappa,\tau) - \tilde{\Lambda}_{g,n}\tilde{u}_n^s(\vartheta,\kappa,\tau) \end{pmatrix}, \tag{5.12}$$

and, for $m_2 \geqslant 2$, we first consider the linear inhomogeneous version of (5.3) with zero initial data, i.e.,

$$\mathcal{L}_n \tilde{U}_n(\tau) = \tilde{\mathcal{N}}_n(\tau), \qquad \tilde{\mathcal{N}}_n \in \big[{}_0\tilde{K}^3_{L^n}(I, m_2 - 2, 2)\big]^2, \qquad \tilde{U}_n|_{\tau=L^{-2}} = 0, \tag{5.13}$$

where moreover for the first component $\tilde{\mathcal{N}}_n^c$ of $\tilde{\mathcal{N}}_n = (\tilde{\mathcal{N}}_n^c, \tilde{\mathcal{N}}_n^s)$ we assume

$$\tilde{\mathcal{N}}_n^c \in K^3(I,2) \quad \text{(in the Fourier sense)}, \quad \text{and} \quad \operatorname{supp} \tilde{\mathcal{N}}_n^c(\tau) \subset I_n = \big\{|\kappa| \leqslant L^n \ell_1/4\big\}. \tag{5.14}$$

Lemma 5.4. *There exists a $C > 0$, independent of $n \in \mathbb{N}$, such that for all $\tilde{\mathcal{N}}_n \in [{}_0\tilde{K}^3_{L^n}(I, m_2 - 2, 2)]^2$ which fulfill* (5.14) *there exists a unique solution of* (5.13) *with*

$$\|\tilde{U}_n\|_{\tilde{K}^3_{L^n}(I,m_2,2)} \leqslant C \|\tilde{\mathcal{N}}_n\|_{{}_0\tilde{K}^3_{L^n}(I,m_2-2,2)}. \tag{5.15}$$

Proof. The first component $\partial_\tau \tilde{u}_n^c(\kappa,\tau) - \tilde{\lambda}_{g,n}(\kappa)\tilde{u}_n^c(\kappa,\tau) = \tilde{\mathcal{N}}_n^c$ is independent of ϑ and thus can be solved by the variation of constant formula using (5.6) (with $b = 0$). For the second component we use resolvent estimates for the solution of

$$(\lambda - \tilde{\Lambda}_{g,n})\tilde{u}_n^s = \tilde{\mathcal{N}}_n^s.$$

There exists a $C > 0$ such that for $m_2 \geqslant 2$, $\tilde{\mathcal{N}}_n^s \in B^{m_1}_{L^n}(m_2 - 2, b)$ all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geqslant 0$ we have

$$\big\|\tilde{u}_n^s\big\|_{B^{m_1}_{L^n}(m_2,b)} + |\lambda|^{m_2/2}\|u\|_{B^{m_1}_{L^n}(0,b)} \leqslant C\big(\big\|\tilde{\mathcal{N}}_n^s\big\|_{B^{m_1}_{L^n}(m_2-2,b)} + |\lambda|^{(m_2-2)}\big\|\tilde{\mathcal{N}}_n^s\big\|_{B^{m_1}_{L^n}(0,b)}\big). \tag{5.16}$$

Similar to Lemma 5.1 this holds since $\tilde{\Lambda}_{g,n}$ is a relatively bounded perturbation of $L^{2n}(\partial_\vartheta + \mathrm{i}\kappa/L^n)^2$ and by construction has spectrum left of $-L^{2n}\gamma_0$. See also [19, Appendix A.2] for an explanation of how to obtain resolvent estimates in weighted spaces. From (5.16) we obtain (5.15) by continuation of $\tilde{\mathcal{N}}_n^s$ for $\tau \in \mathbb{R}$, Laplace transform, and the Paley–Wiener Theorem. In fact, in (5.16) we could choose λ to the right of $-L^{2n}\gamma_0$, but $\operatorname{Re}\lambda \geqslant 0$ is enough to show (5.15) with C independent of n. □

We denote the solution operator of (5.13) by ${}_0\mathcal{L}_n^{-1}$. To solve the nonlinear problem we write $\tilde{U}_n = \tilde{V}_n + \tilde{W}_n$ where $\tilde{V}_n \in \tilde{K}^3_{L^n}(\mathbb{R},3,2)$ is a continuation of $\tilde{U}_n|_{\tau=L^{-2}}$, which exists due to [10, Theorem 4.2.3]. Then $\tilde{W}_n$ fulfills

$$\mathcal{L}_n \tilde{W}_n = G_n(\tilde{W}_n), \quad \tilde{W}_n|_{\tau=L^{-2}} = 0, \quad \text{where } G_n(\tilde{W}_n) = \tilde{\mathcal{N}}_n(\tilde{V}_n + \tilde{W}_n) - \mathcal{L}_n \tilde{V}_n. \tag{5.17}$$

The idea is to show that for $\tilde{W}_n \in {}_0\tilde{K}^3_{L^n}(I,3,2)$ we have $\tilde{G}(\tilde{W}_n) \in {}_0\tilde{K}^3_{L^n}(I,1,2)$ and use Lemma 5.4 and estimates on the nonlinearity to apply the contraction mapping theorem to

$$\Phi(\tilde W) := {}_0\mathcal{L}_n^{-1} G_n(\tilde W_n). \tag{5.18}$$

We set

$$\rho_n := \big\| \big(\tilde u_n^c, \tilde u_n^s\big)\big|_{\tau=1} \big\|_{[B^3_{L^n}(2,2)]^2} \tag{5.19}$$

and obtain the following local existence result, taking into account that $(\tilde u_n^c, \tilde u_n^s)(1/L^2)$ and $(\tilde u_{n-1}^c, \tilde u_{n-1}^s)$ are related by $(\tilde u_n^c, \tilde u_n^s)|_{\tau=L^{-2}} = L\mathcal{R}_{1/L}(\tilde u_{n-1}^c, \tilde u_{n-1}^s)|_{\tau=1}$ and hence, by (4.2),

$$\big\| \big(\tilde u_n^c, \tilde u_n^s\big)\big|_{\tau=L^{-2}} \big\|_{B^3_{L^n}(2,2)} \leqslant CL^{7/2}\rho_{n-1}.$$

Lemma 5.5. *There exist $C_1, C_2 > 0$, independent of n, such that the following holds. If $\rho_{n-1} \leqslant C_1 L^{-7/2}$, then there exists a unique solution $(\tilde u_n^c, \tilde u_n^s) \in [\tilde K^3_{L^n}(I,3,2)]^2$ to (5.3) with*

$$\big\| \big(\tilde u_n^c, \tilde u_n^s\big) \big\|_{[\tilde K^3_{L^n}(I,3,2)]^2} \leqslant C_2 L^{7/2}\rho_{n-1}. \tag{5.20}$$

Moreover, for all $\tau_1 > L^{-2}$ and any $m_2 \in \mathbb{N}$ there exists a C_3, independent of n, such that

$$\big\| \big(\tilde u_n^c, \tilde u_n^s\big) \big\|_{[\tilde K^3_n((\tau_1,1),m_2,2)]^2} \leqslant C_3 L^{7/2}\rho_{n-1}. \tag{5.21}$$

Proof. From standard Sobolev embeddings we have that $\tilde{\mathcal{N}}_n$ is a smooth mapping from $\tilde K^3_{L^n}(I,3,2)$ to $\tilde K^3_{L^n}(I,1,2)$, see also Lemma 5.3. To show that $G_n(\tilde W_n)$ in (5.17) is in ${}_0\tilde K^3_{L^n}(I,1,2)$ we have to fulfill one compatibility condition, namely $G_n(\tilde W_n)|_{\tau=L^{-2}} = 0$, which holds by construction. For sufficiently small ρ_{n-1}, Φ is a contraction since $\tilde{\mathcal{N}}_n$ is quadratic and higher order. In particular, combining Lemma 5.4 with a slight adaption of (5.9) to the time-dependent case we find that C_1, C_2 may be chosen independent of n. The higher regularity follows by a standard bootstrapping argument: for almost all $\tau \in (L^{-2}, 1)$ we have $(\tilde u_n^c, \tilde u_n^s)(\tau) \in B^3_{L^n}(3,2)$. Starting again at such a τ the required compatibility conditions to apply Lemma 5.4 are automatically fulfilled. This yields (5.21). □

5.4. Proof of Theorem 3 (Diffusive stability)

Due to the loss of $L^{7/2}$ in Lemma 5.5 we need to improve (5.20) to iterate (5.4). Given a local solution $(\tilde u_n^c, \tilde u_n^s)$ with the higher regularity (5.21) this will be achieved by using the variation of constant formula and a suitable splitting of $\tilde u_n^c$.

For $\tilde u^c \in \hat H^3(2)$ we define $\Pi \tilde u^c = \partial_\kappa \tilde u^c(0)$, which, by Sobolev embedding, gives a continuous map, i.e. $|\Pi \tilde u_c| \leqslant C\|\tilde u^c\|_{\hat H^3(2)}$. Here, and also in (5.25) below, we need the smoothness in the Bloch wave number, which for $\tilde u^c$ we again identify with the Fourier wave number. To prove Theorem 3 we write

$$\tilde u_n^c(\kappa,1) = \mathrm{i}\psi_n g(\kappa) + r_n^c(\kappa), \qquad \tilde u_n^s(\kappa,\vartheta,1) = r_n^s(\vartheta,\kappa), \tag{5.22}$$

where $g(\kappa) = \kappa \mathrm{e}^{-\alpha\kappa^2}$ and $r_n^c(0) = \partial_\kappa r_n^c(0) = 0$. This makes sense since $\tilde u_n(0,\tau) = 0$ for all $n \in \mathbb{N}$ and all $\tau \in [1/L^2, 1]$ if $\tilde u^c(0,1) = 0$. Substituting (5.22) into (5.3) yields

$$\psi_n - \psi_{n-1} = \Pi I_n^c, \tag{5.23a}$$

$$r_n^c = \mathrm{e}^{(1-L^{-2})\tilde\lambda_{g,n}} L\mathcal{R}_{1/L} r_{n-1} + I_n^c + \mathrm{Res}_n, \tag{5.23b}$$

$$r_n^s = \mathrm{e}^{(1-L^{-2})\tilde\Lambda_g} L\mathcal{R}_{1/L} r_{n-1} + I_{n,0}^s + I_{n,1}^s + I_{n,2}^s, \tag{5.23c}$$

where, using the notation $L^{3n}\tilde{\mathcal{N}}_n^s = \tilde{\mathcal{N}}_{n,0}^s + \tilde{\mathcal{N}}_{n,1}^s + \tilde{\mathcal{N}}_{n,2}^s$ from Lemma 5.3,

$$I_n^c = L^{3n} \int\limits_{1/L^2}^{1} \mathrm{e}^{(1-\tau)\tilde{\lambda}_{g,n}} \tilde{\mathcal{N}}_n^c\big(\tilde{u}_n^c, \tilde{u}_n^s\big)(\tau)\,\mathrm{d}\tau, \qquad I_{n,j}^s = \int\limits_{1/L^2}^{1} \mathrm{e}^{(1-\tau)\tilde{\Lambda}_{g,n}} \tilde{\mathcal{N}}_{n,j}^s\big(\tilde{u}_n^c, \tilde{u}_n^s\big)(\tau)\,\mathrm{d}\tau,$$

and where the residual in (5.23b) is defined by $\mathrm{Res}_n = \mathrm{i}\psi_{n-1}\mathrm{e}^{(1-L^{-2})\tilde{\lambda}_{g,n}} L\mathcal{R}_{1/L} g - \mathrm{i}\psi_n g$. We also define

$$\rho_{n,c} = \big\| r_n^c \big\|_{B^3_{L^n}(2,2)} \quad \text{and} \quad \rho_{n,s} = \big\| r_n^s \big\|_{B^3_{L^n}(2,2)},$$

which gives, cf. (5.19), $\rho_n = \|\tilde{u}_n^c\|_{B^3_{L^n}(2,2)} + \|\tilde{u}_n^s\|_{B^3_{L^n}(2,2)} \leqslant C|\psi_n| + \rho_{n,c} + \rho_{n,s}$.

Now assume that $\rho_{n-1} \leqslant L^{-7/2}$. Then from (5.6), (5.8) we immediately obtain

$$|\psi_n - \psi_{n-1}| \leqslant CL^{-n}\big(C_2 L^{7/2}\rho_{n-1}\big)^2 \tag{5.24}$$

with C_2 from (5.20). Moreover, $\|\mathrm{e}^{(1-L^{-2})\tilde{\lambda}_{g,n}} L\mathcal{R}_{1/L} g - g\|_{B^3_{L^n}(2,2)} \leqslant CL^{-2n}$ and hence $\|\mathrm{Res}_n\|_{B^3_{L^n}(2,2)} \leqslant CL^{-2n}|\psi_{n-1}|$. Next, we have

$$\big\|\mathrm{e}^{(1-L^{-2})\tilde{\lambda}_{g,n}} L\mathcal{R}_{1/L} r_{n-1}^c\big\|_{B^3_{L^n}(2,2)} \leqslant CL^{-1}\big\|r_{n-1}^c\big\|_{B^3_{L^n}(2,2)}. \tag{5.25}$$

This follows from $r_{n-1}^c(\kappa/L) = (\frac{\kappa}{L})^2 \partial_\kappa^2 r_{n-1}^c(\tilde{\kappa})$ for some $\tilde{\kappa}$ between 0 and κ. Here again we need the smoothness in κ. Combining the above estimates we arrive at

$$\rho_{n,c} \leqslant CL^{-1}\rho_{n-1,c} + CL^{-n}\big(C_2 L^{7/2}\rho_{n-1}\big)^2 + CL^{-2n}\big|\psi_{n-1}\big|. \tag{5.26}$$

To estimate $\rho_{n,s}$ first note that

$$\big\|\mathrm{e}^{(1-L^{-2})\tilde{\Lambda}_g} L\mathcal{R}_{1/L} r_{n-1}^s\big\|_{B^3_{L^n}(2,2)} \leqslant C\mathrm{e}^{-\gamma_0 L^{2n}} L^{7/2}\rho_{n-1,s} \leqslant L^{-1}\rho_{n-1,s} \tag{5.27}$$

for L sufficiently large. Next, $I_{n,0}^s$ and $I_{n,1}^s$ can be estimated using (5.7) and (5.9) to

$$\big\|I_{n,0}^s\big\|_{B^3_{L^n}(2,2)} + \big\|I_{n,1}^s\big\|_{B^3_{L^n}(2,2)} \leqslant CL^{-n}\big(L^{7/2}\rho_{n-1}\big)^2. \tag{5.28}$$

However, for the quasi-linear part $I_{n,2}^s$ we have to use the higher regularity (5.21) and split $I_{n,2}^s = \int_{1/L^2}^{1/2} \cdots \mathrm{d}\tau + \int_{1/2}^{1} \cdots \mathrm{d}\tau$ to obtain

$$\begin{aligned} \big\|I_{n,2}^s\big\|_{B^3_{L^n}(2,2)} &\leqslant C\big(C_2 L^{7/2}\rho_{n-1}\big)^2 \int\limits_{1/L^2}^{1/2} (1-\tau)^{-1}\mathrm{e}^{-\gamma_0 L^{2n}(1-\tau)}\,\mathrm{d}\tau + C\big(C_3 L^{7/2}\rho_{n-1}\big)^2 \int\limits_{1/2}^{1} \mathrm{e}^{-\gamma_0 L^{2n}(1-\tau)}\,\mathrm{d}\tau \\ &\leqslant C\big(C_2^2 + C_3^2\big)L^{-n}\big(L^{7/2}\rho_{n-1}\big)^2. \end{aligned} \tag{5.29}$$

This yields

$$\rho_{n,s} \leqslant CL^{-1}\rho_{n-1} + CL^{-n}\big(L^{7/2}\rho_{n-1}\big)^2. \tag{5.30}$$

Now, let $L \geqslant L_0$ with L_0 so large that $CL^{-1} \leqslant L^{-(1-b)}$ and let $\rho_0 = \|(\tilde{u}^c, \tilde{u}^s)\|_{B^3(2,2)} \leqslant L^{-4}$. Then, combining (5.24), (5.26), (5.27) and (5.30), iteration shows that there exists a $\psi_{\lim} \in \mathbb{R}$ such that

$$|\psi_{\lim} - \psi_n| + \rho_{n,c} + \rho_{n,s} \leqslant L^{-n(1-b)} \quad \text{as } n \to \infty, \tag{5.31}$$

where the correction L^{nb} takes care of the powers C^n arising in the iteration. This discrete convergence implies Theorem 3 using $t = L^{2n}\tau$ and the local existence Lemma 5.5.

5.5. *Proof of Theorem 4(i) (Diffusive mixing, Gaussian case)*

The main difference compared to the proof of Theorem 3 are different scalings for $\tilde{u}_n^c$, $\tilde{u}_n^s$, which are now based on (4.4) instead of (4.6). Thus, we introduce

$$\tilde{u}_n^c(\kappa, \tau) = \mathcal{R}_{L^{-n}} \tilde{u}^c\big(\kappa, L^{2n}\tau\big), \qquad \tilde{u}_n^s(\vartheta, \kappa, \tau) = \mathcal{R}_{L^{-n}} \tilde{u}^s\big(\vartheta, \kappa, L^{2n}\tau\big). \tag{5.32}$$

We want to show that

$$\tilde{u}_n^c(\kappa, 1) \to \phi_d e^{-\alpha\kappa^2}, \qquad \tilde{u}_n^s(\kappa, 1) \to 0 \quad \text{as } n \to \infty. \tag{5.33}$$

We obtain

$$\partial_\tau \tilde{u}_n^c(\kappa, \tau) - \tilde{\lambda}_{g,n}(\kappa)\tilde{u}_n^c(\kappa, \tau) = L^{2n}\tilde{\mathcal{N}}_n^c\big(\tilde{u}_n^c, \tilde{u}_n^s\big)(\kappa, \tau), \tag{5.34a}$$

$$\partial_\tau \tilde{u}_n^s(\vartheta, \kappa, \tau) - \tilde{\Lambda}_{g,n}\tilde{u}_n^s(\vartheta, \kappa, \tau) = L^{2n}\tilde{\mathcal{N}}_n^s\big(\tilde{u}_n^c, \tilde{u}_n^s\big)(\vartheta, \kappa, \tau), \tag{5.34b}$$

where $\tilde{\lambda}_{g,n}(\kappa) = L^{2n}\tilde{\lambda}_g(\kappa/L^{2n})$ and $\tilde{\Lambda}_{g,n} = L^{2n}\mathcal{R}_{L^{-n}}\tilde{\Lambda}_g\mathcal{R}_{L^n}$ as before, and

$$\tilde{\mathcal{N}}_n^c\big(\tilde{u}_n^c, \tilde{u}_n^s\big) = \tilde{\eta}\big(\kappa/L^n\big)\mathcal{R}_{L^{-n}}\tilde{\mathcal{N}}\big(\mathcal{R}_{L^n}\tilde{u}_n^c, \mathcal{R}_{L^n}\tilde{u}_n^s\big),$$
$$\tilde{\mathcal{N}}_n^s\big(\tilde{u}_n^c, \tilde{u}_n^s\big) = \mathcal{R}_{L^{-n}}\tilde{P}_{\mathrm{mf}}^s\tilde{\mathcal{N}}\big(\mathcal{R}_{L^n}\tilde{u}_n^c, \mathcal{R}_{L^n}\tilde{u}_n^s\big).$$

The renormalization process reads

$$\text{solve (5.34) for } \tau \in I := \big[L^{-2}, 1\big] \text{ with initial data } \begin{pmatrix} \tilde{u}_n^c \\ \tilde{u}_n^s \end{pmatrix}\bigg|_{\tau=L^{-2}} = \mathcal{R}_{1/L}\begin{pmatrix} \tilde{u}_{n-1}^c \\ \tilde{u}_{n-1}^s \end{pmatrix}\bigg|_{\tau=1}. \tag{5.35}$$

The local existence for (5.34) works exactly as for (5.3). In contrast to Lemma 5.5 with $(\tilde{u}_n^c, \tilde{u}_n^s)(1/L^2) \in [B_{L^n}^3(2,2)]^2$ it suffices here to take $(\tilde{u}_n^c, \tilde{u}_n^s)(1/L^2) \in [B_{L^n}^2(2,2)]^2$ in order to extract the asymptotics (5.33), cf. (5.45). Thus, we set

$$\rho_n := \big\| \big(\tilde{u}_n^c, \tilde{u}_n^s\big)\big|_{\tau=1} \big\|_{[B_{L^n}^2(2,2)]^2}, \tag{5.36}$$

and for $\rho_{n-1} \leqslant C_1 L^{-5/2}$ we obtain a local solution $(\tilde{u}_n^c, \tilde{u}_n^s) \in [\tilde{K}_{L^n}^2(I,3,2)]^2$ to (5.34) with

$$\big\| \big(\tilde{u}_n^c, \tilde{u}_n^s\big) \big\|_{[\tilde{K}_{Ln}^2(I,3,2)]^2} \leqslant C_2 L^{5/2}\rho_{n-1}, \tag{5.37}$$

and for each $\tau_1 > L^{-2}$ and $m_2 \in \mathbb{N}$ there exists a C_3 such that we have the higher regularity

$$\big\| \big(\tilde{u}_n^c, \tilde{u}_n^s\big) \big\|_{[\tilde{K}_{L^n}^2((\tau_1,1),m_2,2)]^2} \leqslant C_3 L^{5/2}\rho_{n-1}. \tag{5.38}$$

The estimates for the linear semigroups work as before, i.e., here

$$\left\| e^{(\tau-\tau')\tilde{\lambda}_{g,n}} \tilde{u}_n^c \right\|_{B^2_{L^n}(2,2)} \leqslant C \max\left\{1, \left(\tau - \tau'\right)^{-b/2}\right\} \left\| \tilde{u}_n^c \right\|_{B^2_{L^n}(2,2-b)}, \tag{5.39}$$

$$\left\| e^{(\tau-\tau')\tilde{\Lambda}_{g,n}} \tilde{u}_n^s \right\|_{B^2_{L^n}(2,2)} \leqslant C \max\left\{1, \left(\tau - \tau'\right)^{-m_2/2}\right\} e^{-\gamma_0 L^{2n}(\tau-\tau')} \left\| \tilde{u}_n^s \right\|_{B^2_{L^n}(2-m_2,2)}. \tag{5.40}$$

The nonlinearities are now estimated as follows.

Lemma 5.6. *Let $\|(\tilde{u}_n^c, \tilde{u}_n^s)\|_{[B^2_{L^n}(2,2)]^2} \leqslant R_n \leqslant 1$. There exists a $C > 0$ such that*

$$L^{2n} \left\| \tilde{\mathcal{N}}_n^c\left(\tilde{u}_n^c, \tilde{u}_n^s\right) \right\|_{B^2_{L^n}(2,1)} \leqslant C L^{-n} R_n^2. \tag{5.41}$$

The term $L^{2n}\tilde{\mathcal{N}}_n^s$ can be split according to the number of ϑ derivatives as $L^{2n}\tilde{\mathcal{N}}_n^s = \tilde{\mathcal{N}}_{n,0}^s + \tilde{\mathcal{N}}_{n,1}^s + \tilde{\mathcal{N}}_{n,2}^s$ such that

$$\left\| \tilde{\mathcal{N}}_{n,i}^s \right\|_{B^2_{L^n}(2-i,2)} \leqslant C R_n^2. \tag{5.42}$$

Proof. Apart from the different power counting, the proof works like the one of Lemma 5.3, with the crucial difference that now the term s_1 from (5.10) vanishes since $\beta = 0$ by assumption. □

Similar to (5.22) we now set

$$\tilde{u}_n^c(\kappa, 1) = \phi_d g(\kappa) + r_n^c(\kappa), \qquad \tilde{u}_n^s(\kappa, \vartheta, 1) = r_n^s(\kappa, \vartheta), \tag{5.43}$$

where $g(\kappa) = e^{-\alpha\kappa^2}$ and $r_n^c(0) = 0$. Here no variables ϕ_n are necessary since, due to the conservation of total phase shift, i.e., $\partial_t \tilde{u}^c(0, t) = 0$ for all t. We obtain

$$r_n^c = e^{(1-L^{-2})\tilde{\lambda}_{g,n}} \mathcal{R}_{1/L} r_{n-1}^c + I_n^c + \mathrm{Res}_n, \tag{5.44a}$$

$$r_n^s = e^{(1-L^{-2})\tilde{\Lambda}_g} \mathcal{R}_{1/L} r_{n-1}^s + I_{n,0}^s + I_{n,1}^s + I_{n,1}^s, \tag{5.44b}$$

where $\mathrm{Res}_n = \phi_d(e^{(1-L^{-2})\tilde{\lambda}_{g,n}} \mathcal{R}_{1/L} g - g)$ and

$$I_n^c = L^{2n} \int_{1/L^2}^{1} e^{(1-\tau)\tilde{\lambda}_{g,n}} \tilde{\mathcal{N}}_n^c\left(\tilde{u}_n^c, \tilde{u}_n^s\right)(\tau)\, d\tau, \qquad I_{n,j}^s = \int_{1/L^2}^{1} e^{(1-\tau)\tilde{\Lambda}_{g,n}} \tilde{\mathcal{N}}_{n,j}^s\left(\tilde{u}_n^c, \tilde{u}_n^s\right)(\tau)\, d\tau,$$

with $L^{2n}\tilde{\mathcal{N}}_n^s = \tilde{\mathcal{N}}_{n,0}^s + \tilde{\mathcal{N}}_{n,1}^s + \tilde{\mathcal{N}}_{n,2}^s$ from Lemma 5.6. Clearly, $\|\mathrm{Res}_n\|_{B^2_{L^n}(2,2)} \leqslant C|\phi_d|L^{-2n}$, and

$$\left\| e^{(1-L^{-2})\tilde{\lambda}_{g,n}} \mathcal{R}_{1/L} r_{n-1}^c \right\|_{B^2_{L^n}(2,2)} \leqslant C L^{-1} \left\| r_{n-1}^c \right\|_{B^2_{L^n}(2,2)} \tag{5.45}$$

which follows by writing $r_{n-1}^c(\kappa/L) = (\frac{\kappa}{L})\partial_\kappa r_{n-1}^c(\tilde{\kappa})$ for some $\tilde{\kappa}$ between 0 and κ. Combining this with (5.39) and (5.41) thus yields

$$\rho_{n,c} \leqslant C L^{-1} \rho_{n-1,c} + C L^{-n}\left(C_2 L^{5/2} \rho_{n-1}\right)^2 + C|\phi_d| L^{-2n}, \tag{5.46}$$

and similarly

$$\rho_{n,s} \leqslant L^{-1}\rho_{n-1,s} + C\big(C_2^2 + C_3^2\big)L^{-n}\big(L^{5/2}\rho_{n-1}\big)^2. \tag{5.47}$$

The proof of Theorem 4(i) now follows by iteration. At this point the assumption $|\phi_d| \leqslant \varepsilon$ is crucial. This can be seen by computing $\rho_{n,c}$ in powers of L for $n = 0, 1, 2, \ldots$ starting with $\rho_{0,c} = 0$. Hence, we need $|\phi_d| \leqslant L^{-d}$ with d sufficiently large.

5.6. Proof of Theorem 4(ii) (Diffusive mixing, Burgers' case)

Essentially, Theorem 4(ii) is again based on the scaling (5.32). However, the crucial difference to the case $\beta = 0$ is that now the analog of (5.41) no longer holds. Therefore, we need to scale $\tilde{u}^s$ differently, i.e., we blow up $\tilde{u}^s_n$ in order to avoid problems with the quadratic terms involving $\tilde{u}^s$ in the critical part, in which the term $\mathrm{i}\ell\beta(\tilde{u}^c * \tilde{u}^c)$ will give the Burgers dynamics for $\tilde{u}^c$. Thus, for small $p > 0$ we introduce

$$\tilde{u}^c_n(\kappa,\tau) = \mathcal{R}_{L^{-n}}\tilde{u}^c\big(\kappa, L^{2n}\tau\big), \qquad \tilde{u}^s_n(\vartheta,\kappa,\tau) = L^{n(1-p)}\mathcal{R}_{L^{-n}}\tilde{u}^s\big(\vartheta,\kappa,L^{2n}\tau\big), \tag{5.48}$$

to obtain

$$\partial_\tau \tilde{u}^c_n(\kappa,\tau) - \tilde{\lambda}_{g,n}(\kappa)\tilde{u}^c_n(\kappa,\tau) = L^{2n}\tilde{\mathcal{N}}^c_n\big(\tilde{u}^c_n,\tilde{u}^s_n\big)(\kappa,\tau), \tag{5.49a}$$

$$\partial_\tau \tilde{u}^s_n(\vartheta,\kappa,\tau) - \tilde{\Lambda}_{g,n}\tilde{u}^s_n(\vartheta,\kappa,\tau) = L^{n(3-p)}\tilde{\mathcal{N}}^s_n\big(\tilde{u}^c_n,\tilde{u}^s_n\big)(\vartheta,\kappa,\tau), \tag{5.49b}$$

where again $\tilde{\lambda}_{g,n}(\kappa) = L^{2n}\tilde{\lambda}_g(\kappa/L^n)$ and $\tilde{\Lambda}_{g,n} = L^{2n}\mathcal{R}_{L^{-n}}\tilde{\Lambda}_g\mathcal{R}_{L^n}$, but now

$$\tilde{\mathcal{N}}^c_n\big(\tilde{u}^c_n,\tilde{u}^s_n\big) = \tilde{\eta}\big(\kappa/L^n\big)\mathcal{R}_{L^{-n}}\tilde{\mathcal{N}}\big(\mathcal{R}_{L^n}\tilde{u}^c_n, L^{-n(1-p)}\mathcal{R}_{L^n}\tilde{u}^s_n\big),$$
$$\tilde{\mathcal{N}}^s_n\big(\tilde{u}^c_n,\tilde{u}^s_n\big) = \mathcal{R}_{L^{-n}}\tilde{P}^s_{\mathrm{mf}}\tilde{\mathcal{N}}\big(\mathcal{R}_{L^n}\tilde{u}^c_n, L^{-n(1-p)}\mathcal{R}_{L^n}\tilde{u}^s_n\big).$$

Accordingly, the renormalization process reads

$$\text{solve (5.49) on } \tau \in I := \big[L^{-2},1\big] \text{ with initial data } \begin{pmatrix}\tilde{u}^c_n\\ \tilde{u}^s_n\end{pmatrix}\bigg|_{\tau=L^{-2}} = \mathcal{R}_{1/L}\begin{pmatrix}\tilde{u}^c_{n-1}\\ L^{1-p}\tilde{u}^s_{n-1}\end{pmatrix}\bigg|_{\tau=1}. \tag{5.50}$$

The estimates for the linear semigroups are again (5.39) and (5.40), while the nonlinear terms are estimated as follows.

Lemma 5.7. *Let $\|(\tilde{u}^c_n,\tilde{u}^s_n)\|_{[B^2_{L^n}(2,2)]^2} \leqslant R_n \leqslant 1$. There exists a $C > 0$ such that $L^{2n}\tilde{\mathcal{N}}^c_n(\tilde{u}^c_n,\tilde{u}^s_n) = s_1 + s_2 + s_3 + s_4$ with $s_1 = \mathrm{i}\beta\kappa(\tilde{u}^c_n * \tilde{u}^c_n)$ and*

$$\|s_2\|_{B^2_{L^n}(2,1)} \leqslant CL^{-n}\big\|\tilde{u}^c_n\big\|^2_{B^2_{L^n}(2,2)},$$
$$\|s_3\|_{B^2_{L^n}(2,1)} \leqslant CL^{-n(1-p)}\big\|\tilde{u}^c_n\big\|_{B^2_{L^n}(2,2)}\big\|\tilde{u}^s_n\big\|_{B^2_{L^n}(2,2)},$$
$$\|s_4\|_{B^2_{L^n}(2,1)} \leqslant CL^{-2n(1-p)}R_n^2. \tag{5.51}$$

The term $L^{n(3-p)}\tilde{\mathcal{N}}^s_n$ can be split according to the number of ϑ derivatives as $L^{n(3-p)}\tilde{\mathcal{N}}^s_n = \tilde{\mathcal{N}}^s_{n,0} + \tilde{\mathcal{N}}^s_{n,1} + \tilde{\mathcal{N}}^s_{n,2}$ such that

$$\big\|\tilde{\mathcal{N}}^s_{n,i}\big\|_{B^2_{L^n}(2-i,2)} \leqslant C\big(L^{n(1-p)}\big\|\tilde{u}^c_n\big\|^2_{B^2_{L^n}(2,2)} + R_n^2\big). \tag{5.52}$$

Proof. The term s_2 contains the quadratic terms in $\tilde{u}_n^c$ except for $\mathrm{i}\beta\kappa(\tilde{u}_n^c * \tilde{u}_n^c)$, i.e., s_2 is of the form $s_2(\kappa,t) = h(\kappa/L^n)(\tilde{u}_n^c * \tilde{u}_n^c)$ with $h(\kappa) = \mathcal{O}(\kappa^2)$. s_3 contains the quadratic interaction of $\tilde{u}_n^c$ and $\tilde{u}_n^s$, and s_4 contains the remaining terms. Then (5.51) follows from the finite support of $\tilde{\mathcal{N}}_n^c$ in Fourier space. It is in s_3, s_4 that the blowup scaling $\tilde{u}_n^s(\vartheta,\kappa,\tau) = L^{n(1-p)}\mathcal{R}_{L^{-n}}\tilde{u}^s(\vartheta,\kappa,L^{2n}\tau)$ is useful. Eq. (5.52) again follows by straightforward power counting. □

The terms involving only $\tilde{u}_n^c$ in (5.52) blow up as $n \to \infty$. However, combining (5.52) with the exponential damping in the stable part, cf. (5.40), we still get local existence for (5.49) with constants independent of n. We let

$$\rho_n := \left\| \left(\tilde{u}_n^c, \tilde{u}_n^s\right)\big|_{\tau=1} \right\|_{[B^2_{L^n}(2,2)]^2}, \tag{5.53}$$

and for $\rho_{n-1} \leqslant C_1 L^{-5/2}$ obtain a local solution $(\tilde{u}_n^c, \tilde{u}_n^s) \in [\tilde{K}_n^2(I,3,2)]^2$ to (5.49) with $\|(\tilde{u}_n^c, \tilde{u}_n^s)\|_{[\tilde{K}^2_{L^n}(I,3,2)]^2} \leqslant C_2 L^{5/2}\rho_{n-1}$, as in (5.37), which moreover enjoys the higher regularity $\|(\tilde{u}_n^c, \tilde{u}_n^s)\|_{[\tilde{K}^2_{L^n}((\tau_1,1),m_2,2)]^2} \leqslant C_3 L^{5/2}\rho_{n-1}$, cf. (5.38).

Similar to (5.22) and (5.43) we now separate from $\tilde{u}_n^c$ the lowest-order asymptotics, now obtained from the Burgers equation. However, due to the contribution $s_1 = \mathrm{i}\beta\kappa(\tilde{u}_n^c * \tilde{u}_n^c)$ of the nonlinearity to the asymptotics here we work out an intermediate step and split $\tilde{u}_n^c$ in a τ-dependent way. In detail, let

$$\tilde{u}_n^c(\kappa,\tau) = \tilde{u}_{n,*}^c(\kappa,\tau) + \tilde{\alpha}_n(\kappa,\tau) \tag{5.54}$$

where

$$\tilde{u}_{n,*}^c(\kappa,\tau) = \chi\left(\kappa/L^n\right)\tilde{u}_*^c(\kappa,\tau) \quad \text{with } \tilde{u}_*^c(\ell,t) = \mathcal{F}\left(\frac{\sqrt{\alpha}}{\beta}\frac{z\mathrm{e}^{-\vartheta^2/(k^2\alpha t)}}{1 + z\,\mathrm{erf}(\vartheta/\sqrt{k\alpha t})}\right)(\ell),$$

cf. (4.11), and χ from (3.17). Consequently $\tilde{\alpha}_n(0,\tau) = 0$ for all n, τ due to the conservation of total phase. Then

$$\partial_\tau \tilde{\alpha}_n^c(\kappa,\tau) - \tilde{\lambda}_{g,n}(\kappa)\tilde{\alpha}_n^c(\kappa,\tau) = L^{2n}\left(\tilde{\mathcal{N}}_n^c\left(\tilde{u}_n^c, \tilde{u}_n^s\right)(\kappa,\tau) - \tilde{\mathcal{N}}_n^c\left(\tilde{u}_{n,*}^c, 0\right)\right) + \mathrm{Res}_n, \tag{5.55a}$$

$$\partial_\tau \tilde{u}_n^s(\vartheta,\kappa,\tau) - \tilde{\Lambda}_{g,n}\tilde{u}_n^s(\vartheta,\kappa,\tau) = L^{n(3-p)}\tilde{\mathcal{N}}_n^s\left(\tilde{u}_n^c, \tilde{u}_n^s\right)(\vartheta,\kappa,\tau), \tag{5.55b}$$

where

$$\mathrm{Res}_n = -\partial_\tau \tilde{u}_{n,*}^c + \tilde{\lambda}_{g,n}(\kappa)\tilde{u}_{n,*}^c + L^{2n}\tilde{\mathcal{N}}_n^c\left(\tilde{u}_{n,*}^c, 0\right).$$

Lemma 5.8. *There exists a* $C > 0$ *such that* $\sup_{L^{-2}\leqslant\tau\leqslant 1} \|\mathrm{Res}_n(\tau)\|_{B^2_{L^n}(2,2)} \leqslant CL^{-n}|\phi_d|$.

Proof. By construction, i.e., since $\tilde{u}_{n,*}^c$ is an exact solution of the Burgers equation,

$$\mathrm{Res}_n(\kappa,\tau) = CL^{-n}\left(\mathcal{O}\left(\kappa^3\right)\tilde{u}_{n,*}^c + \mathcal{O}\left(\kappa^2\left(\tilde{u}_{n,*}^c * \tilde{u}_{n,*}^c\right)\right)\right)$$

which can be estimated in $B^2_{L^n}(2,2)$ by $L^{-n}|\phi_d|$ since $\tilde{u}_*^c(\kappa,\tau)$ is analytic and exponentially decaying. □

Now setting

$$\tilde{u}_n^c(\kappa, 1) = \tilde{u}_{n,*}^c(\kappa, 1) + r_n^c(\kappa), \qquad \tilde{u}_n^s(\kappa, \vartheta, 1) = r_n^s(\kappa, \vartheta), \tag{5.56}$$

the remainder of the proof of Theorem 4(ii) works as the proof of Theorem 4(i) in Section 5.5.

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