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On a conjecture about spectral sets

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Abstract

Let R be a commutative ring with identity. We denote by $\operatorname{Spec}(R)$ the set of prime ideals of R. Call a partial ordered set *spectral* if it is order isomorphic to $(\operatorname{Spec}(R), \subseteq)$ for some R. A longstanding open question about spectral sets (since 1976), is that of Lewis and Ohm [Canad. J. Math. 28 (1976) 820, Question 3.4]: "If $(X, \leqslant)$ is an ordered disjoint union of the posets $(X_\lambda, \leqslant_\lambda)$, $\lambda \in \Lambda$, and if $(X, \leqslant)$ is spectral, then are the $(X_\lambda, \leqslant_\lambda)$ also spectral?".

Let $(X, \leqslant)$ be a poset and $x \in X$. Recall that the *D-component* of x is defined to be the intersection of all subsets of X containing x that are closed under specialization and generization (i.e., under $\leqslant$ and $\geqslant$). Let $(X, \leqslant)$ be a spectral set which is an ordered disjoint union of the posets $(X_\lambda, \leqslant_\lambda)$, $\lambda \in \Lambda$. It is clear that $(X_\lambda, \leqslant_\lambda)$ is a disjoint union of D-components of X. Thus the conjecture of Lewis and Ohm is equivalent to the following question: "Is a D-component of a spectral set spectral?"

This paper deals with topological properties of a D-component of a spectral set, improving the understanding of the conjecture of Lewis and Ohm. The concepts of up-spectral topology and down-spectral topology are introduced and studied.

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Introduction

Let Spec(R) denote the set of prime ideals of a commutative ring R with identity, ordered by inclusion, and call a partial ordered set *spectral* if it is order isomorphic to Spec(R) for some R. The prime spectrum of a general commutative unitary ring, endowed with the Zariski topology, is a well established tool in algebraic geometry. Note also that spectral sets are posets which arise as the prime ideals of distributive lattices with 0 and 1, ordered by inclusion. In the lattice theory literature spectral sets have usually been called *representable posets*. Such spectral sets are of interest not only in (topological) ring and lattice theory, but also in computer science, in particular, in domain theory.

In order that an ordered set $(X, \leqslant)$ be spectral it is necessary (but not sufficient [10]) that it satisfies two conditions:

(K_1): Each nonempty totally ordered subset of $(X, \leqslant)$ has a supremum and an infimum (that is, X is *up-complete* and *down-complete*).

(K_2): For each $a < b$ in X, there exist two adjacent elements $a_1 < b_1$, such that $a \leqslant a_1 < b_1 \leqslant b$ (that is, X is *weakly atomic*).

These properties were noted, for a ring spectrum, by I. Kaplansky (see [7, Theorems 9 and 11]), and then are called respectively the *first condition* and the *second condition* of Kaplansky.

Note that Lewis and Ohm have added another necessary condition (H): for $(X, \leqslant)$ to be spectral;

Let $\mathcal{S} = \{S(x) \mid x \in X\}$, $\mathcal{G} = \{G(x) \mid x \in X\}$, where $S(x) = \{y \in X \mid y \geqslant x\}$ and $G(x) = \{y \in X \mid y \leqslant x\}$.

(H): If $\mathcal{F}$ is a collection of subsets of X such that $\mathcal{F} \subseteq \mathcal{S}$ or $\mathcal{F} \subseteq \mathcal{G}$, then $\bigcap_{F \in \mathcal{F}} F = \emptyset$ implies that there is a finite collection of sets from $\mathcal{F}$ whose intersection is empty.

Following [10], there exists a partially ordered set which satisfies the conditions (K_1), (K_2), and (H) but is not spectral, thus showing that these conditions are not sufficient for a partially ordered set to be spectral.

Recall that if R is a ring, the *Zariski topology* or the *hull-kernel topology* for Spec(R) is defined by letting $C \subseteq \mathrm{Spec}(R)$ be closed if and only if there exists an ideal $\mathfrak{a}$ of R such that

$$C = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{a} \subseteq \mathfrak{p}\}.$$

Although many results about spectral sets have been obtained by Dobbs, Hochster, Fontana, Lewis, Ohm, and others (see [1,2,4–10]), a complete algebraic characterization of spectral sets still seems very far off.

On the other hand, the corresponding topological question of characterizing *spectral spaces* (that is, topological spaces homeomorphic to the prime spectrum of a ring equipped with the Zariski topology) was completely answered by Hochster in [6]. Recall that a closed subset C of a topological space X has a *generic point* if there is some $x \in C$ such that $\overline{\{x\}} = C$. A topological space in which every nonempty irreducible closed subset has a generic point is called a *sober space*. A topology $\mathcal{T}$ on a set X is spectral if and only if the following axioms hold:

(i) *X is a T_0-space.*
(ii) *X is quasi-compact and has a basis of quasi-compact open subsets.*
(iii) *The family of quasi-compact open subsets of X is closed under finite intersections.*
(iv) *Every nonempty irreducible closed subset has a generic point.*

Let R be a ring and $\mathcal{T}$ the Zariski topology on $\mathrm{Spec}(R)$. Then for each $\mathfrak{p} \in \mathrm{Spec}(R)$, we have

$$\overline{\{\mathfrak{p}\}} = S(\mathfrak{p}) = \big\{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mathfrak{p} \subseteq \mathfrak{q}\big\}.$$

According to [10], if $(X, \mathcal{T})$ is a topological space and $\leqslant$ is a partial ordering on X, the topology $\mathcal{T}$ is said to be *compatible with* $\leqslant$, if $\overline{\{x\}} = S(x)$, for each $x \in X$.

Note that $\mathcal{T}$ is compatible with $\leqslant$ if and only if the following conditions hold:

(1) *$S(x)$ is closed for all $x \in X$, and*
(2) *closed sets of $(X, \mathcal{T})$ are closed under specialization.*

It is clear that any order compatible topology is a T_0-topology. Conversely, if $(X, \mathcal{T})$ is a T_0-space, then $\mathcal{T}$ is compatible with the ordering $\leqslant$ defined by: $x \leqslant y$ if and only if $y \in \overline{\{x\}}$. One can obviously see that $(X, \leqslant)$ is spectral if and only if there exists an order compatible spectral topology on X. Thus, the concept of spectral sets may be considered as a merely topological notion.

Now, let us consider the two natural examples of posets $(\mathbb{N}, \leqslant)$ and $(\mathbb{N}, \geqslant)$. Set $\widetilde{\mathbb{N}} = \mathbb{N} \cup \{\omega\}$ and equip $\widetilde{\mathbb{N}}$ with the ordering $\leqslant^{\omega}$ (respectively $\geqslant_{\omega}$) extending $\leqslant$ (respectively $\geqslant$) on $\mathbb{N}$ and making ω the greatest (respectively smallest) element of $(\widetilde{\mathbb{N}}, \leqslant^{\omega})$ (respectively $(\widetilde{\mathbb{N}}, \geqslant_{\omega})$. Then $(\widetilde{\mathbb{N}}, \leqslant^{\omega})$ and $(\widetilde{\mathbb{N}}, \geqslant_{\omega})$ are spectral sets [9, Theorem 3.1].

The two previous examples motivate us to introduce two notions linked to spectral sets. These notions are important in obtaining information concerning a longstanding open question about spectral sets (since 1976) stated by Lewis and Ohm [10, Question 3.4]: if a poset $(X, \leqslant)$ is the disjoint union of posets $\{(X_\lambda, \leqslant_\lambda),\ \lambda \in \Lambda\}$, we shall say that X is the *ordered disjoint union* of the X_λ's if $x \leqslant y$ if and only if there is an λ such that $x, y \in X_\lambda$ and $x \leqslant_\lambda y$.

Theorem 3.1 of [10] says that if $(X_\lambda, \leqslant_\lambda)$ is spectral for each $\lambda \in \Lambda$, then $(X, \leqslant)$ is spectral. However, Lewis and Ohm have not been able to establish the converse of the previous theorem and they raise the following question:

> "If $(X, \leqslant)$ is the ordered disjoint union of posets $(X_\lambda, \leqslant_\lambda)$, $\lambda \in \Lambda$, and if $(X, \leqslant)$ is spectral, then are the $(X_\lambda, \leqslant_\lambda)$ also spectral?".

Let us introduce the two new concepts.

Definition 0.1. Let $(X, \leqslant)$ be a poset, $\omega \notin X$ and $\widetilde{X} = X \cup \{\omega\}$. Equip $\widetilde{X}$ with the partial order $\leqslant^{\omega}$ (respectively $\leqslant_{\omega}$) extending $\leqslant$ on X and making ω the greatest (respectively smallest) element of $(\widetilde{X}, \leqslant^{\omega})$ (respectively $(\widetilde{X}, \leqslant_{\omega})$). The poset $(X, \leqslant)$ is said to be *up-spectral* (respectively *down-spectral*) if $(\widetilde{X}, \leqslant^{\omega})$ (respectively $(\widetilde{X}, \leqslant_{\omega})$) is a spectral set.

Naturally, one can wonder what the topological analog is of each of the notions subsequently introduced. We propose the following definitions.

Definition 0.2. Let X be a topological space.

(1) X is said to be an *up-spectral* space if it satisfies the axioms of a spectral space with the exception that X is not necessarily quasi-compact.
(2) X is said to be a *down-spectral* space if it satisfies the axioms of a spectral space with the exception that X does not necessarily have a generic point when it is irreducible.

It will be shown that a poset $(X, \leqslant)$ is a down-spectral (respectively up-spectral) set if and only if there exists an order compatible down-spectral (respectively up-spectral) topology on $(X, \leqslant)$.

We finish this introduction by stating the main results of this paper.

Result A. *Any disjoint union of down-spectral* (*respectively up-spectral*) *sets is a down-spectral* (*respectively up-spectral*) *set.*

Result B [Proposition 4.1]. *Suppose that* $(X, \mathcal{T})$ *is a disjoint union of topological spaces* $(X_\lambda, \mathcal{T}_\lambda)$ $(\lambda \in \Lambda)$. *Then* X *is up-spectral if and only if each* X_λ *is up-spectral.*

Let $(X, \leqslant)$ be a poset and $x \in X$. Recall that the *D-component* of x is defined to be the intersection of all subsets of X containing x that are closed under specialization and generization.

Let $(X, \leqslant)$ be a spectral set which is an ordered disjoint union of the posets $(X_\lambda, \leqslant_\lambda)$, $\lambda \in \Lambda$. It is clear that $(X_\lambda, \leqslant_\lambda)$ is a disjoint union of D-components of X. Thus the conjecture of Lewis and Ohm is equivalent to the following question: "Is a D-component of a spectral set spectral?"

The aim of this paper is to give some topological properties of a D-component of a spectral set, improving the understanding of the question raised in 1976 about disjoint order unions of posets.

Let $(X, \leqslant)$ be a spectral set, $\mathcal{D}$ a D-component of X and $\mathcal{T}$ an order compatible spectral topology on X.

Lazard has proved in [8], that there exists a family of subsets $(E^n,\ n \in \mathbb{N})$ of X such that:

(i) $\mathcal{D} = \bigcup_{n\in\mathbb{N}} E^n$ (the Lazard's formula for a D-component).
(ii) $E^n \subseteq E^{n+1}$.
(iii) For each $n \in \mathbb{N}$, E^n is a closed subset of X.

We define on $\mathcal{D}$ two topologies $\mathcal{T}_1(\mathcal{D})$ and $\mathcal{T}_2(\mathcal{D})$ by

$(\mathcal{D}, \mathcal{T}_1(\mathcal{D}))$ is the disjoint union of the subspaces E^0, $E^n \setminus E^{n-1}$, $n \in \mathbb{N}\setminus\{0\}$.

Since E^n is a closed subspace of $(X, \mathcal{T})$, it is spectral. Let $\mathcal{T}_n$ be the topology induced by $\mathcal{T}$ on E^n. We define the topology $\mathcal{T}_2(\mathcal{D})$ on $\mathcal{D}$ by letting the closed sets be the F with the property that there exists some $n \in \mathbb{N}$, such that F is a closed subset of $(E^n, \mathcal{T}_n)$. Observe that a set O is open in $(\mathcal{D}, \mathcal{T}_2(\mathcal{D}))$ if and only if it is of the form $O = U \cup (\mathcal{D}\backslash E^k)$, where k is a sufficiently large integer so that $\mathcal{D}\backslash E^k \subseteq O$, and U is open in $(E^k, \mathcal{T}_k)$.

Under the previous notation we have the two following topological results, improving the understanding of the conjecture of Lewis and Ohm.

Result C [Theorem 5.1]. *Let $(X, \leqslant)$ be a spectral set, $\mathcal{D}$ be a D-component of X and $\mathcal{T}$ an order compatible spectral topology on X. Then $(\mathcal{D}, \mathcal{T}_1(\mathcal{D}))$ is an up-spectral topology.*

Result D [Theorem 5.5]. *Let $(X, \leqslant)$ be a spectral set, $\mathcal{D}$ be a D-component of X and $\mathcal{T}$ an order compatible spectral topology on X. Then the space $(\mathcal{D}, \mathcal{T}_2(\mathcal{D}))$ satisfies the following properties*:

(1) *$\mathcal{T}_2(\mathcal{D})$ is compatible with the order induced by $\leqslant$ on $\mathcal{D}$.*
(2) *$\mathcal{T}_2(\mathcal{D})$ is quasi-compact.*
(3) *Each nonempty irreducible closed subset of $(\mathcal{D}, \mathcal{T}_2(\mathcal{D}))$ distinct from $\mathcal{D}$ has a generic point.*

1. Down-spectral topology

We start by describing how to construct down-spectral spaces.

Theorem 1.1. *Let $(X, \mathcal{T})$ be a topological space. Then the following statements are equivalent*:

(i) *$(X, \mathcal{T})$ is a down-spectral space.*
(ii) *There exists a spectral space Y with minimum element ω, such that X is equal to the subspace $Y\backslash\{\omega\}$.*

Proof. (i) $\Rightarrow$ (ii). Let $\omega \notin X$ and $\widetilde{X} = X \cup \{\omega\}$. When $(X, \mathcal{T})$ is reducible (respectively irreducible), we consider the topology $\widetilde{\mathcal{T}}$ on $\widetilde{X}$ such that the closed subsets are $\widetilde{X}$ and the closed subsets of $(X, \mathcal{T})$ (respectively $\widetilde{X}$ and the closed subsets of $(X, \mathcal{T})$ distinct from X).

Clearly, $\overline{\{\omega\}} = \widetilde{X}$, and $(\widetilde{X}, \widetilde{\mathcal{T}})$ is a quasi-compact T_0-space. Thus, ω is the least element of $(\widetilde{X}, \leqslant)$, where $\leqslant$ is the ordering induced by the topology $\widetilde{\mathcal{T}}$ on $\widetilde{X}$.

If $U \subseteq X$ is open, then $U \cup \{\omega\}$ is open in $Y = \widetilde{X}$.

Let O be a nonempty open subset of $(\widetilde{X}, \widetilde{\mathcal{T}})$, then the following properties hold:

(1) There exists an open subset U of $(X, \mathcal{T})$ such that $O = U \cup \{\omega\}$.
(2) $O = U \cup \{\omega\}$ is quasi-compact in $(\widetilde{X}, \widetilde{\mathcal{T}})$ if and only if U is quasi-compact in $(X, \mathcal{T})$.

Hence $(\widetilde{X}, \widetilde{\mathcal{T}})$ has a basis of quasi-compact open subsets which is stable under finite intersections.

Let C be an irreducible closed subset of $(\widetilde{X}, \widetilde{\mathcal{T}})$.

- If $\omega \in C$, then $C = \overline{\{\omega\}} = \widetilde{X}$.
- If $\omega \notin C$, then C is an irreducible closed subset of $(X, \mathcal{T})$. According to the definition of the topology $\widetilde{\mathcal{T}}$, C is necessarily distinct from X. Thus, C has a generic point.

Therefore, $Y = \widetilde{X}$ is a spectral space with minimum element ω and X is equal to the subspace $Y \backslash \{\omega\}$.

(ii) $\Rightarrow$ (i). Let Y be a spectral space with minimum element ω and X is equal to the subspace $Y \backslash \{\omega\}$. Let us prove that X is a down-spectral space.

Let $(U_i,\ i \in I)$ be an open covering of X (with $U_i \neq \emptyset$). Then $(U_i \cup \{\omega\},\ i \in I)$ is an open covering of Y. Since Y is quasi-compact, there exists a finite sub-covering of $(U_i,\ i \in I)$ for X. Thus, X is quasi-compact.

Let $\mathcal{B}$ be a basis of quasi-compact open subsets of Y which is stable under finite intersections. Then $\mathcal{B}' = \{U \backslash \{\omega\} \mid U \in \mathcal{B}\}$ is a basis of quasi-compact open subsets of X which is stable under finite intersections.

Let C be an irreducible closed subset of X such that $C \neq X$. Then C is also an irreducible closed subset of Y. Hence C has a generic point; there exists $x \in C$ such that $C = \overline{\{x\}}^Y = \overline{\{x\}}^X$.

Therefore, X is a down-spectral space. □

Let A be a subset of a topological X. We say that A is *closed under specialization* if $\overline{\{x\}} \subseteq A$, for each $x \in A$.

The next result gives another tool for building down-spectral spaces.

Proposition 1.2. *Let $(X, \mathcal{T})$ be a spectral space and A a subset of X such that $X \backslash A$ is quasi-compact and A is closed under specialization. Let $\mathcal{C}$ be the topology on A, where the closed subsets are A and the closed subsets of $(X,\ \mathcal{T})$ contained in A. Then $\mathcal{C}$ is a down-spectral topology on A.*

Proof. Note that $V \subseteq A$ is open in $\mathcal{C}$ if and only if $V = O \cap A$, for some open set O of X satisfying $X \backslash A \subseteq O$.

Of course, $\overline{\{x\}}^{\mathcal{C}} = \overline{\{x\}}^X$. Thus, $(A,\ \mathcal{C})$ is a T_0-space.

Clearly $(A,\ \mathcal{C})$ is quasi-compact and every irreducible closed subset of $(A, \mathcal{C})$ distinct from A has a generic point.

Let O be a quasi-compact open subset of $(X,\ \mathcal{T})$ such that $X \backslash A \subseteq O$. We prove that $O \cap A$ is quasi-compact in $(A,\ \mathcal{C})$. To do so, let $(O_i,\ i \in I)$ be a family of open subsets of $(X, \mathcal{T})$ such that for each $i \in I$, $X \backslash A \subseteq O_i$ and $(O_i \cap A,\ i \in I)$ is an open covering of $O \cap A$ in $(A,\ \mathcal{C})$. Hence $O \cap A = \bigcup_{i \in I}(O_i \cap A) = (\bigcup_{i \in I} O_i) \cap A$. This forces $O = \bigcup_{i \in I} O_i$. Since O is quasi-compact, there exist finitely many elements $i_1, i_2, \ldots, i_n \in I$ such that $O = O_{i_1} \cup O_{i_2} \cup \cdots \cup O_{i_n}$. Thus, $O \cap A = (O_{i_1} \cap A) \cup (O_{i_2} \cap A) \cup \cdots \cup (O_{i_n} \cap A)$. Therefore, $O \cap A$ is quasi-compact in $(A,\ \mathcal{C})$.

Let U be a quasi-compact open subset of $(A,\ \mathcal{C})$. We may write $U = O \cap A$, where O is an open subset of $(X,\ \mathcal{T})$ such that $X \backslash A \subseteq O$. We claim that O is quasi-compact. Indeed, let $(O_i,\ i \in I)$ be an open covering of O, then $X \backslash A \subseteq \bigcup_{i \in I} O_i$. Thus there exists a finite subset $J \subseteq I$, such that $X \backslash A \subseteq \bigcup_{j \in J} O_j$. Let $V_i = O_i \cup (\bigcup_{j \in J} O_j)$, then $O = \bigcup_{i \in I} V_i$, and $X \backslash A \subseteq V_i$ for each $i \in I$. Thus,

$$U = O \cap A = \bigcup_{i \in I} (V_i \cap A).$$

Hence, there exists a finite subset K of I such that

$$U = \bigcup_{k \in K} (V_k \cap A) = \left(\bigcup_{k \in K} V_k \right) \cap A.$$

Hence $O \cap A = (\bigcup_{k \in K} V_k) \cap A$; and thus $O = \bigcup_{k \in K} V_k = \bigcup_{i \in K \cup J} O_i$. It follows that O is quasi-compact.

Therefore the quasi-compact open subsets of $(A,\ \mathcal{C})$ are the intersections of A with the quasi-compact open subsets of $(X, \mathcal{T})$ containing $X \backslash A$.

Now, it is easily seen that the following properties hold:

- $(A,\ \mathcal{C})$ has a basis of quasi-compact open subsets.
- The collection of quasi-compact open subsets of $(A,\ \mathcal{C})$ is closed under finite intersections.

This proves that $(A, \mathcal{C})$ is a down-spectral space. □

The next result is in the spirit of an analogous one about spectral sets.

Proposition 1.3. *Let $(X, \leqslant)$ be an ordered set. Then $(X, \leqslant)$ is a down-spectral set if and only if there exists an order compatible down-spectral topology on X.*

Proof. *Necessary condition*. Let $\omega \notin X$ and $\widetilde{X} = X \cup \{\omega\}$, where ω is the least element of $\widetilde{X}$. Then $(\widetilde{X},\ \leqslant_\omega)$ is a spectral set. Hence there exists an order compatible spectral topology $\widetilde{\mathcal{T}}$ on $\widetilde{X}$. Let $\mathcal{T}$ be the topology on X, where the closed subsets are X and the closed subsets of $(\widetilde{X}, \widetilde{\mathcal{T}})$ which are contained in X. It is clear, in view of Proposition 1.2, that $\mathcal{T}$ is an order compatible down-spectral topology on X.

Sufficient condition. Let $\mathcal{T}$ be an order compatible down-spectral topology on $(X, \leqslant)$. According to Theorem 1.1, there exists an order spectral topology on $(\widetilde{X} = X \cup \{\omega\}, \leqslant_\omega)$. Therefore, $(\widetilde{X}, \leqslant_\omega)$ is a spectral set, completing the proof. □

Corollary 1.4. *Every spectral set is down-spectral.*

The following result gives an important tool to construct down-spectral sets from spectral sets.

Corollary 1.5. *Let $(X, \leqslant)$ be a spectral set and $x \in X$. Then $(X \backslash G(x), \leqslant)$ is a down-spectral set.*

Proof. Let $\widetilde{\mathcal{T}}$ be an order compatible spectral topology on X. The subset $A = X\backslash G(x)$ is closed under specialization and $X\backslash A$ is quasi-compact. Let $\mathcal{T}$ be the topology on $A = X\backslash G(x)$ such that the closed subsets are $X\backslash G(x)$ and the closed subsets of $(X, \widetilde{\mathcal{T}})$ contained in $X\backslash G(x)$. Hence, according to Proposition 1.2, $(A, \mathcal{T})$ is a down-spectral space. On the other hand, we have $\overline{\{x\}}^{\mathcal{T}} = \overline{\{x\}}^{\widetilde{\mathcal{T}}} = S(x)$. Thus $\mathcal{T}$ is compatible with $\leqslant$. Therefore, $(X\backslash G(x), \leqslant)$ is a down-spectral set by Proposition 1.3. □

The following result indicates that to discuss the spectrality of a down-complete ordered set with finitely many minimal elements, we may suppose, without loss of generality, that it has only one minimum element.

Recall that an ordered set $(X, \leqslant)$ is said to be *down-complete* if each nonempty totally ordered subset of $(X, \leqslant)$ has an infimum.

Proposition 1.6. *Let $(X, \leqslant)$ be a down-complete ordered set with finitely many minimal elements. Then $(X, \leqslant)$ is spectral if and only if $(X, \leqslant)$ is down-spectral.*

Proof. It remains to prove that if $(X, \leqslant)$ is down-spectral, then it is spectral. Let $\mathcal{C}$ be an order compatible down-spectral topology on X. Since X is down-complete and has finitely many minimal elements, $X = S(a_1) \cup S(a_2) \cup \cdots \cup S(a_n)$, where $\{a_1, a_2, \ldots, a_n\} = Min(X)$. Hence X is reducible or has a generic point, in which case $n = 1$, and there is a minimum element. X is spectral in either case. □

2. Up-spectral topology

We start by describing how to construct up-spectral spaces.

Theorem 2.1. *Let $(X, \mathcal{T})$ be a topological space. Then the following statements are equivalent*:

(i) $(X, \mathcal{T})$ *is up-spectral.*

(ii) *There exists a spectral space Y with a maximum element ω, such that X is equal to the subspace $Y\backslash\{\omega\}$.*

Proof. (i) ⇒ (ii). Let $\omega \notin X$ and $\widetilde{X} = X \cup \{\omega\}$. Consider the topology $\widetilde{\mathcal{T}} = \mathcal{T} \cup \{\widetilde{X}\}$ on $\widetilde{X}$. It is easily seen that $(\widetilde{X}, \widetilde{\mathcal{T}})$ is a quasi-compact T_0-space and has a basis of quasi-compact open subsets which is stable under finite intersections. Clearly, ω is the maximum of $\widetilde{X}$ with respect to the ordering induced by the topology $\widetilde{\mathcal{T}}$.

Let C be a nonempty irreducible closed subset of $(\widetilde{X}, \widetilde{\mathcal{T}})$. Then $\omega \in C$ and $C = K \cup \{\omega\}$, where K is a closed subset of $(X, \mathcal{T})$. We discuss two cases:

Case 1. $K = \emptyset$. In this case, $C = \{\omega\}$ has a generic point.

Case 2. $K \neq \emptyset$. Necessarily, K is an irreducible closed subset of $(X, \mathcal{T})$. Hence there exists $x \in X$ such that $C = \overline{\{x\}}^{X} = \overline{\{x\}}^{\widetilde{X}}$.

Therefore, $X = \widetilde{X}\backslash\{\omega\}$ is an up-spectral subspace of $(\widetilde{X}, \widetilde{\mathcal{T}})$.

(ii) $\Rightarrow$ (i). Let $(Y, \mathcal{T})$ be a spectral space with a maximum element ω. Suppose that $X = Y \backslash \{\omega\}$ is a nonempty subspace of Y. Let us prove that X is an up-spectral subspace of Y.

Since Y is the unique open subset of Y containing ω, the topology of X is $\{X\} \cup \{U \in \mathcal{T} \mid \omega \notin U\}$. Hence X has a basis of quasi-compact open subsets which is stable under finite intersections.

Let C be an irreducible closed subset of X. Then $C \cup \{\omega\}$ is an irreducible closed subset of $(Y, \mathcal{T})$. Hence there exists $x \in C$ such that $C \cup \{\omega\} = \overline{\{x\}}^Y$. Thus $C = \overline{\{x\}}^X$, showing that X is a sober space. □

It is natural to have the following.

Proposition 2.2. *Let $(X, \leqslant)$ be an ordered set. Then $(X, \leqslant)$ is an up-spectral set if and only if there exists an order compatible up-spectral topology on X.*

Proof. *Necessary condition*. Let $\omega \notin X$ and $\widetilde{X} = X \cup \{\omega\}$. We define on $\widetilde{X}$ the ordering $\leqslant^\omega$ extending $\leqslant$ on X and making ω the greatest element of $\widetilde{X}$. Since $(X, \leqslant)$ is an up-spectral set, $(\widetilde{X}, \leqslant^\omega)$ is a spectral set. Hence there exists an order compatible spectral topology $\widetilde{\mathcal{T}}$ on $\widetilde{X}$. According to Theorem 2.1[(ii) $\Rightarrow$ (i)], the subspace $X = \widetilde{X} \backslash \{\omega\}$ is up-spectral, its topology is, obviously, compatible with the ordering $\leqslant$ of X.

Sufficient condition. Let $\mathcal{T}$ be an order compatible up-spectral topology on X. Then $\widetilde{X}$ endowed with the topology $\widetilde{\mathcal{T}} = \mathcal{T} \cup \{\widetilde{X}\}$ is a spectral space (cf. the proof of Theorem 2.1(i) $\Rightarrow$ (ii)). Clearly, $\widetilde{\mathcal{T}}$ is compatible with $\leqslant^\omega$. Thus $(\widetilde{X}, \leqslant^\omega)$ is a spectral set, showing that $(X, \leqslant)$ is an up-spectral set. □

According to [6, Proposition 8], if $(X, \leqslant)$ is spectral, then $(X, \geqslant)$ is also spectral.

It follows, immediately, from the definitions and [6, Proposition 8] that we have the following result.

Proposition 2.3. *Let $(X, \leqslant)$ be an ordered set. Then $(X, \leqslant)$ is down-spectral if and only if $(X, \geqslant)$ is up-spectral.*

The previous proposition permits us to state a list of corollaries giving the analogs of results concerning down spectral sets.

Corollary 2.4. *Every spectral set is up-spectral.*

Corollary 2.5. *Let $(X, \leqslant)$ be a spectral set and $x \in X$. Then $(X \backslash S(x), \leqslant)$ is an up-spectral set.*

Corollary 2.6. *Let $(X, \leqslant)$ be an up-complete ordered set with finitely many maximal elements. Then $(X, \leqslant)$ is spectral if and only if $(X, \leqslant)$ is up-spectral.*

3. Ordered disjoint unions of down-spectral sets

In this section we will restrict our attention to ordered disjoint unions of down-spectral sets.

Theorem A. *Any disjoint union of down-spectral sets is a down-spectral set.*

Proof. Let X be the ordered disjoint union of the ordered disjoint sets $((X_\lambda, \leqslant_\lambda),\ \lambda \in \Lambda)$. Suppose that for each $\lambda \in \Lambda$, $(X_\lambda, \leqslant_\lambda)$ is down-spectral. The problem here is to prove that $(\widetilde{X}, \leqslant_\omega)$ is a spectral set. To do this, take $\mathcal{T}_\lambda$ an order compatible spectral topology on $\widetilde{X_\lambda} = X_\lambda \cup \{\omega_\lambda\}$, where ω_λ is least in $\widetilde{X_\lambda}$. Let $Y = \bigcup_{\lambda\in\Lambda} \widetilde{X_\lambda}$ be the ordered disjoint union of $(\widetilde{X}, \leqslant_{\lambda\omega})$. Consider the topology $\mathcal{T}$ on Y, where the open sets are the $\bigcup_{\lambda\in\Lambda} U_\lambda$, with U_λ is an open set of $(\widetilde{X_\lambda}\ , \mathcal{T}_\lambda)$, for λ in a finite subset L of Λ, and $U_\lambda = \widetilde{X_\lambda}$, for each $\lambda \in \Lambda \backslash L$.

Step 3.1. $(Y, \mathcal{T})$ *is a down-spectral space.*

Proof.

Claim 3.1.1. $\mathcal{T}$ *is compatible with* $\leqslant$, *and* $(Y, \mathcal{T})$ *is quasi-compact. Moreover,* $\mathcal{T}$ *has a basis of quasi-compact open subsets which is stable under finite intersections.*

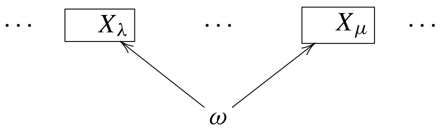
It suffices to observe that if U_λ is quasi-compact and open in X_λ, for each $\lambda \in L \subseteq \Lambda$ with $|L| < \infty$ and $U_\lambda = X_\lambda$, for each $\lambda \in \Lambda\backslash L$, then $U = \bigcup_{\lambda\in\Lambda} U_\lambda$ is quasi-compact.

Claim 3.1.2. *Each irreducible closed subset of* $(Y, \mathcal{T})$ *distinct from* Y *has a generic point.*

Let C be an irreducible closed subset of $(Y, \mathcal{T})$ distinct from Y. Hence there exists a finite subset L of Λ such that $C = \bigcup_{\lambda\in L} C_\lambda$ with C_λ a closed subset of $\widetilde{X_\lambda}$. Thus there exists $\lambda \in L$ such that $C = C_\lambda$, by irreducibility of C. Thus $C = C_\lambda$ is an irreducible closed subset of $(\widetilde{X_\lambda}\, \mathcal{T}_\lambda)$. It follows that C has a generic point, completing the proof of Claim 3.1.2.

On may represent the ordered set $(Y, \leqslant)$ as follows:

Now let us identify all the ω_λ to one, ω. This yields the ordered set $(\widetilde{X}, \leqslant_\omega)$, which looks like

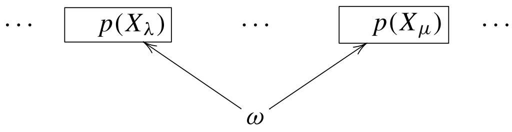


This idea permits to prove that $(\widetilde{X}, \leqslant_\omega)$ is spectral. Consider the equivalence relation $\mathcal{R}$ on Y such that the equivalence classes are given as follows: $\mathrm{cl}_{\mathcal{R}}(x) = \{x\}$, for each $x \in X = \bigcup_{\lambda \in \Lambda} X_\lambda$ and $\mathrm{cl}_{\mathcal{R}}(\omega_\mu) = \{\omega_\lambda \mid \lambda \in \Lambda\}$, for each $\mu \in \Lambda$.

The quotient set $Z = Y/\mathcal{R}$ is ordered by letting:

- for each $x, y \in X_\lambda$, $\mathrm{cl}_{\mathcal{R}}(x) \leqslant \mathrm{cl}_{\mathcal{R}}(y)$ if and only if $x \leqslant_\lambda y$;
- $\omega \leqslant \mathrm{cl}_{\mathcal{R}}(x)$, for each $x \in X = \bigcup_{\lambda \in \Lambda} X_\lambda$.

Let $p : Y \to Z$ be the canonical surjection, then the ordered set $(Z, \leqslant)$ looks like



It is easily seen that $(Z, \leqslant)$ is order isomorphic to $(\widetilde{X}, \leqslant_\omega)$. Thus we are aiming to prove that $(Z, \leqslant)$ is a spectral set.

Let $\mathcal{T}_{\mathcal{R}}$ be the quotient topology on $Z = Y/\mathcal{R}$. To complete the proof of this theorem, it suffices to show that $\mathcal{T}_{\mathcal{R}}$ is an order compatible spectral topology on $(Z, \leqslant)$.

Step 3.2. $(Z, \mathcal{T}_{\mathcal{R}}, \leqslant)$ *is a spectral space.*

The notation of Step 3.1 remains in effect.

Claim 3.2.1. $\mathcal{T}_{\mathcal{R}}$ *is compatible with the ordering* $\leqslant$ *on* Z.

The proof consists of showing that $\overline{\{\alpha\}} = S(\alpha)$, for each $\alpha \in Z$. We need to consider the following two cases:

– Suppose that $\alpha = \omega$. Let V be a nonempty open subset of $Z = Y/\mathcal{R}$, then $p^{-1}(V)$ is a nonempty open subset of $(Y, \mathcal{T})$. Let $x \in p^{-1}(V)$, then there exists $\lambda \in \Lambda$ such that $x \in X_\lambda$. Hence $\omega_\lambda \in G(x) \subseteq p^{-1}(V)$. Therefore, $\omega \in V$, proving that $\overline{\{\omega\}} = S(\omega) = Z$.

– Suppose that $\alpha = p(x)$, where $x \in X = \bigcup_{\lambda \in \Lambda} X_\lambda$. From the definition of the order on Z, it is clear that $p^{-1}(S(\alpha)) = S(x)$. Hence $S(\alpha)$ is a closed subset of Z. It follows that $\overline{\{\alpha\}} \subseteq S(\alpha)$. Conversely, let $p(y) \in S(\alpha)$ and V an open subset of Z containing $p(y)$. There exists a saturated (under $\mathcal{R}$) open subset U of Y such that $V = p(U)$. Hence $x \in G(y) \subseteq U$, so that $\alpha = p(x) \in V$. Thus $p(y) \in \overline{\{\alpha\}}$. Therefore, $\overline{\{\alpha\}} = S(\alpha)$.

Claim 3.2.2. $(Z, \mathcal{T}_{\mathcal{R}})$ *is quasi-compact.*

Since $(Y, \mathcal{T})$ is quasi-compact and p is continuous, $Z = p(Y)$ is also quasi-compact.

Claim 3.2.3. $(Z, \mathcal{T}_{\mathcal{R}})$ *has a basis of quasi-compact open subsets which is stable under finite intersections.*

Let $\mathcal{B}_{\mathcal{R}} = \{p(U) \mid U \in \mathcal{B}$ and $\{\omega_\lambda \mid \lambda \in \Lambda\} \subseteq U\}$. Any $U \in \mathcal{B}$ such that $\{\omega_\lambda \mid \lambda \in \Lambda\} \subseteq U$ is a saturated open subset of Y, so that $p(U)$ is an open subset of $Z = Y/\mathcal{R}$. It is clear that $p(U)$ is quasi-compact since U is.

Now, let V be a nonempty open subset of Z. Then $\omega \in V$. Hence $U = p^{-1}(V)$ is a nonempty open subset of $(Y, \mathcal{T})$ containing $\{\omega_\lambda \mid \lambda \in \Lambda\}$. There exists a finite subset L of Λ such that

$$U = p^{-1}(V) = \bigcup_{\lambda \in \Lambda} U_\lambda,$$

with U_λ an open set of $(\widetilde{X_\lambda}\ \mathcal{T}_\lambda)$, for each $\lambda \in L$, and $U_\lambda = \widetilde{X_\lambda}$, for each $\lambda \in \Lambda \backslash L$. Let $\mathcal{B}_\lambda$ be a basis of quasi-compact open subsets of $\widetilde{X_\lambda}$. Then for each $\lambda \in \Lambda$, U_λ is a union of some nonempty elements of $\mathcal{B}_\lambda$ each of them contains ω_λ (since $\overline{\{\omega_\lambda\}} = \widetilde{X_\lambda}$). Hence $V = p(U)$ is a union of some elements of $\mathcal{B}_\mathcal{R}$, proving that $\mathcal{B}_\mathcal{R}$ is a basis of quasi-compact open subsets of $(Z, \mathcal{T}_\mathcal{R})$.

Let $p(U)$ and $p(V) \in \mathcal{B}_\mathcal{R}$. Since U and V are saturated, we get

$$p(U) \cap p(V) = p(U \cap V),$$

showing that $\mathcal{B}_\mathcal{R}$ is stable under finite intersections.

Claim 3.2.4. $(Z, \mathcal{T}_\mathcal{R})$ *is sober.*

Let G be an irreducible closed subset of Z. We discuss two cases.

Case 1. $\omega \in G$. In this case, $G = \overline{\{\omega\}} = Z$. Thus G has a generic point.

Case 2. Suppose that $\omega \notin G$. Then there exists a finite subset L of Λ such that $p^{-1}(G) = \bigcup_{\lambda \in L} F_\lambda$, where F_λ is a closed subset of $(\widetilde{X_\lambda},\ \mathcal{T}_\lambda)$ and $\omega_\lambda \notin F_\lambda$, for each $\lambda \in L$, in this case the F_λ are saturated. Hence the $p(F_\lambda)$ are closed subsets of Z. Since $G = \bigcup_{\lambda \in L} p(F_\lambda)$ and G is irreducible, there is some $\lambda \in L$, with $G = p(F_\lambda)$ and F_λ is necessarily irreducible in $(\widetilde{X_\lambda},\ \mathcal{T}_\lambda)$. Thus there exists $x \in Y \backslash \{\omega_\lambda \mid \lambda \in \Lambda\}$ such that $p^{-1}(G) = F_\lambda = \overline{\{x\}} = S(x)$. By the definition of the ordering on Y, we have $p(S(x)) = S(p(x))$, for each $x \in Y \backslash \{\omega_\lambda \mid \lambda \in \Lambda\}$. Therefore, $G = S(p(x))$, proving that G has a generic point. □

The following result is an immediate consequence of Theorem A and Proposition 2.3.

Corollary 3.3. *Any disjoint unions of up-spectral sets is an up-spectral set.*

4. Disjoint union of up-spectral spaces

Let $(X, \leqslant)$ be an ordered disjoint union of the posets $(X_\lambda, \leqslant_\lambda)$, $\lambda \in \Lambda$. Our concern in this section is to give necessary and sufficient topological conditions on X in order to get all the $(X_\lambda, \leqslant_\lambda)$ spectral.

Let $\mathcal{T}$ be a topology on X. We denote by $(X/\Lambda, \mathcal{T}/\Lambda)$ the quotient space of the topological space $(X, \mathcal{T})$ by the equivalence relation $\backsim$ on X whose equivalence classes are the X_λ.

Proposition 4.1. *Suppose that* $(X, \mathcal{T})$ *is a disjoint union of topological spaces* $(X_\lambda, \mathcal{T}_\lambda)$ $(\lambda \in \Lambda)$. *Then* X *is up-spectral if and only if each* X_λ *is up-spectral.*

Proof. Suppose that $(X, \mathcal{T})$ is an up-spectral space. Since X_λ is a clopen subset of $(X, \mathcal{T})$, it is an up-spectral space.

Suppose that each $(X_\lambda, \mathcal{T}_\lambda)$ is an up-spectral space. The disjoint union $(X, \mathcal{T})$ is easily a T_0-space.

Let $\mathcal{B}_\lambda$ be a basis of quasi-compact open subsets of $((X_\lambda, \mathcal{T}_\lambda))$ which is stable under finite intersections. Then clearly, $\mathcal{B} = \bigcup_{\lambda \in \Lambda} \mathcal{B}_\lambda$ is a basis of quasi-compact open subsets of $(X, \mathcal{T})$ stable under finite intersections.

Let C be a nonempty irreducible closed subset of X. Then $C = \bigcup_{\lambda \in \Lambda} C_\lambda$, where C_λ is a closed subset of $(X_\lambda, \mathcal{T}_\lambda)$. For each $\lambda \in \Lambda$, X_λ is a clopen subset of $(X, \mathcal{T})$. Hence the C_λ are closed in $(X, \mathcal{T})$. Thus there exists $\lambda \in L$ such that $C = C_\lambda$. So that C is a nonempty irreducible closed subset of X_λ. Accordingly, there exists $x \in X_\lambda$ such that $\overline{\{x\}}^{X_\lambda} = C$. Since X_λ is a closed subset of $(X, \mathcal{T})$, $\overline{\{x\}}^{X} = \overline{\{x\}}^{X_\lambda} = C$. Thus, $(X, \mathcal{T})$ is sober. □

Corollary 4.2. *Let $(X, \leqslant)$ be an ordered disjoint union of the posets $(X_\lambda, \leqslant_\lambda)$, $\lambda \in \Lambda$, such that Λ. Then the following statements are equivalent*:

(i) *For each $\lambda \in \Lambda$ $(X_\lambda, \leqslant_\lambda)$ is an up-spectral set.*
(ii) *There exists an order compatible up-spectral topology $\mathcal{T}$ on X such that $(X/\Lambda, \mathcal{T}/\Lambda)$ is a discrete space.*

Corollary 4.3. *Let $(X, \leqslant)$ be an ordered disjoint union of $(X_\lambda, \leqslant_\lambda)$, $\lambda \in \Lambda$, such that Λ is finite. Then the following statements are equivalent*:

(i) *For each $\lambda \in \Lambda$ $(X_\lambda, \leqslant_\lambda)$ is a spectral set.*
(ii) *There exists an order compatible spectral topology $\mathcal{T}$ on X such that $(X/\Lambda, \mathcal{T}/\Lambda)$ is a T_1-space.*
(iii) *There exists an order compatible spectral topology $\mathcal{T}$ on X such that $(X/\Lambda, \mathcal{T}/\Lambda)$ is a discrete space.*

5. *D*-components of a spectral space

Let $(X, \leqslant)$ be a spectral set, $\mathcal{D}$ a D-component of X and $\mathcal{T}$ an order compatible spectral topology on X. In the introduction we have recalled the Lazard's formula for a D-component.

We define on $\mathcal{D}$ two topologies $\mathcal{T}_1(\mathcal{D})$ and $\mathcal{T}_2(\mathcal{D})$ by:

$\big(\mathcal{D}, \mathcal{T}_1(\mathcal{D})\big)$ is the disjoint union of the subspaces E^0, $E^n \backslash E^{n-1}$, $n \in \mathbb{N}\backslash\{0\}$.

Since E^n is a closed subspace of $(X, \mathcal{T})$, it is spectral. Let $\mathcal{T}_n$ be the topology induced by $\mathcal{T}$ on E^n. We define the topology $\mathcal{T}_2(\mathcal{D})$ on $\mathcal{D}$ by letting the closed sets be the F with the property that there exists some $n \in \mathbb{N}$, such that F is a closed subset of $(E^n, \mathcal{T}_n)$. Observe that a set O is open in $(\mathcal{D}, \mathcal{T}_2(\mathcal{D}))$ if and only if it is of the form $O = U \cup (\mathcal{D}\backslash E^k)$, where k is a sufficiently large integer so that $\mathcal{D}\backslash E^k \subseteq O$, and U is open in $(E^k, \mathcal{T}_k)$.

Under the previous notation, we have the two following topological results, shedding some light on the conjecture of Lewis and Ohm.

Theorem 5.1. *Let $(X, \leqslant)$ be a spectral set, $\mathcal{D}$ be a D-component of X and $\mathcal{T}$ an order compatible spectral topology on X. Then $(\mathcal{D}, \mathcal{T}_1(\mathcal{D}))$ is an up-spectral topology.*

We need three lemmata.

Lemma 5.2 [3, Proposition 7, p. 122]. *Let X be a topological space and U a nonempty open subset of X. Then the mapping $V \mapsto \overline{V}$ defines a bijection from the set of irreducible nonempty closed subsets of U onto the set of irreducible nonempty closed subsets of X meeting U. The inverse bijection is $Z \mapsto Z \cap U$.*

Lemma 5.3. *Let X be a sober space and U a nonempty open subset of X. Then U is sober.*

Proof. Let F be a nonempty irreducible closed subset of U. According to Lemma 5.2, $\overline{F}$ is an irreducible closed subset of X. Hence $\overline{F}$ has a generic point x. Let us write $\overline{F} = \overline{\{x\}}^X$, and note that $x \in U$. Clearly, $x \in F = \overline{F} \cap U$. Thus $F = \overline{\{x\}}^U$. □

Let U be an open subset of a topological space X. If X has a basis $\mathcal{B}$ of quasi-compact open subsets closed under finite intersections, then $\mathcal{B}_U = \{O \in \mathcal{B} \mid O \subseteq U\}$ is a basis of quasi-compact open subsets of U which is closed under finite intersections. Therefore, according to Lemma 5.3, the following lemma is easily checked.

Lemma 5.4. *Any open subset of an up-spectral space is up-spectral.*

Proof of Theorem 5.1. For each $n \in \mathbb{N}$, $(E^n, \mathcal{T}_n)$ is a spectral space. On the other hand $E^n \backslash E^{n-1}$ is an open subset of E^n, hence $E^n \backslash E^{n-1}$ is up-spectral (cf. Lemma 5.4). Therefore, $(\mathcal{D}, \mathcal{T}_1(\mathcal{D}))$ is an up-spectral space, by Proposition 4.1. □

Theorem 5.5. *Let $(X, \leqslant)$ be a spectral set, $\mathcal{D}$ be a D-component of X and $\mathcal{T}$ an order compatible spectral topology on X. Then the space $(\mathcal{D}, \mathcal{T}_2(\mathcal{D}))$ satisfies the following properties*:

(1) *$\mathcal{T}_2(\mathcal{D})$ is compatible with the order induced by $\leqslant$ on $\mathcal{D}$.*
(2) *$\mathcal{T}_2(\mathcal{D})$ is quasi-compact.*
(3) *Each nonempty irreducible closed subset of $(\mathcal{D}, \mathcal{T}_2(\mathcal{D}))$ distinct from $\mathcal{D}$ has a generic point.*

(*Thus what is missing in the conclusion of this theorem to make the topology $\mathcal{T}_2(\mathcal{D})$ down-spectral is that it has a basis of quasi-compact open subsets closed under finite intersections.*)

Proof. (1) Let $x \in \mathcal{D} = \bigcup_{n \in \mathbb{N}} E^n$. There exists $n \in \mathbb{N}$, such that $x \in E^n$. Hence $S(x)$ is a closed subset of E^n. Thus, $S(x)$ is a closed subset of $(\mathcal{D}, \mathcal{T}_2(\mathcal{D}))$. On the other hand, it is easily seen that any closed subset of $(\mathcal{D}, \mathcal{T}_2(\mathcal{D}))$ is closed under specialization, since each closed subset of E^n has this property. Therefore, $\mathcal{T}_2(\mathcal{D})$ is compatible with the ordering $\leqslant$ on $\mathcal{D}$.

(2) Let $(\mathcal{U}_i, i \in I)$ be a collection of open subsets of $(\mathcal{D}, \mathcal{T}_2(\mathcal{D}))$ such that $\mathcal{D} = \bigcup_{i \in I} \mathcal{U}_i$. $\mathcal{U}_i$ has the form $\mathcal{U}_i = U^i_{n_i} \cup (\mathcal{D} \backslash E^{n_i})$, and $U^i_{n_i}$ is an open subset of $(E^{n_i}, \mathcal{T}_{n_i})$. Let $p = \min\{n_i \colon i \in I\}$. Of course, $E^p = \bigcup_{i \in I} (\mathcal{U}_i \cap E^p)$ and $\mathcal{U}_i \cap E^p$ is an open subset of E^p. Thus, there exists a finite subsets J of I, such that $E^p = \bigcup_{i \in J} (\mathcal{U}_i \cap E^p)$. The equality $\mathcal{D} = E^p \cup (\mathcal{D} \backslash E^p)$ implies that $\mathcal{D} = \bigcup_{i \in J} \mathcal{U}_i$, proving that $(\mathcal{D}, \mathcal{T}_2(\mathcal{D}))$ is quasi-compact.

(3) Any nonempty irreducible closed subset of $(\mathcal{D}, \mathcal{T}_2(\mathcal{D}))$ distinct from $\mathcal{D}$ is necessarily of the form $F = F_n$, where $n \in \mathbb{N}$, and F_n is an irreducible closed subset of $(E^n, \mathcal{T}_n)$. Therefore, F has a generic point. □

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