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LINEAR ALGEBRA AND ITS APPLICATIONS

ρ-Contraction and 2×2 matrix

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Abstract

In this paper, the following is proved. When $|a| \leqslant 1$ and $|b| \leqslant 1$,

$$A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$$

is a ρ-contraction if and only if

$$|c|^2 + |a-b|^2 \leqslant \inf_{\zeta \in D} \left| \frac{\{\rho + (1-\rho)\bar{a}\bar{\zeta}\}\{\rho + (1-\rho)b\zeta\} - \bar{a}b|\zeta|^2}{\rho\zeta} \right|^2,$$

where D is the open unit disc. The result is then extended to quadratic operators. Several special cases of the result are analysed in detail.

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1. Introduction

For $\rho > 0$, a bounded linear operator A on a Hilbert space $\mathscr{H}$ is a ρ-contraction if and only if the powers of A admit a representation

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$$A^n h = \rho P U^n h \quad (h \in \mathscr{H}; n = 1, 2, \ldots),$$

where U is a unitary operator (called a unitary ρ-dilation) on some Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ as a subspace, and where P is the projection from $\mathscr{K}$ to $\mathscr{H}$. A is a contraction, that is, the norm $\|A\| \leqslant 1$ if and only if A is a 1-contraction. This is a theorem of Sz. -Nagy [5]. The numerical radius $w(A)$ of A is not bigger than 1 if and only if A is a 2-contraction. This is a theorem of Berger [1]. Sz.-Nagy and Foias [6] gave intrinsic characterizations of operators of ρ-contractions. Unfortunately, even if $\mathscr{H}$ is of two dimension, it is difficult to determine when A is a ρ-contraction except $\rho = 1$. In this paper, we assume that $\mathscr{H}$ is a two dimensional Hilbert space. Assuming A is triangular, we determine when A is a ρ-contraction. In fact, for $\rho = 2$, the numerical radius is given.

In this paper, D is the open unit disc and $\overline{D}$ is its closure. $\mathscr{H}ol(\overline{D}, \alpha, \overline{D})$ denotes a set of all holomorphic functions from $\overline{D}$ into $\overline{D}$ which vanish at $\alpha \in D$ and $\mathscr{H}ol(\overline{D})$ denotes a set of all holomorphic contractions on $\overline{D}$.

In the proof of Theorem, we use three elementary well known results.

(1) $\sup\{|f'(\alpha)|; f \in \mathscr{H}ol(\overline{D}, \alpha, \overline{D})\} = 1/(1 - |\alpha|^2)$.

(2) $\sup\{|f(\beta)|; f \in \mathscr{H}ol(\overline{D}, \alpha, \overline{D})\} = \left|\frac{\alpha - \beta}{1 - \bar{\alpha}\beta}\right|$ when $\beta \in D$.

(3) $f(A)$ is a contraction for any $f \in \mathscr{H}ol(\overline{D})$ if and only if $f(A)$ is a contraction for any $f \in \mathscr{H}ol(\overline{D}, \alpha, \overline{D})$.

Proof. (1) If $f \in \mathscr{H}ol(\overline{D}, \alpha, \overline{D})$ then $|(f \circ \phi_{-\alpha})'(0)| \leqslant 1$ by the Schwarz's lemma where $\phi_{-\alpha}(z) = (z + \alpha)/(1 + \bar{\alpha}z)$. Hence the supremum is less than equal to $1/(1 - |\alpha|^2)$. The supremum is attained by $\phi_\alpha(z) = (z - \alpha)/(1 - \bar{\alpha}z)$. (2) If $f \in \mathscr{H}ol(\overline{D}, \alpha, \overline{D})$ then $f(z) = \phi_\alpha(z)g(z)$ and $|g(z)| \leqslant 1$. Hence the supremum is attained by $\phi_\alpha(z)$. (3) The 'only if' part is clear. If $f \in \mathscr{H}ol(\overline{D})$ and $f(\alpha) = \beta, g = \phi_\beta \circ f$ belongs to $\mathscr{H}ol(\overline{D}, \alpha, \overline{D})$ and it follows that $\|g(A)\| \leqslant 1$ or, equivalently $\|f(A)\| = \|\phi_{-\beta}(g(A))\| \leqslant 1$ (see [4]). □

2. Basic result

In order to give a Theorem, we use a theorem of Misra ([4], Theorem 1.1) and a result of Ando and the second author [7]. Corollary 1 is known. Corollary 2 is new and with a theorem of Berger [1], it gives a necessary and sufficient condition for $w(A) \leqslant 1$.

Theorem. *Suppose a and b are complex numbers in the closed unit disc $\overline{D}$. Then,*

$$A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$$

it is a ρ-contraction if and only if

$$|c|^2 + |a-b|^2 \leqslant \inf_{\zeta \in D} \left| \frac{\{\rho + (1-\rho)\bar{a}\bar{\zeta}\}\{\rho + (1-\rho)b\zeta\} - \bar{a}b|\zeta|^2}{\rho\zeta} \right|^2.$$

Proof. By [7], A is a ρ-contraction if and only if for any $\zeta \in D, g_\zeta(A)$ is a contraction where $g_\zeta(z) = z\zeta/\{\rho + (1-\rho)z\zeta\}$. If $a = b$, by Theorem 1.1 of [4], $g_\zeta(A)$ is a contraction if and only if

$$|c|\,|g_\zeta'(a)| \leqslant (\sup\{|f'(g_\zeta(a))|; f \in \mathscr{H}ol(\bar{D}, g_\zeta(a), \bar{D})\})^{-1}.$$

Hence by (1)

$$|c| \leqslant \frac{1 - |g_\zeta(a)|^2}{|g_\zeta'(a)|} = \frac{|\rho + (1-\rho)a\zeta|^2 - |a\zeta|^2}{|\zeta\rho|}.$$

Thus A is a ρ-contraction if and only if

$$|c| \leqslant \inf_{\zeta \in D} \left| \frac{|\rho + (1-\rho)a\zeta|^2 - |a\zeta|^2}{\zeta\rho} \right|.$$

This implies the theorem when $a = b$.

Suppose $a \neq b$. Note that $g_\zeta(a) \neq g_\zeta(b)$ for $\zeta \neq 0$. Hence A is a ρ-contraction if and only if

$$\begin{bmatrix} f \circ g_\zeta(a) & c\,\dfrac{f \circ g_\zeta(a) - f \circ g_\zeta(b)}{a-b} \\ 0 & f \circ g_\zeta(b) \end{bmatrix}$$

is a contraction for any $f \in \mathscr{H}ol(\bar{D}, g_\zeta(b), \bar{D})$ and for any $\zeta \in \bar{D}$ by the von Neumann's inequality and (3). As in Theorem 1.1 of [4], this is equivalent to

$$|c|^2/|a-b|^2 \leqslant \frac{1}{|f \circ g_\zeta(a)|^2} - 1$$

for any $f \in \mathscr{H}ol(\bar{D}, g_\zeta(b), \bar{D})$ and for any $\zeta \in \bar{D}$. It is easy to see that by (2)

$$\sup\{|f \circ g_\zeta(a)|; f \in \mathscr{H}ol(\bar{D}, g_\zeta(b), \bar{D})\} = \left| \frac{g_\zeta(a) - g_\zeta(b)}{1 - \overline{g_\zeta(a)}g_\zeta(b)} \right|^{-1}$$

$$= \left| \frac{\{\rho + (1-\rho)\bar{a}\bar{\zeta}\}\{\rho + (1-\rho)b\zeta\} - \bar{a}b|\zeta|^2}{(a-b)\rho\zeta} \right|.$$

This implies the theorem when $a \neq b$. □

Corollary 1. *In Theorem, A is a* 1*-contraction if and only if*

$$|c|^2 \leqslant \inf_{\zeta \in D} \left| \frac{1 - \bar{a}b|\zeta|^2}{\zeta} \right|^2 - |a-b|^2 = (1 - |a|^2)(1 - |b|^2).$$

Corollary 2. *In Theorem, A is a 2-contraction if and only if*

$$|c|^2 \leqslant \inf_{|\zeta|=1} |2 - (\bar{a}\bar{\zeta} + b\zeta)|^2 - |a - b|^2 = 4 \inf_{|\zeta|=1} \{(1 - \mathrm{Re}(a\zeta))(1 - \mathrm{Re}(b\zeta))\}.$$

3. Extension to quadratic operators

An operator A on $\mathscr{H}$ is called a quadratic operator if A satisfies quadratic polynomial, that is, $A^2 + rA + sI = 0$ for some complex numbers r and s. In the finite dimensional case, if $t^2 + rt + s = (t - a)(t - b)$ then A is unitarily similar to a matrix of the form

$$\begin{bmatrix} aI_l & C \\ 0 & bI_m \end{bmatrix}, \qquad (*)$$

where C is an $l \times m$ matrix. For a reference on the structure theorem of quadratic operator, see [2]. Since the ρ-contraction is invariant under the unitary similarity, we may assume that a quadratic operator A has a form of $(*)$. Then we can show A is a ρ-contraction if and only if

$$\|C\|^2 + |a - b|^2 \leqslant \inf_{\zeta \in D} \left| \frac{\{\rho + (1-\rho)\bar{a}\bar{\zeta}\}\{\rho + (1-\rho)b\zeta\} - \bar{a}b|\zeta|^2}{\rho\zeta} \right|^2.$$

In fact, by the singular value decomposition of C(cf. [3]), there is an $l \times m$ matrix $\Sigma = [\sigma_{ij}]$ where $\sigma_{ij} = 0$ for all $i \neq j$, and $\sigma_{11} \geqslant \sigma_{22} \geqslant \cdots \geqslant \sigma_{qq} \geqslant 0$ ($q = \min\{l, m\}$) and there are unitary matrices $U \in M_l$, $V \in M_m$ such that $UCV^* = \Sigma$. Here $\sigma_{11}, \sigma_{22}, \ldots, \sigma_{qq}$ are the decreasing ordered singular values of A, specially, $\sigma_{11} = \|C\|$. Then A is unitarily similar to a matrix

$$\begin{bmatrix} aI_l & \Sigma \\ 0 & bI_m \end{bmatrix}$$

and this matrix is unitarily similar to the direct sum of 2×2 matrices of form

$$C_i = \begin{bmatrix} a & \sigma_{ii} \\ 0 & b \end{bmatrix}$$

and possibly with aI_k or bI_k with $k = |l - m|$. It follows that A is a ρ-contraction if and only if C_1 is. From the Theorem in this paper our assertion is led.

4. Several special cases

In the rest of this paper, let

$$A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}.$$

For some special a and b, we compute the infimums in Theorem and Corollary 2 as that in Corollary 1.

1. When $|a| = 1$ or $|b| = 1$, A is a ρ-contraction if and only if $c = 0$.
2. When $a = b$, A is a ρ-contraction if and only if $|c| \leqslant (\rho - 2)|a|^2 + 2(1 - \rho)|a| + \rho$.
3. When $b = 0$, A is a ρ-contraction if and only if $|c|^2 \leqslant \rho(\rho - 2)|a|^2 + 2\rho(1 - \rho)|a| + \rho^2$.
4. When $a \geqslant 0$ and $b \geqslant 0$, A is a ρ-contraction if and only if $|c|^2 \leqslant \{\rho + (1 - \rho)(a + b) + (\rho - 2)ab\}^2 - (a - b)^2$.
5. When $a = -b$ is real, A is a ρ-contraction if and only if $|c|^2 + 4a^2 \leqslant \{\rho - (2 - \rho)a^2\}^2$.

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