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FROM FUZZY DATA TO FUNCTIONAL RELATIONSHIPS

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Abstract—The paper begins with a discussion and proposals of how to specify fuzzy data, especially from pseudo-exact point-shaped observations. Then the problems in finding a local approximation of the unknown functional relationship assumed to be valid are considered. For the next stage of the investigation, a family of functional relationships is chosen, and several methods are presented to estimate the parameter of this family using the specified fuzzy data. The methods are illustrated by an example.

1. INTRODUCTION

The results obtained by measurements or observations, shortly data, are mostly recorded as points. But they are, admittedly, subject to inaccuracy and uncertainty. Usually, they are considered as realizations of random elements, on which certain assumptions are to be specified, being necessary for the application and interpretation of the statistical procedures used.

To investigate if and what functional relationship is, approximately, satisfied by the data, we have, as a rule, to specify families of functions, so-called setups. Using the data, we have to choose one of the setups (problem of discrimination) and a special function (problem of estimation) out of the chosen setup to be used as an approximation of the functional relationship under study.

The alternative way of dealing with the inaccuracy and uncertainty of the data is via their *representation as a fuzzy set* [1–6].

In the next section we will present some proposals as to how to specify fuzzy sets as results of experiments or observations. Then we turn to the problem of evaluating the desired functional relationship. In Section 3 we consider the case that it does not make sense to specify a setup (local approximation). In Section 4 we handle the problem of "estimation" of the unknown parameter of the chosen setup (global approximation). The methods suggested in Section 4 are illustrated by an example in Section 5.

In a certain sense, the functional relationship is filtered out of the fuzzy data. But, simultaneously, their inaccuracy and uncertainty remain present in the fuzzy character of reasoning. This seems to be one of the advantages of the approach.

2. SPECIFICATION OF FUZZY DATA

A fuzzy set M is given by its membership function m, i.e.

$$M{:}\, m \,|\, X \to [0, 1], \tag{1}$$

where X denotes the universe of discourse.

For functional relationships, X is the observation field $B \times B_y \subset R^k \times R^1$ with $\mathbf{x} \in B \subset R^k$ and $y \in B_y \subset R^1$. Mainly, we consider explicit functional relationships $y = f(\mathbf{x})$. Sometimes we will give modifications for the implicit case: $y = x_0$, $f_{\text{imp}}(\mathbf{x}, x_0) = 0$.

For the specification of m we should use available prior knowledge with regard to the practical problem, which the data are taken from. This includes known precision of measurement, ranges of data, or estimates thereof, not necessarily obtained by statistical methods. Moreover, even experts' opinions themselves can be taken and used as fuzzy data in the framework of our concept. Besides, the data may here come from quite different sources. This is, as a rule, not allowed when using a statistical treatment. The value $m(\mathbf{x}, y)$ responds to the degree of membership that this point $(\mathbf{x}, y)$ belongs to the observation M and will be met as a point of the functional relationship $y = f(\mathbf{x})$. In the plane, for example, we can imagine such fuzzy observations as grey-tone pictures, where $m(\mathbf{x}, y)$ corresponds to the degree of blackness.

Experimenters, being accustomed to specify limits of variability for their observations, may be willing to specify grey-tone borders instead of those limits. But, in general, data are recorded as points, which we will refer to as *pseudo-exact* results. Hence, it seems necessary to sketch some proposals as to how to arrive at fuzzy observations from pseudo-exact results. We consider here only the *two-dimensional case.* The generalization to k variables is obvious. More details and methods can be found in Refs [3, 7].

For each given pseudo-exact datum (x_i, y_i), $i = 1(1)n$, we specify a *domain of influence* D_i. The set D_i contains all points which can belong to the ith observation, when we take into account the inaccuracy and uncertainty of the datum (x_i, y_i). Over this set D_i as a support, we specify a function m_i valuing the degree of membership within D_i. For example, we may specify the ellipse D_i,

$$D_i = \{(x, y) \in X; c_i(x - x_i)^2 + d_i(y - y_i)^2 \leqslant 1\}, \tag{2}$$

with suitably chosen positive numbers c_i and d_i. Correspondingly, m_i is then a segment of an elliptical paraboloid,

$$m_i(x, y) = [1 - c_i(x - x_i)^2 - d_i(y - y_i)^2]^+, \tag{3}$$

where $[v]^+ = \max\{v, 0\}$. An alternative is a rectangular $D_i = \{(x, y) \in X; |x - x_i| \leqslant b_x, |y - y_i| \leqslant b_y\}$, with suitably chosen positive numbers b_x and b_y and a pyramidal section

$$m_i(x, y) = [1 - c_x|x - x_i| - c_y|y - y_i|]^+ \tag{4}$$

with c_x and c_y possibly different from b_x and b_y, respectively. The proposal to work with formulae (2) and (3) is inspired by mathematical statistics: level lines of normal distributions are ellipses. The alternative based on formula (4) is adapted from interval mathematics techniques.

Both proposals have been used in previous examples [2, 4, 6]. As can also be gathered from theoretical considerations, the results obtained show only slight dependence on the analytical form of the specified membership function provided that *local monotonicity* is guaranteed, i.e. if for two different specifications m' and m'' the following relation holds:

$$\forall(x_1, y_1), \quad (x_2, y_2) \in X \times X : m'(x_1, y_1) \geqslant m'(x_2, y_2) \Leftrightarrow m''(x_1, y_1) \geqslant m''(x_2, y_2). \tag{5}$$

Hence we can choose the structure most convenient for computation and without essential influence on the results.

For instance, following the procedure used in an example in Ref. [3], a very rough specification choosing two simultaneous radii of influence, say $r_{(1)}$ and $r_{(2)}$, with $r_{(1)} < r_{(2)}$, $\forall i = 1(1)n$, and the membership functions

$$m_i(x, y) = \begin{cases} 1 & \text{for } (x - x_i)^2 + (y - y_i)^2 \leqslant r_{(1)}^2 \\ 0.5 & \text{for } r_{(1)}^2 < (x - x_i)^2 + (y - y_i)^2 \leqslant r_{(2)}^2 \\ 0 & \text{for } r_{(2)}^2 < (x - x_i)^2 + (y - y_i)^2 \end{cases} \tag{6}$$

already leads to useful results.

The support of M_i is the disc with radius $r_{(2)}$ around (x_i, y_i) and coincides with the 0.5-cut of M_i; whereas the disc with radius $r_{(1)}$ forms the 1-cut.

Finally, the fuzzy sets M_i, defined by m_i, can be collected in different manners (cf. Dubois and Prade [8])—we consider two of them here.

If we look, for example, for points which belong to at least one of the fuzzy observations (and hence will be possibly met as points of the functional relationship), then we have to join the M_i:

$$M = M_i, \underset{i}{\cup} : m(\mathbf{x}, y) = \max_i m_i(\mathbf{x}, y). \tag{7}$$

If we want to use each observation at every point with equal rights (e.g. when having used statistical data in specifying the fuzzy observations), then we collect them by summing up and renormalization, i.e.

$$M: m(\mathbf{x}, y) = \mathrm{c} \sum_{i=1}^{n} m_i(\mathbf{x}, y). \tag{8}$$

The constant c in equation (8) is to be chosen such that $m(\mathbf{x}, y) \leqslant 1$, $\forall(\mathbf{x}, y)$.

Using the specification (6), we can profit from certain procedures known from mathematical morphology [9] to easily obtain the summation (8). Let G denote a *structure element*, e.g. a disc of radius r, and $G_{(x,y)}(r)$ this element when arranged at (x, y) with a certain characterizing point, e.g. the centre. Then we can compute

$$\mathrm{c_B}(x, y; r) = \mathrm{card}\{(x_i, y_i) \in G_{(x,y)}(r)\}, \tag{9}$$

for $r = r_{(1)}$ and $r = r_{(2)}$, and obtain

$$m(x, y) = \mathrm{c_O}[\mathrm{c_B}(x, y; r_{(1)}) + \mathrm{c_B}(x, y; r_{(2)})], \tag{10}$$

where $\mathrm{c_O}$ is some suitable constant.

3. LOCAL FUZZY APPROXIMATIONS

There are many cases where it does not make sense to specify setups, at least at an early stage of the investigation. Then, with the given data specified as a fuzzy set M, it is the aim of an *exploratory analysis* (see, for example, Ref. [10] for the concept of exploratory data analysis) to obtain a first impression of the possible graph of the functional relationship which is assumed to be valid.

In the case of an *explicit* relationship, $y = f(\mathbf{x})$, a suitable *local approximation* is obtained by forming the *maximum trace*

$$\hat{f}_{\mathrm{F}}(\mathbf{x}) = \arg\sup_{y} m(\mathbf{x}, y), \tag{11}$$

i.e. for every value of $\mathbf{x}$ we take the values of y having the maximum grade of possibility according to the fuzzy set M. In general, the maximum trace will be neither unique nor continuous, but will show the behaviour of a natural range of mountains as we may see in topographical maps. This, however, is no disadvantage: for a first impression it suffices to recognize the occurring trends, moreover, premature smoothing and pressing for uniqueness may inadmissibly alter the information inherent in the data.

If the maximum trace branches out into several clearly separated ranges of y, or, if such separated ranges occur simultaneously in different regions of y, then we should inspect the pseudo-exact points for gross errors (outliers), reconsider the specification of the fuzzy data, or, finally, reconsider the possibility that the functional relationship has an implicit form.

For such *implicit* functional relationships we will suggest taking the *ridges* of the membership function surface m as an approximation of the unknown graph. This can be accomplished, for example, by means of watershed algorithms, which have been developed even for higher dimensions [9, 11].

In addition to the maximum trace or ridges, resp., we can consider the *level lines* of the fuzzy data, i.e.

$$l(\alpha, m) = \{(\mathbf{x}, y): m(\mathbf{x}, y) = \alpha\}, \tag{12}$$

which give us an impression of the absolute grade of possibility that a certain point is met as a point of the functional relationship. This method is illustrated by a real-data numerical example in Ref. [3].

4. GLOBAL FUZZY APPROXIMATIONS

Usually, it is necessary or desirable to have a closed-form analytical representation for the functional relationship within the whole region of interest. Then we have to specify a setup, either from the prior knowledge offered by the branch of science that has posed the problem or inspired by the local approximation obtained according to the procedures presented in the preceding section. In either case, a *setup* is a family of functions $\{g(\cdot, \mathbf{a})\}_{\mathbf{a}\in A}$; $A \subseteq R^r$; by which the unknown functional relationship is to be represented or approximated.

The origin of the setup, however, can influence the choice of a procedure to "estimate" the parameter $\mathbf{a}$. In any case, the "information" contained in the fuzzy data is to be transferred into A. For this presentation we restrict ourselves to the case of an explicit functional relationship, the case of an implicit functional relationship will be handled analogously.

We consider, at first, the case that the point $\mathbf{x} \in B$, where the relationship will be of interest, can be taken as random. As an example we mention the dependence of daily growth on the condition of climate. The daily values of temperature, pressure, humidity etc. come out randomly, in particular they are independent of the conception of the observer. Hence we propose to use, in this case, the usual expectation value known from probability theory. Let P be a suitable probability measure over B, then we consider, $\forall \mathbf{a} \in A$, the membership function m along the graph $\{(\mathbf{x}, g(\mathbf{x}, \mathbf{a}))\}$ of the relationship taking the expectation value

$$m_{\mathrm{E}}(\mathbf{a}) = \mathrm{E}m(\mathbf{x}, g(\mathbf{x}, \mathbf{a})) = \int_B m(\mathbf{x}, g(\mathbf{x}, \mathbf{a}))\,\mathrm{d}P(\mathbf{x}). \tag{13}$$

For every $\mathbf{a}$, the value $m_{\mathrm{E}}(\mathbf{a})$ represents the *expected cardinality* (cf. Dubois and Prade [8]) of M along the graph $\{(\mathbf{x}, g(\mathbf{x}, \mathbf{a}))\}$; $\mathbf{x} \in B$. In the sense of Zadeh [12], $m_{\mathrm{E}}(\mathbf{a})$ is the probability of the following event: when realizing an $\mathbf{x} \in B$ according to the measure P the point $(\mathbf{x}, g(\mathbf{x}, \mathbf{a}))$ belongs to the observation M. Hence, the function $m_{\mathrm{E}} | A \to [0, 1]$ defines a fuzzy set over A valuing the fuzzy data M from the standpoint of the frequency characterized by P. For application, it is not necessary to specify a random variable associated with P, the measure P can be taken merely as an additional possibility to value within B.

Alternatively, we can start with the specification of a fuzzy measure $\tilde{P}$ over B expressing the importance which the observer attributes to the subsets of B with respect to the relationship. As an example, we consider the situation where we have to design suitable or desirable working regions. Then we propose to use the *fuzzy expectation* value with respect to $\tilde{P}$. It is given by the special Sugeno integral [cf. 8, 13],

$$m_{\mathrm{F}}(\mathbf{a}) = \fint_B m(\mathbf{x}, g(\mathbf{x}, \mathbf{a})) \circ \tilde{P}(\cdot), \tag{14}$$

of the membership function along the graph of the relationship $\{(\mathbf{x}, g(\mathbf{x}, \mathbf{a}))\}$ with respect to the fuzzy measure $\tilde{P}$. The function m_{F} defines a fuzzy set in A. Both m_{E} and m_{F} were suggested and used in Ref. [2].

Another principle of transfer was suggested in Ref. [4]. It starts with the fuzzification of the statement: "There is a point, say $(\bar{\mathbf{x}}, \bar{y})$, in a crisp set C with $\bar{y} = f(\bar{\mathbf{x}})$". The degree that "the fuzzy observation M_i contains a point of the functional relationship $y = g(\mathbf{x}, \mathbf{a})$" is then given by

$$m_{\mathrm{J}}(\mathbf{a}; M_i) = \sup_{\mathbf{x}\in B} m_i(\mathbf{x}, g(\mathbf{x}, \mathbf{a})). \tag{15}$$

The value $m_J(\mathbf{a}; M_i)$ is called the *fuzzy grade of validity* of $g(\cdot, \mathbf{a})$ in M_i. From equation (15) we obtain the degree that "each of the fuzzy observations M_i contains a point of the functional relationship $y = g(\mathbf{x}, \mathbf{a})$" by

$$m_J(\mathbf{a}) := m_J(\mathbf{a}; M_1, \ldots, M_n) = \min_i m_J(\mathbf{a}; M_i). \tag{16}$$

The values of m_J are called *joint grades of validity* of the functional relationship given the fuzzy data.

Hence

$$m_J(\mathbf{a}) = \min_i \sup_{\mathbf{x} \in B} m_i(\mathbf{x}, g(\mathbf{x}, \mathbf{a})) \tag{17}$$

is a certain *fuzzy minimax principle* and uses the fuzzy data without any previous summation.

Each of the three membership functions m_E, m_F and m_J defines a certain fuzzy set over A, which we call $\hat{A}_E$, $\hat{A}_F$ and $\hat{A}_J$, respectively. They are *fuzzy estimates* of the parameter of the functional relationship reflecting the fuzzy character of the data, irrespective of whether the parameter is assumed crisp or fuzzy. The latter case connects the problem of linear regression with fuzzy models, as considered in Ref. [14]. As in numerical approximations, the methods differ in their properties. They remind us of the well-known differences between two main principles in mathematics: averaging with respect to a given measure (m_E, m_F) and the minimax principle (m_J). The method of expected cardinality is *robust* with respect to the specification of the M_i. In particular, variations of the membership functions have only a slight influence on the obtained results. However, to guarantee that the method supplies useful results we must take care that M contains useful information with respect to the measure P. If

$$\operatorname{supp} P \cap \operatorname{supp} M = \varnothing, \tag{18}$$

then M does not contain any useful information and m_E vanishes all over A. In this case we have to reconsider our measure of interest or look for other useful observations. Note that we do not meet this case when working with a uniform distribution over B.

Moreover, it can happen that only a subset of all given observations M_i operates when calculating the fuzzy set over A. This seems to be the price for the desirable robustness of the method. With respect to the fuzzy expectation method the properties are quite similar.

The joint grades of validity method handles the fuzzy observations M_i individually. Its degree of robustness is much smaller than that of the methods mentioned before. In particular, it is sensitive to "outliers". We can say that it responds with an m_J vanishing all over A if $\nexists \mathbf{a} \in A$ and $(\mathbf{x}_i, y_i) \in \operatorname{supp} M_i$ with $y_i = g(\mathbf{x}_i, a)$, for each $i \in \{1, \ldots, n\}$. This is a property which is more crucial than property (18). It suffices one "error" for a breakdown. However, this property fits the method for use in model discrimination where several families of functional relationships are confronted with the given fuzzy observations. Although the whole fuzzy set $\hat{A}$ acts as the desired estimate, it will be of interest which subsets or single points can be favoured in further investigations. First of all, the level lines (12) are useful for this purpose when interpreted in a similar manner—as known from statistical confidence analysis. Moreover, but with due caution, maximum points also can be used sometimes to represent the estimate $\hat{A}$, e.g.

$$\hat{a}_E := \{\mathbf{a} \in A : m_E(\mathbf{a}) = \sup_{\mathbf{b} \in A} m_E(\mathbf{b})\}. \tag{19}$$

Note that $\hat{a}_E$ may contain more than one point. In general we will use a together with the appropriate level lines of $\hat{A}$.

Remark. Usually the local fuzzy approximation $\hat{f}_F$ serves only as a hint in the choice of an appropriate setup to be applied in the global approximation. But we can also use $\hat{f}_F$ as a starting point for another approach.

If we explain $1 - m(\cdot,\cdot)$ as the *loss* incurred by uncertainty, then, for example,

$$\int_B (1 - m(\mathbf{x}, \hat{f}_F(\mathbf{x})))\, \mathrm{d}P(\mathbf{x}) \tag{20}$$

is the unavoidable *risk* when using the best [cf. 11] estimating graph $\{(\mathbf{x}, \hat{f}_F(\mathbf{x}))\};\ \mathbf{x} \in B$ of the unknown functional relationship. Hence,

$$h_E(\mathbf{a}) = \int_B (m(\mathbf{x}, \hat{f}_F(\mathbf{x})) - m(\mathbf{x}, g(\mathbf{x}, \mathbf{a})))\, \mathrm{d}P(\mathbf{x}) \tag{21}$$

represents the *regret* for using $g(\cdot, \mathbf{a})$ instead of $\hat{f}_F$. As can be easily seen, $\hat{a}_E$ also solves the minimum regret problem

$$h_E(\mathbf{a}) \overset{!}{=} \inf_{\mathbf{a} \in A}. \tag{22}$$

These conclusions remain valid when using, in expression (20), the Sugeno integral with respect to a fuzzy measure $\tilde{P}$. With

$$h_J(\mathbf{a}) = \max_i \sup_{\mathbf{x} \in B} (m_i(\mathbf{x}, \hat{f}_F(\mathbf{x})) - m_i(\mathbf{x}, g(\mathbf{x}, \mathbf{a}))), \tag{23}$$

however, the minimum regret problem has solutions which are probably not contained in $\hat{a}_J$.

5. EXAMPLE

To illustrate the proposed methods we consider four fuzzy observations of the type

$$M_i : m_i(x, y) = [1 - r_i^{-2}((x - x_i)^2 + (y - y_i)^2)]^+, \tag{24}$$

$i = 1, 2, 3, 4$; where

$$\begin{array}{lll} x_1 = 0.2 & y_1 = 0.2 & r_1 = 0.1 \\ x_2 = 0.4 & y_2 = 0.45 & r_2 = 0.1 \\ x_3 = 0.7 & y_3 = 0.6 & r_3 = 0.12 \\ x_4 = 0.9 & y_4 = 0.8 & r_4 = 0.08. \end{array} \tag{25}$$

This is a special case of the specification in expressions (2) and (3). The supports of formulae (24) and (25) are discs around the pseudo-exact data (x_i, y_i). The membership functions describe paraboloids of revolution.

We want to transfer $M = V_i M_i$ to the parameter set of the explicit functional relationship

$$y = a_0 + a_1 x; x \in [0, 1]. \tag{26}$$

To have the parameter set bounded we assume that we are given determinate prior knowledge: $y \in [0, 1]$. Then we obtain the parameter set of interest

$$A := \{(a_0, a_1) : a_0 \in [0, 1] \wedge a_1 \in [-a_0, 1 - a_0]\}.$$

As the (necessary) measure over $[0, 1]$ we choose the uniform one $P(\mathrm{d}x) = \tilde{P}(\mathrm{d}x) = \mathrm{d}x$.

The example was treated according to the expected cardinality method and the fuzzy expectation method in Ref. [2], according to the joint grades of validity method in Ref. [15].

The numerical computations are elementary and therefore have been omitted. The membership functions of the resulting sets $\hat{A}_E$, $\hat{A}_F$ and $\hat{A}_J$ are represented in Figs 1–3, respectively, by some level lines.

The obtained surfaces are similar in shape, but different in height. The similarity is not surprising in the considered all-clear problem. The smaller heights of m_E and m_F are due to the fact that there are subsets of the x-axis, over which the observations have low or vanishing membership values. Modification of P would compensate for this. The robustness properties mentioned in the foregoing section would become visible, if we were to modify the fuzzy observations. An example of this kind can be found in Ref. [2].

6. CONCLUDING REMARKS

The concept of fuzzy data analysis with respect to functional relationships, as presented in the preceding sections, can be and will be developed in several directions in forthcoming papers. As an example, we will discuss a Bayes-like procedure. Let there be a prior valuation of the parameter by a fuzzy set A_P defined by the membership function m_P. The investigation according to one of the principles of Section 4 leads to an estimate $\hat{A}_L$ ($L = E, F, J$), with membership function m_L. Then

$$\min\{m_P(\mathbf{a}), m_L(\mathbf{a})\} \tag{27}$$

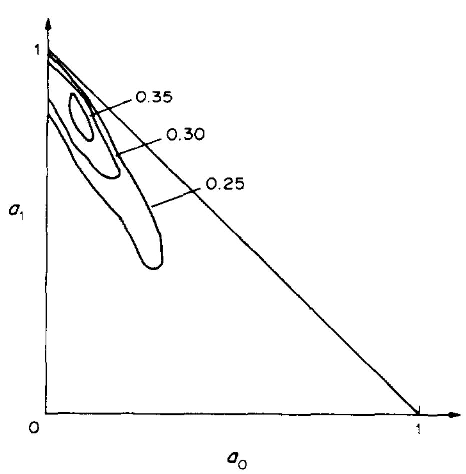

Fig. 1. Level lines of m_E.

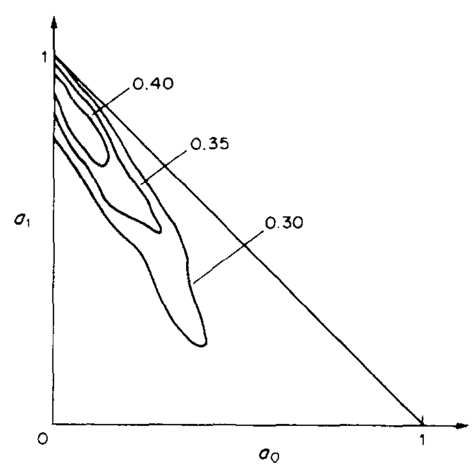

Fig. 2. Level lines of m_F.

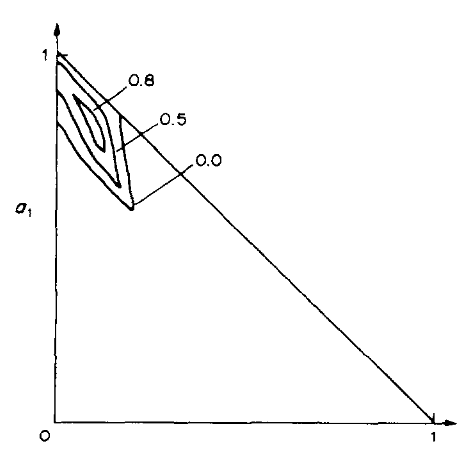

Fig. 3. Level lines of m_J.

evaluates the parameter simultaneously according to both fuzzy sets. Renormalization of expression (27) yields the so-called *a posteriori* valuation

$$m(\mathbf{a} \mid M) = \frac{\min\{m_P(\mathbf{a}), m_L(\mathbf{a})\}}{\sup\limits_{\mathbf{b}\in A} \min\{m_P(\mathbf{b}), m_L(\mathbf{b})\}} \tag{28}$$

when the fuzzy data M are taken into consideration, via m_L. This is a fuzzy analogue of the well-known Bayes formula: if we interpret the membership functions as possibility density functions, by formula (28) a possibilistic prior distribution is coupled with an actual possibility distribution on A, yielding possibilistic posterior information.

Remark. What is defined by formula (28) is, strictly speaking, the conditioning of a possibility measure by a fuzzy event. Note that in SMETS [16] a Bayes formula of similar structure is given, but there a possibilistic prior is combined with probabilistic sample information, resulting in a possibilistic posterior information.

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