ELSEVIER Discrete Applied Mathematics 56 (1995) 87–91

DISCRETE APPLIED MATHEMATICS

Note

Probabilities within optimal strategies for tournament games

David C. Fisher, Jennifer Ryan*

Department of Mathematics, Campus Box 170, University of Colorado at Denver, P.O. Box 173364, Denver, CO 80217-3364, USA

Received 26 October 1993; revised 28 March 1994

Abstract

Let T be a tournament. The *tournament game* on T is: Two *players independently pick a node. If both pick the same node, the game is tied. Otherwise, the player whose node is at the tail of the arc connecting the two nodes wins.* Fisher and Ryan showed that for any tournament T, the tournament game on T has a unique optimal strategy. If one node beats all others, the optimal strategy always picks that node. Otherwise, we show the probability that a node is picked in the optimal strategy is at most 1/3. We also find bounds on the minimum nonzero probability of a node in the optimal strategy.

0. Introduction

Fisher and Ryan [2, 3] studied a generalization of "Scissors, Paper and Stone" called *Tournament Games*. Given a tournament, two players independently pick a node. If both pick the same node, the game is tied. Otherwise, the player whose node is at the tail of the arc connecting the two nodes wins. A strategy is a vector of probabilities on the nodes, so the strategy $(x_1, x_2, \ldots, x_n)$ for an n node tournament dictates that node 1 is played with probability x_1, node 2 with probability x_2, etc. A strategy is *optimal* if it maximizes the expected winnings. Since the game is fair, both players have the same optimal strategy and the value of the game is 0.

Fisher and Ryan and Laffond et al. [5] independently showed that there is a unique optimal strategy for this game (see Fig. 1, where the nodes are labelled with the probability of playing that node in an optimal strategy).

*Corresponding author.

SSDI 0166-218X(94)00075-1

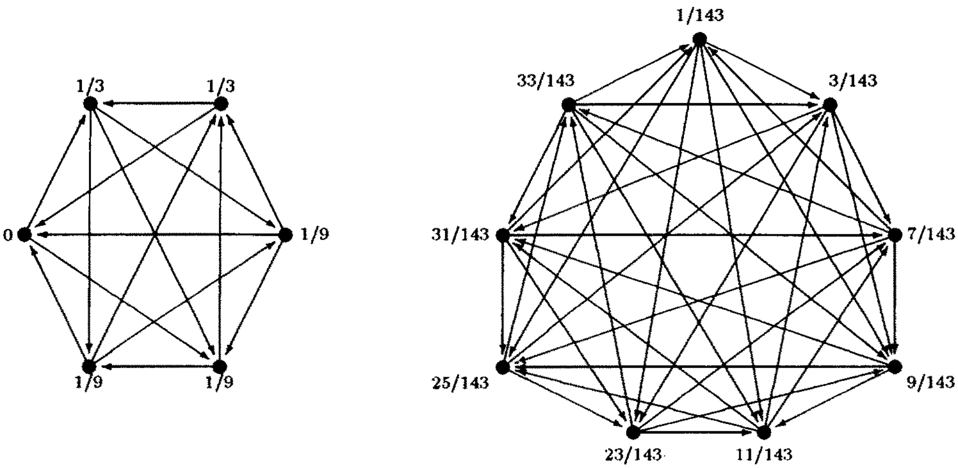

Fig. 1. Tournament games on various tournaments. In each of these games, two players simultaneously choose one of the nodes. If the nodes are the same, the game is a tie. Otherwise, the player picking the node at the tail of the arc connecting the selected node wins. The nodes are labelled with the probability of playing that node in an optimal strategy.

If one node beats all other nodes, then the optimal strategy is to always play that node. However if there is no such node, the opponent can defeat such a pure strategy by playing a node that beats the node which is always being played. So if each node has in-degree of 1 or more, the optimal strategy will be a mixed strategy with several nodes being played with positive probability. *What is the maximum noncertain probability of a node in the optimal strategy for a tournament game?* Section 1 answers this question.

Conversely, a node may not be used in the optimal strategy (Fisher and Reeves [1] showed that an average of half the nodes in the optimal strategy on a random tournament have probability zero). Sometimes a node will be used but its probability will be small. *What is the minimum nonzero probability for a node in the optimal strategy for a tournament game?* Section 2 addresses this question.

To answer these questions, we will use a matrix which differs from the usual tournament matrix. The *payoff matrix* $K(T)$ of an n node tournament T is defined by the $n \times n$ skew-symmetric matrix whose ij element is

$$k_{ij} = \begin{cases} 1 & \text{if } i \to j, \\ -1 & \text{if } j \to i, \\ 0 & \text{if } i = j. \end{cases}$$

Fisher and Ryan [2] showed that the vector of nonzero probabilities in an optimal strategy is the unique null vector of the payoff matrix of the corresponding subtournament of T.

1. The maximum noncertain probability

In Fig. 1, the largest probability in the optimal strategy for the left tournament is $\frac{1}{3}$, and for the right, it is $\frac{33}{143} \approx 0.23077$. *What is the largest possible probability less than one in an optimal strategy?* Theorem 1 shows the answer is $\frac{1}{3}$. The following simple lemma is given in [3].

Lemma (Fisher and Ryan [3]). *Let p be the optimal strategy of a tournament game on T and let i be a node with $p_i > 0$. Then $[K(T)p]_i = 0$.*

Theorem 1. *Let p_i be the probability of node i in an optimal strategy. Then either $p_i = 1$ or $p_i \leqslant \frac{1}{3}$.*

Proof. Partition the nodes except i into two subsets: let L be the nodes that lose to i, and B be the nodes that beat i. Let $l = \sum_{j \in L} p_j$ and $b = \sum_{j \in B} p_j$. Since L, B and i partition the nodes, we have $p_i + l + b = 1$. Also, by the lemma, if $p_i > 0$ then $l - b = [K(T)p]_i = 0$, i.e., $l = b$. Hence if $p_i \neq 1$, $l > 0$ and $b > 0$.

For any $j \in L$, let $r_j = \sum_{k \in L} K(T)_{jk} p_k$. Since $K(T)$ is skew-symmetric,

$$\sum_{j \in L} p_j r_j = \sum_{j \in L} p_j \sum_{k \in L} K(T)_{jk} p_k = \sum_{j \in L} \sum_{k \in L} K(T)_{jk} p_j p_k = 0.$$

Since $p_j > 0$ for some $j \in L$, we have that for one such j, $r_j < 0$. Then by the lemma

$$0 = [K(T)p]_j = -p_i + r_j + \sum_{k \in B} K_{jk} p_j \leqslant -p_i + \sum_{k \in B} p_j = -p_i + b.$$

Hence $b = l \geqslant p_i$. As $p_i + b + l = 1$, we can conclude $p_i \leqslant \frac{1}{3}$. □

Theorem 1 is exact in that for all odd numbers n, we can construct an n node tournament T whose optimal strategy uses all nodes and that has two nodes with probability $\frac{1}{3}$. Let T be the tournament where node 1 beats node 2, nodes i beat node 1 and loses to node 2 for all $i > 2$, and with a regular subtournament (a subtournament where each node has indegree equal to outdegree) on the nodes $3, 4, \ldots, n$. Then in the optimal strategy, nodes 1 and 2 have probability $\frac{1}{3}$, and node i has probability $1/(3(n-2))$ for all $i > 2$.

2. The minimum nonzero probability

We now address the question: *What is the minimum nonzero probability of a node in the optimal strategy on an n node tournament game?* Let a_n be the answer to the question. Fisher and Ryan [3] enumerated all fractional winners on 7 or less nodes. This showed that $a_1 = a_2 = 1$, $a_3 = a_4 = \frac{1}{3}$, $a_5 = a_6 = \frac{1}{9}$, and $a_7 = a_8 = \frac{1}{35}$.

Fisher and Ryan [4] studied L_n, the maximum value of the least common denominator of the probabilities in an optimal strategy for an n node tournament game. They showed that $L_n \leqslant n^{(n+1)/4}$. Since $a_n \geqslant 1/L_n$, we have $a_n \geqslant n^{-(n+1)/4}$. Further, an exhaustive search showed that $L_9 = 189$. However, in these tournaments, the smallest numerator is 9. The same exhaustive search shows that the tournament on the right side of Fig. 1 is one of the 9 node tournaments with a node of the smallest nonzero probability. So $a_9 = a_{10} = \frac{1}{149}$.

Theorem 2. *Let a_n denote the minimum possible nonzero probability in the optimal strategy of a tournament on n nodes. Then $a_{m+n-1} \leqslant a_m a_n$.*

Proof. Let R be an m node tournament with optimal strategy $\boldsymbol{x} = (x_1, x_2, \ldots, x_m)^{\mathrm{T}}$ so that $x_1 = a_m$. Let S be an n node tournament with optimal strategy $\boldsymbol{y} = (y_1, y_2, \ldots, y_m)^{\mathrm{T}}$ so that $y_1 = a_n$. Then create a tournament T on $m + n - 1$ nodes by replacing node 1 of S by the entire tournament R (so the nodes in R beat and lose to nodes in S according to how node 1 did). The optimal strategy for T is the vector $\boldsymbol{z} = (y_1 \boldsymbol{x}^{\mathrm{T}}, y_2, \ldots, y_n)^{\mathrm{T}}$. Thus the probability of node 1 of T is $a_m a_n$ and hence $a_{m+n-1} \leqslant a_m a_n$. □

A random search found an 11 node tournament game with a node with probability $\frac{1}{835}$ in the optimal strategy. Its payoff matrix and optimal strategy are

$$K(T) = \begin{bmatrix} 0 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 0 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \\ -1 & 1 & 0 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 0 & -1 & -1 & 1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 & 0 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 & 0 & 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 1 & 1 & -1 & 0 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 0 \end{bmatrix}$$

and Optimal Strategy $= \frac{1}{835}(1\ 53\ 61\ 61\ 69\ 81\ 83\ 95\ 97\ 111\ 123)^{\mathrm{T}}$.

The random search also revealed a 13 node tournament game with a node with probability $\frac{1}{4201}$ in the optimal strategy. Its payoff matrix and optimal strategy are

$$K(T)=\begin{bmatrix} 0 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & 0 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 0 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 0 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 & 0 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 & -1 & 1 & 0 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & 0 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 0 & 1 & 1 & -1 & -1 \\ -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & 0 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 0 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 0 \end{bmatrix}$$

and Optimal Strategy $= \frac{1}{4201}(1\ 15\ 69\ 85\ 315\ 353\ 395\ 397\ 401\ 499\ 523\ 563\ 585)^{\mathrm{T}}$.

The following table shows values for L_n and p that have been determined empirically.

n	1,2	3,4	5,6	7,8	9,10	11,12	13,14
L_n	1	3	9	35	183	$\geqslant 979$	$\geqslant 4635$
a_n	1	1/3	1/9	1/35	1/143	$\leqslant 1/835$	$\leqslant 1/4201$

References

[1] D.C. Fisher and R.B. Reeves, Optimal strategies for random tournament games, Linear Algebra Appl., to appear.

[2] D.C. Fisher and J. Ryan, Optimal strategies for a generalized "scissors, paper and stone" game, Amer. Math. Monthly 99 (1992) 935–942.

[3] D.C. Fisher and J. Ryan, Tournament games and positive tournaments, J. Graph Theory, to appear.

[4] D.C. Fisher and J. Ryan, Tournament games and condorcet voting, Linear Algebra Appl., to appear.

[5] G. Laffond, J.F. Laslier and M. Le Breton, The bipartisan set of a tournament game, Games Econom. Behavior 5 (1993) 182–201.