Discrete Mathematics 55 (1985) 181–184
North-Holland

INTERVAL GRAPHS AND SEARCHING

Lefteris M. KIROUSIS
Department of Mathematics, University of Patras, Patras, Greece, and Department of Computer Science, National Technical University, Athens, Greece

Christos H. PAPADIMITRIOU
Department of Mathematics, University of Patras, Patras, Greece, and Department of Computer Science, National Technical University, Athens, Greece

Received 3 November 1983
Revised 5 September 1984

The interval thickness of a graph G is the minimum clique number over any interval supergraph of G. The node-search number is the least number of searchers required to clear the 'contaminated' edges of a graph. The clearing is accomplished by concurrently having searchers on both of its endpoints.

We prove that for any graph, these two parameters coincide.

1. Introduction

The *searching game* was introduced by Parsons [5]. In the original version, sometimes called *edge searching*, an undirected graph G is considered as a system of tunnels in which a swift and cunning fugitive is hidden. The *search number* of G is defined as the least number of *searchers* (or pebbles) which guarantees the capture of the fugitive.

It was shown in [3] that there is always a strategy for searching G in which the least possible number of searchers is used and moreover no tunnel is searched twice. In other words *recontamination* does not help in searching a graph. This implies that the problem of computing the search number of a graph is in NP. It was shown in [4] that it is NP-complete, whereas it can be solved efficiently for trees.

In [2] a slightly different version of searching was introduced. In this new version, called *node searching*, the clearing of an edge takes place once both its endpoints simultaneously carry a searcher.

Formally, node searching is a one-player game played on an undirected graph G, using pebbles called searchers or guards. A *searching strategy* S is a sequence of moves where the player either places a searcher on a node of the graph that carries no searcher or deletes the searcher of a guarded node.

The edges of the graph are initially considered contaminated by a gas. The object of a searching strategy is to clear all edges. The clearing of an edge is accomplished once both its endpoints concurrently carry a searcher. A clear edge may be recontaminated once there appears a path that carries no searchers and that connects this edge with a contaminated one. The appearance of such an edge is due to deletion of searchers separating a contaminated from a clear edge.

The complexity measure of a searching strategy S is the maximum number of searchers that appear concurrently on the graph at any point. A strategy is called *optimal* if this number takes its least possible value. This least value is called the *node-search number* of G, and is denoted by $\mathrm{ns}(G)$.

In [2] it was proved that recontamination does not help in node-searching either. This is equivalent to asserting that for any graph there is an optimal node-searching strategy during which no node is visited twice by a searcher. It was also shown that the problem of computing $\mathrm{ns}(G)$ is NP-complete.

The interest in node-searching arises from its relation with certain other important graph parameters like, for example, the pebble demand or the vertex separator of a graph [2].

In this paper we show that, rather surprisingly, inverval thickness and node-search number coincide.

2. The main result

As is well known, an *interval graph* is one that has an interval model, that is a set of intervals of the real line, one for each vertex, such that two intervals intersect if and only if the corresponding nodes are adjacent.

Every graph G is a subgraph of an interval graph in a trivial way. We just consider the clique with the same number of nodes as G. In some sense, however, this is an undesirable answer, because the corresponding interval model is 'thick', with many intervals overlapping at the same point.

Definition. The *interval thickness* of a graph G, denoted by $\theta(G)$, is the smallest max-clique over all interval supergraphs of G.

For example, the 4-cycle has interval thickness three; actually, its node-search number is also three.

Theorem. *For any graph G,* $\mathrm{ns}(G) = \theta(G)$.

Before we give the proof of the theorem, we state and prove a lemma.

Lemma. *Let G be a graph and let S be a strategy of placing and deleting searchers from the nodes of the graph subject to the following conditions:*

(i) *A node accepts a searcher only once,*

(ii) *the deleting of a searcher from a vertex v takes place after the placing of searchers on all adjacent vertices of v.*

Under the above conditions, S is a recontamination-free node-searching strategy for G.

Proof. From condition (ii), it easily follows that every edge of G is cleared once. It remains to be proved that recontamination of a cleared edge never takes place. Suppose, to the contrary, that a recontamination occurs. That can only happen because at a step t_0 of S we removed a searcher from an endpoint of a contaminated edge e. But because of condition (ii), edge e was cleared at a step preceding t_0. This could only happen if at a step $t_1 < t_0$, we removed a searcher from an endpoint of a contaminated edge. The same argument can be repeated and, therefore, we obtain a strictly decreasing sequence $t_0 > t_1 > t_2 > \cdots$ of nonnegative integers, a contradiction. □

Proof of the theorem. Let us first prove that $\mathrm{ns}(G) \leqslant \theta(G)$. Let G' be an interval supergraph of G such that the maximum clique of G' has $\theta(G)$ elements. Without loss of generality we may suppose that the endpoints of the intervals that represent G' are nonnegative integers and the least of them is zero. Moreover, we may suppose that these intervals are closed, and that none of them and no intersection of them is a singleton. That is permissible, since we have the freedom of magnifying the intervals and moving their endpoints a little without destroying the intersection relations. We define now the following strategy for placing and deleting searchers from the nodes of G. If the corresponding interval of a node v is $[i, j]$, then place a searcher on v at time i and remove it at time j. Since two adjacent nodes of G are represented by intersecting intervals, the strategy we described satisfies the conditions of the lemma, and therefore, is a recontamination-free node-searching strategy for G. It remains to be proved that the maximum number of searchers it requires is $\leqslant \theta(G)$. From the definition of the searching strategy it follows that if two searchers appear concurrently on G, the corresponding intervals of the carrying nodes intersect, and therefore, these nodes are adjacent in G'. So, there can never be more than $\theta(G)$ searchers concurrently on G.

Let us come now to the converse, namely $\theta(G) \leqslant \mathrm{ns}(G)$. Let S be a recontamination-free node-searching strategy of G using the least possible number of searchers. We assign to every node v of G the interval $[i, j]$, where node v is visited by a searcher at the ith step of the strategy and this searcher is removed at the jth step. Since recontamination is not allowed, this interval is uniquely defined. Moreover, since S is a node-searching strategy, for any two adjacent nodes of G there must be a time when they concurrently carry a searcher. Consequently, the corresponding intervals intersect and, so, the interval graph we defined is a supergraph of G. It remains to be proved that its maximum clique has $\leqslant \mathrm{ns}(G)$ nodes. Consider any clique on this supergraph. The corresponding intervals intersect pairwise. But it is well known (and intuitively obvious) that if the elements of a finite set of intervals intersect pairwise, the intersection of all of them is not empty. Therefore, there must be an instant when all the nodes of this clique concurrently carry a searcher. Therefore, $\theta(G) \leqslant \mathrm{ns}(G)$. □

As an immediate corollary to the above theorem we get, first, that the problem of computing the interval thickness of a graph is NP-complete, and, second, that the polynomial-time dynamic programming algorithm for checking whether the node-search number of a graph is less than a constant [1] applies also in the case of interval thickness.

References

[1] J.A. Ellis, I.H. Sudborough and J.S. Turner, Graph separation and search number, Preprint, Northwestern Univ. and Washington Univ. (1984).

[2] L.M. Kirousis and C.H. Papadimitriou, Searching and pebbling, Theoret. Comput. Sci., Submitted.

[3] A. LaPaugh, Technical Report, Princeton Univ. (1983).

[4] N. Megiddo, S.L. Hakimi, M.R. Garey, D.S. Johnson and C.H. Papadimitriou, The complexity of searching a graph (preliminary version), Proc. 22nd IEEE Found. Comp. Sci. Symp. (1981) 376–381.

[5] T.D. Parsons, Pursuit-evasion in a graph, in: Y. Alani and D.R. Lick, eds., Theory and application of graphs, (Springer, Berlin, 1976) 426–441.